Monte Carlo Integration—Theory

Max Turgeon

18/09/2020

Antithetic variables

As discussed in class, the usefulness of antithetic variables is predicated on the idea that monotone functions preserve the negative correlation of variables. Let's prove this.

Theorem

Let f, g be increasing functions and X a random variable. Assuming all expectations exist, we have

$$E(f(X)g(X)) \geq E(f(X))E(g(X)).$$

Proof

Let $x, y \in \mathbb{R}$. Since f, g are increasing, both $f(x) - f(y)$ and $g(x) - g(y)$ have the same sign (positive or negative), and therefore

$$\left(f(x) - f(y)\right)\left(g(x) - g(y)\right) \geq 0.$$

From this, we immediately get that

$$E\left[\left(f(X) - f(Y)\right)\left(g(X) - g(Y)\right)\right] \geq 0,$$

where Y is independent of X and follows the same distribution.

We can expand the product and use the linearity of the expectation to get:

$$E\left[f(X)g(X)\right] + E\left[f(Y)g(Y)\right] \geq E\left[f(X)g(Y)\right] + E\left[f(Y)g(X)\right].$$

Since X, Y have the same distribution, we have

$$E\left[f(X)g(X)\right] + E\left[f(Y)g(Y)\right] = 2E\left[f(X)g(X)\right].$$

And since X, Y are independent, we have

$$\begin{aligned} E\left[f(X)g(Y)\right] + E\left[f(Y)g(X)\right] &= E\left[f(X)\right]E\left[g(Y)\right] + E\left[f(Y)\right]E\left[g(X)\right] \\ &= E\left[f(X)\right]E\left[g(X)\right] + E\left[f(X)\right]E\left[g(X)\right] \\ &= 2E\left[f(X)\right]E\left[g(X)\right]. \end{aligned}$$

Putting all this together, we get

$$E(f(X)g(X)) \geq E(f(X))E(g(X)),$$

which is what we wanted to show. $\square$

Corrolary

If f is monotone and $U \sim U(0,1)$, then $f(U)$ and $f(1-U)$ are negatively correlated.

Proof

Assume f is increasing. Then $g(x) = -f(1-x)$ is also increasing. Using the Theorem above, we have

$$\begin{aligned} E(f(U)g(U)) \geq E(f(U))E(g(U)) &\Rightarrow E(-f(U)f(1-U)) \geq -E(f(U))E(-f(1-U)) \\ &\Rightarrow E(f(U)f(1-U)) \leq E(f(U))E(f(1-U)). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \operatorname{Cov}(f(U), f(1-U)) &= E(f(U)f(1-U)) - E(f(U))E(f(1-U)) \\ &\leq 0. \end{aligned}$$

Finally, if f is decreasing, repeat the comptutations above by replacing f with $-f$. $\square$