UNIVERSITY OF TORONTO, DEPARTMENT OF ECE

ECE1762 - Homework 1

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1 PROBLEM I

Sort the following functions from asymptotically smallest to asymptotically largest.

$$2^{log_{10}n} \quad log_{lgn}n \quad lg(nlogn) \quad n \quad (\sqrt{2})^{lgn} \quad (lglgn)^{lglgn}$$

Solution:

We could find that

$$\lim_{n\to\infty}\frac{\sqrt{2}^{lgn}}{n} = \lim_{n\to\infty}\frac{n^{0.5}}{n} = \lim_{n\to\infty}\frac{1}{n^{0.5}} = 0$$

So

$$\sqrt{2}^{lgn} = o(n)$$

Similarly we have

$$\lim_{n\to\infty}\frac{2^{log_{10}n}}{\sqrt{2}^{lgn}} = \lim_{n\to\infty}\frac{n^{log_{10}2}}{n^{0.5}} = \lim_{n\to\infty}\frac{1}{n^{0.5-log_{10}2}}$$

since $0.5 - log_{10}2 > 0$

$$\lim_{n\to\infty}\frac{1}{n^{0.5-log_{10}2}} = 0$$

So

$$2^{log_{10}n} = o(\sqrt{2}^{lgn})$$

To compare $2^{log_{10}n}$ and $(lglgn)^{lglgn}$,
We apply $log()$ on both side
LHS:

$$lg(2^{log_{10}n}) = lg_{10}2lgn$$

RHS:

$$lg((lglgn)^{lglgn}) = lglgn \cdot lglglgn < (lglgn)^2$$

Then we have:

$$\lim_{n\to\infty} \frac{(lglgn)^2}{lg_{10}2lgn} = \lim_{n\to\infty} \frac{2lglgn}{log_{10}2ln2lgn} = \lim_{n\to\infty} \frac{2}{log_{10}2ln^2 2lgn} = 0$$

Then we have

$$(lglgn)^2 = o(2^{log_{10}n})$$

And $lglgn \cdot lglglgn < (lglgn)^2$, so

$$(lglgn)^{lglgn} = o(2^{log_{10}n})$$

To compare $(lglgn)^{lglgn}$ and $lg(nlogn)$,
We apply $log()$ on both side
LHS:

$$log(lg(nlogn)) = lg(lgn + lglgn) < lg(2lgn)$$

RHS:

$$log((lglgn)^{lglgn}) = lglgn \cdot lglglgn$$

Then we have:

$$\lim_{n\to\infty} \frac{lg(2lgn)}{lglgn \cdot lglglgn} = \lim_{n\to\infty} \frac{1 + lglgn}{lglgn \cdot lglglgn} = \lim_{n\to\infty} \frac{1}{lglgn \cdot lglglgn} + \frac{1}{lglglgn} = 0$$

Then we have

$$lg(2lgn) = o(lglgn \cdot lglglgn)$$

And $lg(lgn + lglgn) < lg(2lgn)$, so

$$lg(nlogn) = o((lglgn)^{lglgn})$$

To compare $log_{lgn}n$ and $lg(nlogn)$,
LHS:

$$log_{lgn}n = \frac{lgn}{lglgn}$$

RHS:

$$lg(nlogn) = lgn + lglgn > lgn$$

Then we have:

$$\lim_{n\to\infty} \frac{\frac{lgn}{lglgn}}{lgn} = 0$$

So,

$$log_{lgn}n = \frac{lgn}{lglgn} = o(lgn)$$

Since,

$$lgn + lglgn > lgn$$

Therefore,

$$log_{lgn}n = o(lgn + lglgn) = o(lg(nlogn))$$

In conlusion,

$$n >> (\sqrt{2})^{lgn} >> 2^{log_{10}n} >> (lglgn)^{lglgn} >> lg(nlogn) >> log_{lgn}n$$

2 PROBLEM II

Show that any sequence of $n^3 + 1$ numbers contains either

- a strictly-increasing subsequence of length $n + 1$.
- a strictly-decreasing subsequence of length $n + 1$, or
- $n + 1$ elements with the same value.

Proof:
Let $a_1, a_2, ..., a_{n^3+1}$ be a sequence of $n^3 + 1$ distinct real numbers. Let $f(n)$ be a function such that $f(a_k) = (i_k, d_k, s_k)$ with i_k being the length of the longest increasing subsequence starting at a_k and d_k being the length of the longest decreasing subsequence starting at a_k and s_k being the lenght of longest identical subsequence starting at a_k.

Suppose that the longest length of i_k, d_k and s_k is n. Then each i_k, d_k and s_k must between 1 and n

This means there are n^3 possible $f(a_k) = (i_k, d_k, s_k)$. Since k ranges from 1 to $n^3 + 1$, by the PHP theorem at least two of the $f(a_k)$ must contain the same values.

This meas there must exist s < t such that $f(a_s) = f(a_t)$. So we have $i_s = i_t$, $d_s = d_t$ and $s_s = s_t$.
we have 3 cases:

- if $a_s < a_t$ then a_s could be added to the first of increasing sequence begin with a_t, which means $i_s \neq i_t$. Which is a contradiction.
- if $a_s > a_t$ then a_s could be added to the first of decreasing sequence begin with a_t, which means $d_s \neq d_t$. Which is a contradiction.
- if $a_s = a_t$ then a_s could be added to the identical sequence begin with a_t, which means $s_s \neq s_t$. Which is a contradiction.

Therefore, there must be an increasing, decreasing or identical subsequence of length $n + 1$.

3 PROBLEM III

Solve the following recurrences. State tight asymptotic bounds for each function in the form $\Theta(f(n))$ for some recognizable function $f(n)$. Prove your answer. Assume reasonable but nontrivial base cases if none are supplied.

(a) $A(n) = 2A(n/4) + nloglogn$

(b) $B(n) = B(n/2) + logn$

(c) $C(n) = 3C(n/2) + nlogn$

(d) $F(n) = F(\lfloor logn \rfloor) + logn$

Solution:

(a) $A(n) = 2A(n/4) + nloglogn$

We can apply master theorem for $A(n)$

$A(n) = 2A(n/4) + nloglogn$. This is a divid-and-conquer recurrence with $a = 2$, $b = 4$, $f(n) = nloglogn$. $n^{log_b^a} = n^{0.5}$.

Since

$$f(n) = \Omega(n^{0.5+\epsilon})$$

and

$$af(\frac{n}{b}) = 2f(\frac{n}{4}) = \frac{n}{4}lglg\frac{n}{4} \le \frac{n}{2}lglgn$$

Then, $af(\frac{n}{b}) \le cf(n)$, when $\frac{1}{2} \le c < 1$. Which satisfied the condition in case 3 of master theorem. So, $A(n) = \Theta(f(n)) = \Theta(nlglgn)$

(b) $B(n) = B(n/2) + logn$

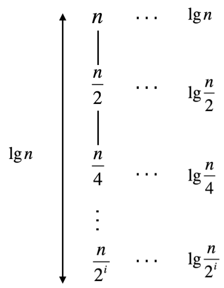


Figure 3.1: recursion tree of $B(n)$

The recurrsion tree of $B(n)$ is shown in Figure 3.1. We get a guess $B(n) = \Theta(lg^2 n)$

First we prove $B(n) = O(lg^2 n)$ part by induction. The inductive hypothesis is $B(\frac{n}{2}) \leq clog^2 \frac{n}{2}$ for some constant $c > 0$
Then we have

$$\begin{aligned} B(n) &\leq clog^2 \frac{n}{2} + logn \\ &= c[log^2 n - 2logn + 1] + logn \\ &= clog^2 n + c + (1-2c)logn \\ &\leq clog^2 n \end{aligned}$$

if $c + (1-2c)logn \leq 0$. This condition holds when $n \geq 2$ and $c \geq 1$. Thus $B(n) = O(log^2 n)$

For the lower bound, $B(n) = \Omega(log^2 n)$, we use inductive hypothesis that $B(\frac{n}{2}) \geq clog^2 \frac{n}{2}$ for some constant $c > 0$
Similarly we have

$$\begin{aligned} B(n) &\geq clog^2 \frac{n}{2} + logn \\ &= c[log^2 n - 2logn + 1] + logn \\ &= clog^2 n + c + (1-2c)logn \\ &\geq clog^2 n \end{aligned}$$

if $c + (1-2c)logn \geq 0$. This condition holds when $n \geq 1$ and $c = \frac{1}{4}$. So, $B(n) = \Omega(log^2 n)$

Thus, $B(n) = O(log^2 n)$ and $B(n) = \Omega(log^2 n)$. So we conclude that $B(n) = \Theta(log^2 n)$

(c) $C(n) = 3C(n/2) + nlogn$
We can apply master theorem for $C(n)$

$C(n) = 3A(n/2) + nlogn$. This is a divid-and-conquer recurrence with $a = 3$, $b = 2$, $f(n) = nlogn$. $n^{log_b a} = n^{log_2 3}$.

Since

$$f(n) = \Omega(n^{log_2 3 - \epsilon}), 0 < \epsilon \leq log_2 3 - 1$$

Which satisfied the condition in case 1 of master theorem. So $C(n) = \Theta(n^{log_b a}) = \Theta(n^{log_2 3})$

(d) $F(n) = F(\lfloor logn \rfloor) + logn$
Since $F(n) = F(\lfloor logn \rfloor) + logn$, $F(n) = \Omega(logn)$

Dismiss flooring we have

$$\begin{aligned} F(n) &= F(logn) + logn \\ F(lgn) &= F(lglgn) + lglgn \\ ... &= ... \\ F(lglg...lgn) &= F(lglg...lglgn) + lglg...lgn \end{aligned}$$

Thus we have:

$$\begin{aligned} F(n) &= lgn + lglgn + lglglgn + ... + lglg...lgn \\ &\leq clgn, wherec = log*(n) \end{aligned}$$

So, $F(n) = O(logn)$. Thus, $F(n) = O(logn)$ and $F(n) = \Omega(logn)$. So $F(n) = \Theta(logn)$

4 PROBLEM IV

m balls are thrown into n bins (independently) so that each ball is equally likely to fall into any of the bins. Estimate as precisely as you can the smallest number m (as a function of n) so that the probability of all balls falling into different bins is smaller than $\frac{1}{n^c}$, for a fixed constant $c > 0$.

Proof:

Suppose that x: all balls falling into different bins

$$\begin{aligned} Pr(x) &= \frac{P(n,m)}{n^m} \\ &= \frac{n(n-1)(n-2)...(n-m+1)}{n^m} \\ &= \frac{(n-1)(n-2)...(n-m+1)}{n^{m-1}} \\ &= \frac{\prod_{i=1}^{i=m-1} n-i}{n^{m-1}} \\ &\leq \frac{(n-1)^{m-1}}{n^{m-1}} \\ &= (1-\frac{1}{n})^{m-1} \end{aligned}$$

Since,

$$1-\beta \leq e^{-\beta}$$

we have

$$(1-\frac{1}{n})^{m-1} \leq e^{-\frac{m-1}{n}}$$

if

$$e^{-\frac{m-1}{n}} \leq \frac{1}{n^c}$$

then

$$e^{\frac{m-1}{n}} \geq n^c$$

Apply $ln()$ on both sides,

$$\frac{m-1}{n} \geq clnn$$

then we have,

$$m-1 \geq cnlnn$$

Therefore,

$$m \geq cnlnn+1$$

5 Problem V

Give a combinatorial argument to prove that

$$\sum_{i=0}^{n} \binom{n}{i} 2^i = 3^n$$

Solution:

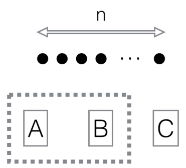


Figure 5.1: Picture of putting balls into bins

As shown in Figure 5.1, n balls are thrown into 3 bins name A, B and C. The number of ways for n balls falling into 3 bins is 3^n

Also, the number of ways for n balls not falling into bin C for i times is: $\binom{n}{i}2^i$. Then , we have the possibility of n balls not falling into bin C for i times is:

$$\frac{\binom{n}{i}2^i}{3^n}$$

As $0 \leq i \leq n$, the sum of possibilities of $\frac{\binom{n}{i}2^i}{3^n}$ from $i=0$ to $i=n$ should be 1, as it contains all the possible results for throwing n balls into 3 bins.

Therefore,

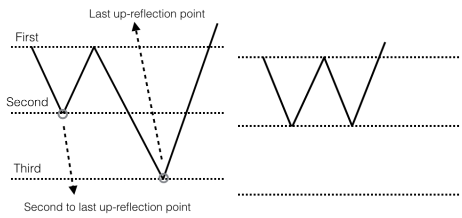
$$\frac{\sum_{i=0}^{n}\binom{n}{i}2^i}{3^n} = 1$$

$$\sum_{i=0}^{n}\binom{n}{i}2^i = 3^n$$

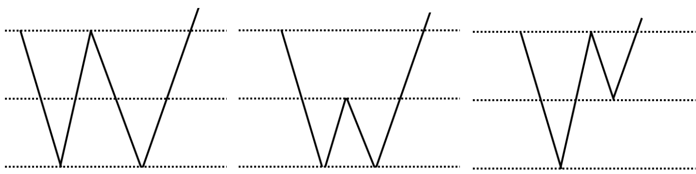
6 Problem VI

Consider a light ray entering two adjacent planes of glass on a table. At any meeting surface (between the two planes of glass, or between the top glass and the air) the light may either reflect (bounce) or continue straight through (refract). In the example below, the light ray bounces 7 times before it leaves the glass plane. The light always reflects between the bottom glass and the table. How many different paths can a light ray take if it bounces n times before it leaves the top glass plane? Give a recurrence relation to answer this question.

Solution:
According to the Figure of problem, we can notice that for any n is even, $F(n) = 0$.
If n is an odd, we will have following 5 basic paths below for the last two reflections before it leaves table:



(a) Path case 1 (b) Path case 2



(c) Path case 3 (d) Path case 4 (e) Path case 5

Figure 6.1: 5 basic paths for the last two reflections

As shown in Figure 6.1, we can note that if the second to last up-reflection point is on the second level from the top, there are two kinds possible paths to leave table as shown in Figure 6.1a and Figure 6.1b. We can also notice that if the second to last up-reflection point is on the second level from top, the last up-reflection point will have $\frac{1}{2}$ possiblity to be on the second level from top as shown in Figure 6.1a and $\frac{1}{2}$ possiblity to be on the third level from top as shown in Figure 6.1b. However, if the second to last up-reflection point is on the third level from top, there are three kinds of possible paths to leave table as shown in Figure 6.1c, Figure 6.1d and Figure 6.1e. Similarly, we could note that if the second to last up-reflection point is on the third level from top, the last up-reflection point will have $\frac{1}{3}$ possiblity to be on the second level from top as shown in Figure 6.1e and $\frac{2}{3}$ possiblity to be on the third level from top as shown in Figure 6.1c as well as Figure 6.1d.

Based on the discussion above, we could analysis possible paths for $F(n)$. To get the recurrence relation of $F(n)$, we need to analyze $F(n-2)$ and $F(n-4)$.

we could assume that there are p_1 numbers of paths in $F(n-4)$ whose last up-reflection point is on the third level from top and p_2 numbers of paths in $F(n-4)$ whose last up-reflection point is on the second level from top. Thus, we have:

$$F(n-4) = p_1 + p_2$$

Also, according to the disscussion and Figure 6.1 above, we have:

$$F(n-2) = 3p_1 + 2p_2$$

$$\begin{aligned} F(n) &= 3[(3p_1)\cdot\frac{2}{3} + (2p_2)\cdot\frac{1}{2}] + [(3p_1)\cdot\frac{1}{3} + (2p_2)\cdot\frac{1}{2}] \\ &= 8p_1 + 5p_2 \end{aligned}$$

Based one this three equations, we can get $p_1 = F(n-2) - 2F(n-4)$, $p_2 = 3F(n-4) - F(n-2)$. Thus, we have

$$F(n) = \begin{cases} 0 & n \text{ is even} \\ 3F(n-2) - F(n-4) & n \text{ is odd} \end{cases}$$