Formula sheet - Quantum Mechanics

Table 1: The first few radial wave functions for hydrogen, $R_{nl}(r)$.

$R_{10} = 2a^{-3/2}\exp(-r/a)$
$R_{20} = \frac{1}{\sqrt{2}}a^{-3/2}\left(1-\frac{1}{2}\frac{r}{a}\right)\exp(-r/2a)$ $R_{21} = \frac{1}{\sqrt{24}}a^{-3/2}\frac{r}{a}\exp(-r/2a)$
$R_{30} = \frac{2}{\sqrt{27}}a^{-3/2}\left(1-\frac{2}{3}\frac{r}{a}+\frac{2}{27}\left(\frac{r}{a}\right)^2\right)\exp(-r/3a)$ $R_{31} = \frac{8}{27\sqrt{6}}a^{-3/2}\left(1-\frac{1}{6}\frac{r}{a}\right)\left(\frac{r}{a}\right)\exp(-r/3a)$ $R_{32} = \frac{4}{81\sqrt{30}}a^{-3/2}\left(\frac{r}{a}\right)^2\exp(-r/3a)$
$R_{40} = \frac{1}{4}a^{-3/2}\left(1-\frac{3}{4}\frac{r}{a}+\frac{1}{8}\left(\frac{r}{a}\right)^2-\frac{1}{192}\left(\frac{r}{a}\right)^3\right)\exp(-r/4a)$ $R_{41} = \frac{\sqrt{5}}{16\sqrt{3}}a^{-3/2}\left(1-\frac{1}{4}\frac{r}{a}+\frac{1}{80}\left(\frac{r}{a}\right)^2\right)\frac{r}{a}\exp(-r/4a)$ $R_{42} = \frac{1}{64\sqrt{5}}a^{-3/2}\left(1-\frac{1}{12}\frac{r}{a}\right)\left(\frac{r}{a}\right)^2\exp(-r/4a)$ $R_{43} = \frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3\exp(-r/4a)$

Table 2: The first few spherical harmonics, $Y_l^m(\theta,\phi)$

$Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$	$Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$
$Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$	$Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2}\left(5\cos^3\theta-3\cos\theta\right)$
$Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$	$Y_3^{\pm 1} = \mp\left(\frac{21}{64\pi}\right)^{1/2}\sin\theta\left(5\cos^2\theta-1\right)e^{\pm i\phi}$
$Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2}\left(3\cos^2\theta-1\right)$	$Y_3^{\pm 2} = \left(\frac{105}{32\pi}\right)^{1/2}\sin^2\theta\cos\theta e^{\pm 2i\phi}$
$Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi}$	$Y_3^{\pm 3} = \mp\left(\frac{35}{64\pi}\right)^{1/2}\sin^3\theta e^{\pm 3i\phi}$

$$Y_l^m(\theta,\phi) = \epsilon\sqrt{\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}}e^{im\phi}P_l^m(\cos\theta),$$
$$\epsilon = (-1)^m \text{ for } m \geq 0 \text{ and } \epsilon = 1 \text{ for } m \leq 0$$

$$\int_0^{2\pi}\int_0^{\pi}\left[Y_l^m(\theta,\phi)\right]^*\left[Y_{l'}^{m'}(\theta,\phi)\right]\sin\theta\, d\theta\, d\phi = \delta_{ll'}\delta_{mm'}$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}, \qquad a = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = \frac{\hbar}{\alpha m_e c}$$

$$E_n = -\frac{m_e e^4}{2\left(4\pi\epsilon_0\right)^2\hbar^2 n^2} = \frac{E_1}{n^2} = -\frac{\alpha^2 mc^2}{2n^2} = \frac{-\hbar^2}{2ma^2n^2} \quad (n = 1,2,3,\ldots)$$

The Harmonic Oscillator

$$\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p}+m\omega x) \qquad [\hat{a}_-,\hat{a}_+] = 1$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_+ + \hat{a}_-) \qquad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}_+ - \hat{a}_-)$$

$$\hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1} \qquad \hat{a}_-\psi_n = \sqrt{n}\psi_{n-1}$$

$$\psi_n = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\psi_0 \qquad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right) E_n = \left(n+\frac{1}{2}\right)\hbar\omega$$

Nondegenerate Pertubation Theory

$$E_n^1 = \left\langle\psi_n^0\left|H'\right|\psi_n^0\right\rangle \qquad \psi_n^1 = \sum_{m\neq n}\frac{\left\langle\psi_m^0\left|H'\right|\psi_n^0\right\rangle}{(E_n^0-E_m^0)}\psi_m^0$$

$$E_n^2 = \sum_{m\neq n}\frac{\left|\left\langle\psi_m^0\left|H'\right|\psi_n^0\right\rangle\right|^2}{(E_n^0-E_m^0)}$$

Degeneracy

$$\begin{pmatrix} W_{aa} & W_{ab} \\ W_{ba} & W_{bb}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = E^1\begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \quad W_{ij} = \left\langle\psi_i^0\left|H'\right|\psi_j^0\right\rangle$$

Time-Dependent Perturbation Theory

$$c_f(t) = \delta_{fi} - \frac{i}{\hbar}\int_0^t dt' e^{i\left(E_f^{(0)}-E_i^{(0)}\right)t'/\hbar}\left\langle E_f^{(0)}\left|H_1(t')\right|E_i^{(0)}\right\rangle + \cdots$$

General formulae

$$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\left\langle[\hat{H},\hat{Q}]\right\rangle + \left\langle\frac{\partial Q}{\partial t}\right\rangle \qquad i\hbar\frac{\partial\Psi}{\partial t} = \hat{H}\Psi$$

$$\sigma_A\sigma_B \geq \left|\frac{1}{2i}\left\langle[\hat{A},\hat{B}]\right\rangle\right| \qquad [x,\hat{p}] = i\hbar \qquad \hat{\mathbf{p}} = -i\hbar\nabla$$

$$[\hat{L}_x,\hat{L}_y] = i\hbar\hat{L}_z, \quad [\hat{L}_y,\hat{L}_z] = i\hbar\hat{L}_x, \quad [\hat{L}_z,\hat{L}_x] = i\hbar\hat{L}_y$$

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\hat{L}^2 f_l^m = \hbar^2 l(l+1)f_l^m, \quad \hat{L}_z f_l^m = \hbar m f_l^m$$

$$l = 0,\ 1/2,\ 1,\ 3/2$$
$$m = -l,\ -(l-1),\ \ldots,\ (l-1),\ l$$

$$\hat{H} = -\mu\cdot\mathbf{B}$$

Position & momentum space transformations

$$\Phi(p,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{-ipx/\hbar}\Psi(x,t)\,dx$$
$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}e^{ipx/\hbar}\Phi(p,t)\,dp$$

Ehrenfest's theorem

$$\langle p\rangle = m\frac{d\langle x\rangle}{dt}, \quad \frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle$$

Integral formulas

$$\int_0^\infty x^n e^{-x/a}\,dx = a^{n+1}n! \qquad \begin{array}{l}\int\delta(x-a)\,dx = 1\\ \int f(x)\delta(x-a)\,dx = f(a)\end{array}$$