Problem Set 2–STAT 7200

1. Prove the result on Slide 33 of the notes on the Multivariate Normal distribution.

2. Let $\mathbf{Y} = (Y_1, Y_2, Y_3)$ be a multivariate normal random vector with

$$\mu = (3, 0, 6), \qquad \Sigma = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{pmatrix},$$

and let $\mathbf{U} = (3Y_1 - 2Y_2 + Y_3, Y_2 - 2Y_3)$.

   (a) Write $\mathbf{U} = (U_1, U_2)$ as $\mathbf{U} = A\mathbf{Y}$ for a suitable matrix A.
   (b) Find the distribution of $\mathbf{U}$.
   (c) Find a two-dimensional vector $\mathbf{w} = (w_1, w_2)$ such that

$$Y_2, \qquad Y_2 - \mathbf{w}^T \begin{pmatrix} Y_1 \\ Y_3 \end{pmatrix}$$

   are jointly independent.
   (d) Find the conditional distribution of Y_3 given $Y_1 = 3$ and $Y_2 = 1$.

3. Let Y_1 be univariate standard normal $N(0,1)$, and let

$$Y_2 = \begin{cases} -Y_1 & -1 \le Y_1 \le 1, \\ Y_1 & \text{otherwise.} \end{cases}$$

Show that

   (a) Y_2 also follows a standard normal distribution;
   (b) (Y_1, Y_2) does *not* follow a bivariate normal distribution.

4. Let $\mathbf{Y}$ be a random vector defined by

$$\mathbf{Y} = X\beta + Z\mathbf{B} + \mathbf{E},$$

where X is $p \times q$, Z is $p \times r$, both are non-random; β is a q-dimensional parameter vector; and $\mathbf{B} \sim N_r(0, \Omega)$, $\mathbf{E} \sim N_p(0, \sigma^2 I_p)$, and both are independent. Show that

(a) $\mathbf{Y} \sim N_p(X\beta, \Sigma)$, where $\Sigma = Z\Omega Z^T + \sigma^2 I_p$.

(b) $\begin{pmatrix} \mathbf{Y} \\ \mathbf{B} \end{pmatrix} \sim N_{p+r}\left(\begin{pmatrix} X\beta \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma & Z\Omega \\ \Omega Z^T & \Omega \end{pmatrix}\right)$.

(c) $E(\mathbf{B} \mid \mathbf{Y}) = \Omega Z \Sigma^{-1} \left(\mathbf{Y} - X\beta\right)$.

(d) $\mathbf{Y} \mid \mathbf{B} \sim N_p(X\beta + Z\mathbf{B}, \sigma^2 I_p)$.

5. Let $\mathbf{Y} \sim N_p(\mu, \Sigma)$. Compute the characteristic function of $\mathbf{Y}^T A \mathbf{Y}$, where A is a non-random matrix.

6. Let $\mathbf{Y}$ be such that $E(\mathbf{Y}) = \mu$ and $\mathrm{Cov}(\mathbf{Y}) = \Sigma$. Show that

$$\min_{\mathbf{c}} E\left((\mathbf{Y} - \mathbf{c})^T (\mathbf{Y} - \mathbf{c})\right) = \mathrm{tr}\Sigma,$$

and that the minimum is attained at $\mathbf{c} = \mu$.

7. Assume $\mathbf{Y} \sim t_{p,\nu}(\mu, \Lambda)$ follows a multivariate t distribution. Let $\mathbf{Y}$ be partitioned as $\mathbf{Y} = \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix}$, with $\mathbf{Y}_i$ of dimension p_i and $p = p_1 + p_2$. Demonstrate the following:

(a) The expected value is $E(\mathbf{Y}) = \mu$, and the covariance is $\mathrm{Cov}(\mathbf{Y}) = [\nu/(\nu - 2)]\Lambda$, $\nu > 2$.

(b) The quadratic form $p^{-1}(\mathbf{Y} - \mu)^T \Lambda^{-1} (\mathbf{Y} - \mu)$ follows an F distribution $F(p, \nu)$.

(c) The marginal distribution is $\mathbf{Y}_2 \sim t_{p_2,\nu}(\mu_2, \Lambda_{22})$, where

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix},$$

and

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}.$$