Generating Random Variates—Theory

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Inverse-Transform method

The first method we discussed in the lecture is based on transforming a uniform variate by using the quantile function. The success of this method relies on the following theorem:

Theorem

Let X be a random variable with CDF F_X. Let $U \sim Unif(0,1)$ be uniform on the interval $(0,1)$. Then $Y = F_X^{-1}(U)$ has the same distribution as X.

Proof

Recall the definition of the *quantile function* F_X^{-1}:

$$F_X^{-1}(u) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq u\}.$$

Fix $u \in (0,1)$. Because $F_X^{-1}(u)$ is an infimum for the set $\{x \in \mathbb{R} \mid F_X(x) \geq u\}$, we know that

$$F_X\left(F_X^{-1}(u)\right) \geq u.$$

Conversely, for $x \in \mathbb{R}$, we have

$$F_X^{-1}\left(F_X(x)\right) = \inf\{y \in \mathbb{R} \mid F_X(y) \geq F_X(x)\}.$$

But clearly $x \in \{y \in \mathbb{R} \mid F_X(y) \geq F_X(x)\}$, so it must be greater than the infimum: $F_X^{-1}\left(F_X(x)\right) \leq x$.

Therefore, if we fix $x \in \mathbb{R}$, we can set up an equality between two sets:

$$\left\{u \mid F_X^{-1}(u) \leq x\right\} = \left\{u \mid F_X(x) \geq u\right\}.$$

By taking the probability of each set with respect to the distribution of U, we get an equality of probability statements:

$$P(F_X^{-1}(U) \leq x) = P(U \leq F_X(x)) = F_X(x),$$

where the last equality follows from the definition of uniform distribution. $\square$

Accept-Reject algorithm

Another method that was discussed is the *accept-reject* method. The idea is to sample from an "easy" distribution Y, and then decide if we accept or reject that proposal as a sample from our distribution of interest X. It may not be clear at first why it works, so let's look at a proof.

Theorem

Let X have density f and Y have density g.[1] Further suppose that there exists a constant $c > 1$ such that

$$\frac{f(t)}{g(t)} \leq c$$

for all t such that $f(t) > 0$. The algorithm described in the notes produces a variable X distributed according to f.

Proof

Recall from the lecture: we need to sample y from g, $u \sim Unif(0,1)$, and compute the ratio $r := \frac{f(y)}{cg(y)}$. If $u < r$, we accept the sample. Therefore, we need to compute our probabilities *conditional on* $U < r$. More precisely, we want to show:

$$P\left(Y \leq x \mid U < \frac{f(Y)}{cg(Y)}\right) = P(X \leq x).$$

Note that we have

$$P\left(Y \leq x \mid U < \frac{f(Y)}{cg(Y)}\right) = \frac{P\left(Y \leq x, U < \frac{f(Y)}{cg(Y)}\right)}{P\left(U < \frac{f(Y)}{cg(Y)}\right)}.$$

We will compute both probabilities as double integrals (note that the probabilities are with respect to the joint distribution (U, Y)). First, we have

$$\begin{aligned}
P\left(Y \leq x, U < \frac{f(Y)}{cg(Y)}\right) &= \int_{-\infty}^{x} \int_{0}^{\frac{f(y)}{cg(y)}} 1 \cdot g(y) du dy \\
&= \int_{-\infty}^{x} \frac{f(y)}{cg(y)} g(y) dy \\
&= \frac{1}{c} \int_{-\infty}^{x} f(y) dy \\
&= \frac{1}{c} P(X \leq x).
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
P\left(Y \leq x, U < \frac{f(Y)}{cg(Y)}\right) &= \int_{-\infty}^{\infty} \int_{0}^{\frac{f(y)}{cg(y)}} 1 \cdot g(y) du dy \\
&= \int_{-\infty}^{\infty} \frac{f(y)}{cg(y)} g(y) dy \\
&= \frac{1}{c} \int_{-\infty}^{\infty} f(y) dy \\
&= \frac{1}{c},
\end{aligned}$$

where the last equality follows from the fact that densities integrate to 1.

[1] We will assume that we have continuous distributions, but the proof works *mutatis mutandis* if we replace the densities by probability mass functions.

Putting this all together, we can see that

$$\frac{P\left(Y \leq x, U < \frac{f(Y)}{cg(Y)}\right)}{P\left(U < \frac{f(Y)}{cg(Y)}\right)} = P(X \leq x).$$

which is what we needed to prove. $\square$

From the proof, we see why the constant c can be *almost* arbitrary; but can you see where we used the assumption that c must uniformly bound the ratio of the densities? *Hint*: what is the support of the uniform random variable?