UNIVERSITY OF TORONTO, DEPARTMENT OF ECE

ECE1762 - Homework 1

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1 PROBLEM I

Sort the following functions from asymptotically smallest to asymptotically largest.

$$2^{log_{10}n} \quad log_{lgn}n \quad lg(nlogn) \quad n \quad (\sqrt{2})^{lgn} \quad (lglgn)^{lglgn}$$

Solution:

We could find that

$$\lim_{n\to\infty}\frac{\sqrt{2}^{lgn}}{n} = \lim_{n\to\infty}\frac{n^{0.5}}{n} = \lim_{n\to\infty}\frac{1}{n^{0.5}} = 0$$

So

$$\sqrt{2}^{lgn} = o(n)$$

Similarly we have

$$\lim_{n\to\infty}\frac{2^{log_{10}n}}{\sqrt{2}^{lgn}} = \lim_{n\to\infty}\frac{n^{log_{10}2}}{n^{0.5}} = \lim_{n\to\infty}\frac{1}{n^{0.5-log_{10}2}}$$

since $0.5 - log_{10}2 > 0$

$$\lim_{n\to\infty}\frac{1}{n^{0.5-log_{10}2}} = 0$$

So

$$2^{log_{10}n} = o(\sqrt{2}^{lgn})$$

To compare $2^{log_{10}n}$ and $(lglgn)^{lglgn}$,
We apply $log()$ on both side
LHS:

$$lg(2^{log_{10}n}) = lg_{10}2lgn$$

RHS:

$$lg((lglgn)^{lglgn}) = lglgnlglglgn < (lglgn)^2$$

Then we have:

$$\lim_{n\to\infty} \frac{(lglgn)^2}{lg_{10}2lgn} = \lim_{n\to\infty} \frac{2lglgn}{log_{10}2ln2lgn} = \lim_{n\to\infty} \frac{2}{log_{10}2ln^2 2lgn} = 0$$

Then we have

$$(lglgn)^2 = o(2^{log_{10}n})$$

And $lglgnlglglgn < (lglgn)^2$, so

$$(lglgn)^{lglgn} = o(2^{log_{10}n})$$

To compare $(lglgn)^{lglgn}$ and $lg(nlogn)$,
We apply $log()$ on both side
LHS:

$$log(lg(nlogn)) = lg(lgn + lglgn) < lg(2lgn)$$

RHS:

$$log((lglgn)^{lglgn}) = lglgnlglglgn$$

Then we have:

$$\lim_{n\to\infty} \frac{lg(2lgn)}{lglgnlglglgn} = \lim_{n\to\infty} \frac{1+lglgn}{lglgnlglglgn} = \lim_{n\to\infty} \frac{1}{lglgnlglglgn} + \frac{1}{lglglgn} = 0$$

Then we have

$$lg(2lgn) = o(lglgnlglglgn)$$

And $lg(lgn + lglgn) < lg(2lgn)$, so

$$lg(nlogn) = o((lglgn)^{lglgn})$$

To compare $log_{lgn}n$ and $lg(nlogn)$,
LHS:

$$log_{lgn}n = \frac{lgn}{lglgn}$$

RHS:

$$lg(nlogn) = lgn + lglgn > lgn$$

Then we have:

$$\lim_{n\to\infty} \frac{\frac{lgn}{lglgn}}{lgn} = 0$$

So,

$$log_{lgn}n = \frac{lgn}{lglgn} = o(lgn)$$

Since,

$$lgn + lglgn > lgn$$

Therefore,

$$log_{lgn}n = o(lgn + lglgn) = o(lg(nlogn))$$

In conlusion,

$$n >> (\sqrt{2})^{lgn} >> 2^{log_{10}n} >> (lglgn)^{lglgn} >> lg(nlogn) >> log_{lgn}n$$

2 PROBLEM II

Show that any sequence of $n^3 + 1$ numbers contains either

- a strictly-increasing subsequence of length $n + 1$.
- a strictly-decreasing subsequence of length $n + 1$, or
- $n + 1$ elements with the same value.

Proof:
Let $a_1, a_2, ..., a_{n^3+1}$ be a sequence of $n^3 + 1$ distinct real numbers. Let $f(n)$ be a function such that $f(a_k) = (i_k, d_k, s_k)$ with i_k being the length of the longest increasing subsequence starting at a_k and d_k being the length of the longest decreasing subsequence starting at a_k and s_k being the lenght of longest identical subsequence starting at a_k.

Suppose that the longest length of i_k, d_k and s_k is n. Then each i_k, d_k and s_k must between 1 and n

This means there are n^3 possible $f(a_k) = (i_k, d_k, s_k)$. Since k ranges from 1 to $n^3 + 1$, by the PHP theorem at least two of the $f(a_k)$ must contain the same values.

This meas there must exist s < t such that $f(a_s) = f(a_t)$. So we have $i_s = i_t$, $d_s = d_t$ and $s_s = s_t$.
we have 3 cases:

- if $a_s < a_t$ then a_s could be added to the first of increasing sequence begin with a_t, which means $i_s \neq i_t$. - a contradiction
- if $a_s > a_t$ then a_s could be added to the first of decreasing sequence begin with a_t, which means $d_s \neq d_t$. - a contradiction
- if $a_s = a_t$ then a_s could be added to the identical sequence begin with a_t, which means $s_s \neq s_t$. - a contradiction

Therefore, there must be an increasing, decreasing or identical subsequence of length $n + 1$.

3 PROBLEM III

Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. Prove your answer. Assume reasonable but nontrivial base cases if none are supplied.

(a) $A(n) = 2A(n/4) + nloglogn$

(b) $B(n) = B(n/2) + logn$

(c) $C(n) = 3C(n/2) + nlogn$

(d) $F(n) = F(\lfloor logn \rfloor) + logn$

4 PROBLEM IV

m balls are thrown into n bins (independently) so that each ball is equally likely to fall into any of the bins. Estimate as precisely as you can the smallest number m (as a function of n) so that the probability of all balls falling into different bins is smaller than $\frac{1}{n^c}$, for a fixed constant $c > 0$.

Proof:

Suppose that x: all balls falling into different bins

$$\begin{aligned} Pr(x) &= \frac{P(n,m)}{n^m} \\ &= \frac{n(n-1)(n-2)...(n-m+1)}{n^m} \\ &= \frac{(n-1)(n-2)...(n-m+1)}{n^{m-1}} \\ &= \frac{\prod_{i=1}^{i=m-1} n-i}{n^{m-1}} \\ &\leq \frac{(n-1)^{m-1}}{n^{m-1}} \\ &= (1-\frac{1}{n})^{m-1} \end{aligned}$$

Since,

$$1-\beta \leq e^{-\beta}$$

we have

$$(1-\frac{1}{n})^{m-1} \leq e^{-\frac{m-1}{n}}$$

if

$$e^{-\frac{m-1}{n}} \leq \frac{1}{n^c}$$

then

$$e^{\frac{m-1}{n}} \geq n^c$$

Apply $ln()$ on both sides,

$$\frac{m-1}{n} \geq clnn$$

then we have,

$$m-1 \geq cnlnn$$

Therefore,

$$m \geq cnlnn + 1$$

5 Problem V

Give a combinatorial argument to prove that

$$\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$$

Solution:

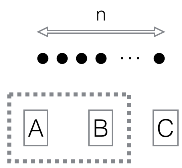

Figure 5.1: Problem 5

As shown in Figure 5.1, n balls are thrown into 3 bins name A, B and C. The number of ways for n balls falling into 3 bins is 3^n

Also, the number of ways for n balls not falling into bin C for i times is: $\binom{n}{i}2^i$. Then , we have the possibility of n balls not falling into bin C for i times is:

$$\frac{\binom{n}{i}2^i}{3^n}$$

As $0 \leq i \leq n$, the sum of possibilities of $\frac{\binom{n}{i}2^i}{3^n}$ from $i = 0$ to $i = n$ should be 1, as it contains all the possible results for throwing n balls into 3 bins.

Therefore,

$$\frac{\sum_{i=0}^{n}\binom{n}{i}2^i}{3^n} = 1$$

$$\sum_{i=0}^{n}\binom{n}{i}2^i = 3^n$$

6 Problem VI

Consider a light ray entering two adjacent planes of glass on a table. At any meeting surface (between the two planes of glass, or between the top glass and the air) the light may either reflect (bounce) or continue straight through (refract). In the example below, the light ray bounces 7 times before it leaves the glass plane. The light always reflects between the bottom glass and the table. How many different paths can a light ray take if it bounces n times before it leaves the top glass plane? Give a recurrence relation to answer this question.