Test for Covariances

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STAT 7200–Multivariate Statistics

Objectives

- Review general theory of likelihood ratio tests
- Tests for structured covariance matrices
- Test for equality of multiple covariance matrices

Likelihood ratio tests i

- We will build our tests for covariances using likelihood ratios.
  - Therefore, we quickly review the asymptotic theory for regular models.
- Let $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ be a random sample from a density p_θ with parameter $\theta \in \mathbb{R}^d$.
- We are interested in the following hypotheses:

$$H_0 : \theta \in \Theta_0, \qquad H_1 : \theta \in \Theta_1,$$

where $\Theta_i \subseteq \mathbb{R}^d$.

Likelihood ratio tests ii

- Let $L(\theta) = \prod_{i=1}^n p_\theta(\mathbf{Y}_i)$ be the likelihood, and define the likelihood ratio

$$\Lambda = \frac{\max_{\theta \in \Theta_0} L(\theta)}{\max_{\theta \in \Theta_0 \cup \Theta_1} L(\theta)}.$$

- **Recall**: we reject the null hypothesis H_0 for small values of Λ.

Likelihood ratio tests iii

Theorem (Van der Wandt, Chapter 16)

Assume Θ_0, Θ_1 are *locally linear*. Under regularity conditions on p_θ, we have

$$-2 \log \Lambda \to \chi^2(k),$$

where k is the difference in the number of free parameters between the null model Θ_0 and the unrestricted model $\Theta_0 \cup \Theta_1$.

- Therefore, in practice, we need to count the number of free parameters in each model and hope the sample size n is large enough.

Tests for structured covariance matrices i

- We are going to look at several tests for structured covariance matrix.
- Throughtout, we assume $\mathbf{Y}_1, \ldots, \mathbf{Y}_n \sim N_p(\mu, \Sigma)$ with Σ positive definite.
    - Like other exponential families, the multivariate normal distribution satisfies the regularity conditions of the theorem above.
    - Being positive definite implies that the unrestricted parameter space is *locally linear*, i.e. we are staying away from the boundary where Σ is singular.

Tests for structured covariance matrices ii

- A few important observations about the unrestricted model:
  - The number of free parameters is equal to the number of entries on and above the diagonal of Σ, which is $p(p+1)/2$.
  - The sample mean $\bar{\mathbf{Y}}$ maximises the likelihood **independently of the structure of Σ**.
  - The maximised likelihood for the unrestricted model is given by

$$L(\hat{\mathbf{Y}}, \hat{\Sigma}) = \frac{\exp(-np/2)}{(2\pi)^{np/2}|\hat{\Sigma}|^{n/2}}.$$

Specified covariance structure i

- We will start with the simplest hypothesis test:

$$H_0 : \Sigma_0.$$

- Note that there is no free parameter in the null model.
- Write $V = n\hat{\Sigma}$. Recall that we have

$$L(\hat{\mathbf{Y}}, \Sigma) = (2\pi)^{-np/2}|\Sigma|^{-n/2} \exp\left(-\frac{1}{2}\mathrm{tr}(\Sigma^{-1}V)\right).$$

Specified covariance structure ii

- Therefore, the likelihood ratio is given by

$$\begin{aligned}\Lambda &= \frac{(2\pi)^{-np/2}|\Sigma_0|^{-n/2}\exp\left(-\frac{1}{2}\mathrm{tr}(\Sigma_0^{-1}V)\right)}{\exp(-np/2)(2\pi)^{-np/2}|\hat{\Sigma}|^{-n/2}}\\ &= \frac{|\Sigma_0|^{-n/2}\exp\left(-\frac{1}{2}\mathrm{tr}(\Sigma_0^{-1}V)\right)}{\exp(-np/2)|n^{-1}V|^{-n/2}}\\ &= \left(\frac{e}{n}\right)^{np/2}|\Sigma_0^{-1}V|^{n/2}\exp\left(-\frac{1}{2}\mathrm{tr}(\Sigma_0^{-1}V)\right).\end{aligned}$$

- In particular, if $\Sigma_0 = I_p$, we get

$$\Lambda = \left(\frac{e}{n}\right)^{np/2}|V|^{n/2}\exp\left(-\frac{1}{2}\mathrm{tr}(V)\right).$$

Test for sphericity i

- *Sphericity* means the different components of $\mathbf{Y}$ are **uncorrelated** and have the **same variance**.
  - In other words, we are looking at the following null hypothesis:

$$H_0 : \Sigma = \sigma^2 I_p, \qquad \sigma^2 > 0.$$

  - Note that there is one free parameter.
- We have

$$\begin{aligned} L(\hat{\mathbf{Y}}, \sigma^2 I_p) &= (2\pi)^{-np/2} |\sigma^2 I_p|^{-n/2} \exp\left(-\frac{1}{2}\mathrm{tr}((\sigma^2 I_p)^{-1} V)\right) \\ &= (2\pi\sigma^2)^{-np/2} \exp\left(-\frac{1}{2\sigma^2}\mathrm{tr}(V)\right). \end{aligned}$$

Test for sphericity ii

- Taking the derivative of the logarithm and setting it equal to zero, we find that $L(\hat{\mathbf{Y}}, \sigma^2 I_p)$ is maximised when

$$\widehat{\sigma^2} = \frac{\mathrm{tr}V}{np}.$$

- We then get

$$\begin{aligned} L(\hat{\mathbf{Y}}, \widehat{\sigma^2} I_p) &= (2\pi\widehat{\sigma^2})^{-np/2} \exp\left(-\frac{1}{2\widehat{\sigma^2}}\mathrm{tr}(V)\right) \\ &= (2\pi)^{-np/2} \left(\frac{\mathrm{tr}V}{np}\right)^{-np/2} \exp\left(-\frac{np}{2}\right). \end{aligned}$$

Test for sphericity iii

- Therefore, we have

$$\begin{aligned}\Lambda &= \frac{(2\pi)^{-np/2}\left(\frac{\mathrm{tr}V}{np}\right)^{-np/2}\exp\left(-\frac{np}{2}\right)}{\exp(-np/2)(2\pi)^{-np/2}|\hat{\Sigma}|^{-n/2}}\\ &= \frac{\left(\frac{\mathrm{tr}V}{np}\right)^{-np/2}}{|n^{-1}V|^{-n/2}}\\ &= \left(\frac{|V|}{(\mathrm{tr}V/p)^p}\right)^{n/2}.\end{aligned}$$

- Decompose $\mathbf{Y}_i$ into k blocks:

$$\mathbf{Y}_i = (\mathbf{Y}_{1i}, \ldots, \mathbf{Y}_{ki}),$$

where $\mathbf{Y}_{1i} \sim N_{p_k}(\mu_k, \Sigma_{kk})$ and $\sum_{j=1}^k p_j = p$.
- This induces a decomposition on Σ and V:

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \cdots & \Sigma_{1k} \\ \vdots & \ddots & \vdots \\ \Sigma_{k1} & \cdots & \Sigma_{kk} \end{pmatrix}, \qquad V = \begin{pmatrix} V_{11} & \cdots & V_{1k} \\ \vdots & \ddots & \vdots \\ V_{k1} & \cdots & V_{kk} \end{pmatrix}.$$

Test for independence ii

- We are interested in testing for independence between the different blocks $\mathbf{Y}_{1i}, \ldots, \mathbf{Y}_{ki}$. This equivalent to

$$H_0 : \Sigma = \begin{pmatrix} \Sigma_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Sigma_{kk} \end{pmatrix}.$$

    - Note that there are $\sum_{j=1}^{k} p_j(p_j + 1)/2$ free parameters.
- Under the null hypothesis, the likelihood can be decomposed into k likelihoods that can be maximised independently.

- This gives us

$$\begin{aligned}\max L(\hat{\mathbf{Y}}, \Sigma) &= \prod_{j=1}^k \frac{\exp(-np_j/2)}{(2\pi)^{np_j/2}|\widehat{\Sigma_{jj}}|^{n/2}} \\ &= \frac{\exp(-np/2)}{(2\pi)^{np/2}\prod_{j=1}^k|\widehat{\Sigma_{jj}}|^{n/2}}.\end{aligned}$$

- Putting this together, we conclude that

$$\Lambda = \left(\frac{|V|}{\prod_{j=1}^k|V_{jj}|}\right)^{n/2}.$$