Multivariate Random Variables

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STAT 7200–Multivariate Statistics

Joint distributions

- Let X and Y be two random variables.
- The *joint distribution function* of X and Y is

$$F(x, y) = P(X \leq x, Y \leq y).$$

- More generally, let $Y_1, \ldots, Y_p$ be p random variables. Their *joint distribution function* is

$$F(y_1, \ldots, y_p) = P(Y_1 \leq y_1, \ldots, Y_p \leq y_p).$$

Joint densities

- If F is absolutely continuous almost everywhere, there exists a function f called the *density* such that

$$F(y_1, \ldots, y_p) = \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_p} f(u_1, \ldots, u_p) du_1 \cdots du_p.$$

- The *joint moments* are defined as follows:

$$E(Y_1^{n_1} \cdots Y_p^{n_p}) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_1^{n_1} \cdots u_p^{n_p} f(u_1, \ldots, u_p) du_1 \cdots du_p.$$

- **Exercise**: Show that this is consistent with the univariate definition of $E(Y_1^{n_1})$, i.e. $n_2 = \cdots = n_p = 0$.

Marginal distributions i

- From the joint distribution function, we can recover the *marginal distributions*:

$$F_i(x) = \lim_{\substack{y_j \to \infty \\ j \neq i}} F(y_1, \ldots, y_p).$$

- More generally, we can find the joint distribution of a subset of variables by sending the other ones to infinity:

$$F(y_1, \ldots, y_r) = \lim_{\substack{y_j \to \infty \\ j > r}} F(y_1, \ldots, y_p), \quad r < p.$$

Marginal distributions ii

- Similarly, from the joint density function, we can recover the *marginal densities*:

$$f_i(x) = \int_{-\infty}^{\infty} f(u_1, \ldots, u_p) du_1 \cdots \widehat{du_i} \cdots du_p.$$

- In other words, we are integrating *out* the other variables.

Example i

- Let $R = [a_1, b_1] \times \cdots \times [a_p, b_p] \subseteq \mathbb{R}^p$ be a hyper-rectangle, with $a_i < b_i$, for all i.
- If $\mathbf{Y} = (Y_1, \ldots, Y_p)$ is **uniformly distributed** on R, then its density is given by

$$f(y_1, \ldots, y_p) = \begin{cases} \prod_{i=1}^p \frac{1}{b_i - a_i} & (y_1, \ldots, y_p) \in R, \\ 0 & \text{else.} \end{cases}$$

- For convenience, we can also use the indicator function:

$$f(y_1, \ldots, y_p) = \prod_{i=1}^p \frac{I_{[a_i, b_i]}(y_i)}{b_i - a_i}.$$

Example ii

- We then have

$$
\begin{aligned}
F(y_1, \ldots, y_p) &= \int_{-\infty}^{y_1} \cdots \int_{-\infty}^{y_p} f(u_1, \ldots, u_p) du_1 \cdots du_p \\
&= \prod_{i=1}^{p} \left(\frac{y_i - a_i}{b_i - a_i} I_{[a_i, b_i]}(y_i) + I_{[b_i, \infty)}(y_i) \right).
\end{aligned}
$$

- Finally, note that we recover the *univariate* uniform distribution by sending all components but one to infinity:

$$
F_i(x) = \lim_{\substack{y_j \to \infty \\ j \neq i}} F(y_1, \ldots, y_p) = \frac{x - a_i}{b_i - a_i} I_{[a_i, b_i]}(x) + I_{[b_i, \infty)}(x).
$$

Introduction to Copulas i

- **Copula theory** provides a general and powerful way to model general multivariate distributions.
- The main idea is that we can decouple (and recouple) the *marginal* distributions and the *dependency structure* between each component.
    - Copulas capture this dependency structure.
    - Sklar's theorem tells us about how to combine the two.

Introduction to Copulas ii

Definition
A p-dimensional copula is a function $C : [0,1]^p \to [0,1]$ that arises as the distribuction function (CDF) of a random vector whose marginal distributions are all uniform on the interval $[0,1]$.

In particular, we have

$$C(1, \ldots, u_i, \ldots, 1) = u_i, \qquad u_i \in [0,1].$$

Introduction to Copulas iii

Probability integral transform
If Y is a continuous (univariate) random variable with CDF F_Y, then

$$F_Y(Y) \sim U(0,1).$$

Proof

$$\begin{aligned} P(F_Y(Y) \leq x) &= P(Y \leq F_Y^{-1}(x)) \\ &= F_Y(F_Y^{-1}(x)) \\ &= x. \end{aligned}$$

$\square$

Sklar's Theorem i

- Using the Probability integral transform, we can prove one part of Sklar's theorem.
- More precisely, let $\mathbf{Y} = (Y_1, \ldots, Y_p)$ be a continuous random vector with CDF F, and let $F_1, \ldots, F_p$ be the CDFs of the marginal distributions.
- We know that $F_1(Y_1), \ldots, F_p(Y_p)$ are uniformly distributed on $[0, 1]$, and therefore the CDF of their joint distribution is a copula C.

Sklar's Theorem ii

$$
\begin{aligned}
C(u_1, \ldots, u_p) &= P(F_1(Y_1) \leq u_1, \ldots, F_p(Y_p) \leq u_p) \\
&= P(Y_1 \leq F_1^{-1}(u_1), \ldots, Y_p \leq F_p^{-1}(u_p)) \\
&= F(F_1^{-1}(u_1), \ldots, F_p^{-1}(u_p)).
\end{aligned}
$$

- By taking $u_i = F_i(y_i)$, we get

$$F(y_1, \ldots, y_p) = C(F_1(y_1), \ldots, F_p(y_p)).$$

Sklar's Theorem iii

Theorem
Let $\mathbf{Y} = (Y_1, \ldots, Y_p)$ be any random vector with CDF F, and let $F_1, \ldots, F_p$ be the CDFs of the marginal distributions. There exist a copula C such that

$$F(y_1, \ldots, y_p) = C(F_1(y_1), \ldots, F_p(y_p)). \tag{1}$$

If the marginal distributions are absolutely continuous, then C is unique.

Conversely, given a copula C and univariate CDFs $F_1, \ldots, F_p$, then Equation 1 defines a valid CDF for a p-dimensional random vector.

Examples i

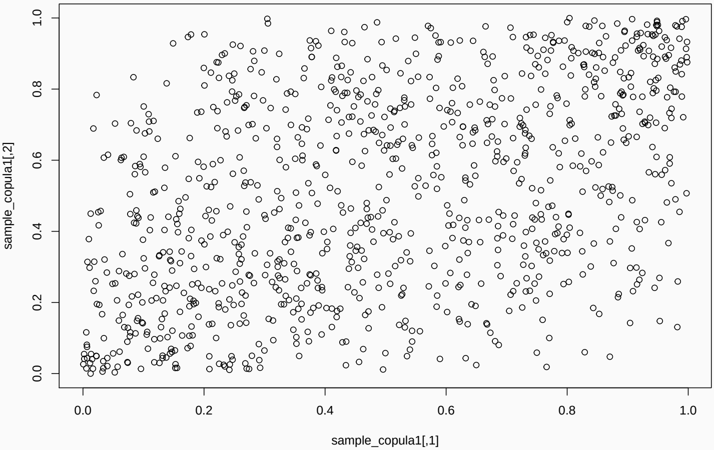
- **Gaussian copulas**: Let Φ be the CDF of the standard univariate normal distribution, and let Φ_Σ be the CDF of multivariate normal distribution with mean 0 and covariance matrix Σ. The Gaussian copula C_G is defined as

$$C_G(u_1, \ldots, u_p) = \Phi_\Sigma(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_p)).$$

Examples ii

```r
library(copula)

# Gaussian copula where correlation is 0.5
gaus_copula <- normalCopula(0.5, dim = 2)
sample_copula1 <- rCopula(1000, gaus_copula)

plot(sample_copula1)
```

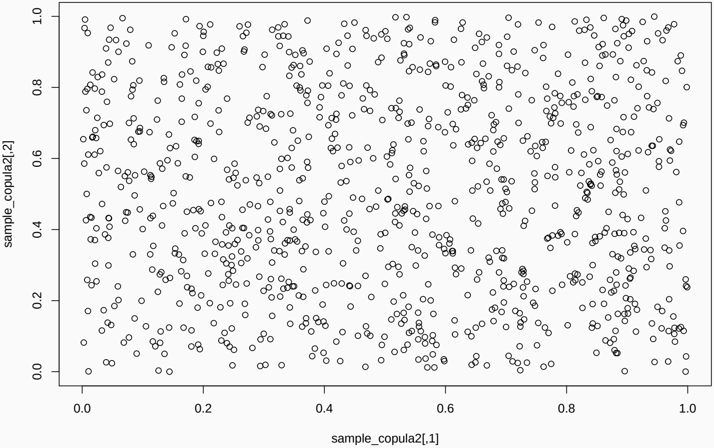
Examples iii



Examples iv

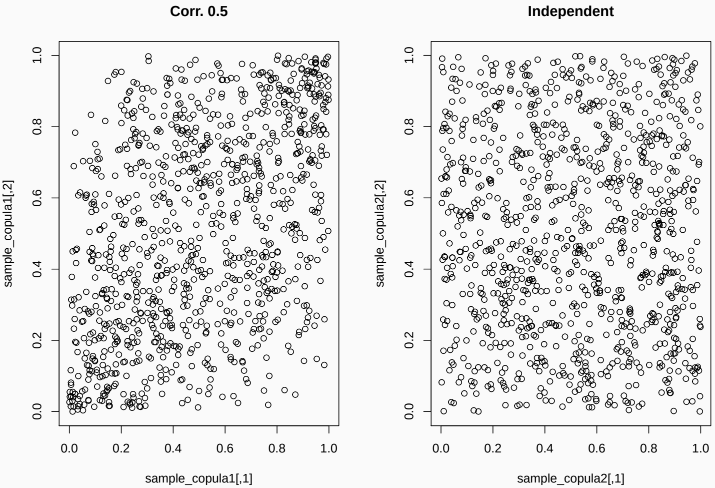
```r
# Compare with independent copula,
# i.e. two independent uniform variables.
gaus_copula <- normalCopula(0, dim = 2)
sample_copula2 <- rCopula(1000, gaus_copula)
plot(sample_copula2)
```

Examples v



1.0
0.8
0.6
0.4
0.2
0.0
sample_copula2[,2]
0.0
0.2
0.4
0.6
0.8
1.0
sample_copula2[,1]

Examples vi



Corr. 0.5
Independent
sample_copula1[,2]
sample_copula1[,1]
sample_copula2[,2]
sample_copula2[,1]
0.0
0.2
0.4
0.6
0.8
1.0

Examples vii

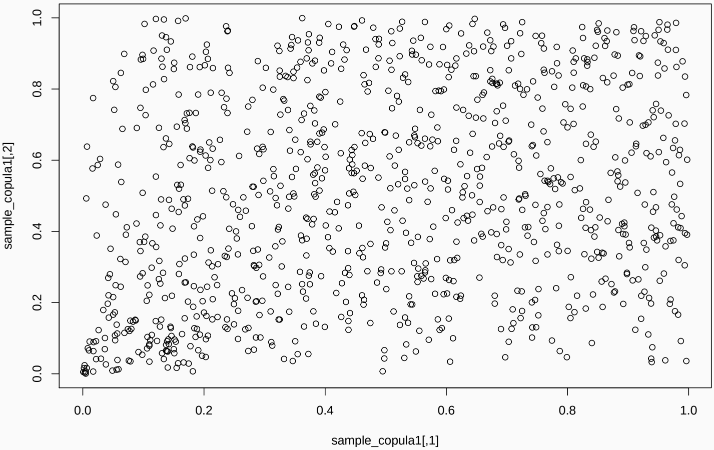
For a properly chosen θ:

Name	$C(u,v)$
Ali-Mikhail-Haq	$\frac{uv}{1-\theta(1-u)(1-v)}$
Clayton	$\max\left((u^{-\theta}+v^{-\theta}-1)^{1/\theta},0\right)$
Independence	uv

Examples viii

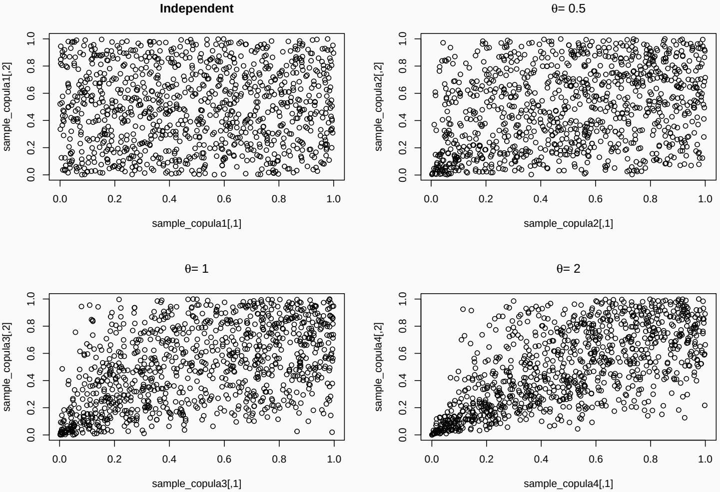
```r
# Clayton copula with theta = 0.5
clay_copula <- claytonCopula(param = 0.5)
sample_copula1 <- rCopula(1000, clay_copula)

plot(sample_copula1)
```

Examples ix



1.0
0.8
0.6
0.4
0.2
0.0
sample_copula1[,2]
0.0
0.2
0.4
0.6
0.8
1.0
sample_copula1[,1]

Examples x



Independent
θ= 0.5
θ= 1
θ= 2
sample_copula1[,1]
sample_copula1[,2]
sample_copula2[,1]
sample_copula2[,2]
sample_copula3[,1]
sample_copula3[,2]
sample_copula4[,1]
sample_copula4[,2]
0.0
0.2
0.4
0.6
0.8
1.0

Conditional distributions

- Let f_1, f_2 be the densities of random variables Y_1, Y_2, respectively. Let f be the joint density.
- The *conditional density* of Y_1 given Y_2 is defined as

$$f(y_1|y_2) := \frac{f(y_1, y_2)}{f_2(y_2)},$$

  whenever $f_2(y_2) \neq 0$ (otherwise it is equal to zero).
- Similarly, we can define the conditional density in $p > 2$ variables, and we can also define a conditional density for $Y_1, \ldots, Y_r$ given $Y_{r+1}, \ldots, Y_p$.

Expectations

- Let $\mathbf{Y} = (Y_1, \ldots, Y_p)$ be a random vector.
- Its *expectation* is defined entry-wise:

$$E(\mathbf{Y}) = (E(Y_1), \ldots, E(Y_p)).$$

- **Observation**: The dependence structure has no impact on the expectation.

Covariance and Correlation i

- The multivariate generalization of the variance is the *covariance matrix*. It is defined as

$$\mathrm{Cov}(\mathbf{Y}) = E\left((\mathbf{Y} - \mu)(\mathbf{Y} - \mu)^T\right),$$

  where $\mu = E(\mathbf{Y})$.
- **Exercise**: The (i, j)-th entry of $\mathrm{Cov}(\mathbf{Y})$ is equal to

$$\mathrm{Cov}(Y_i, Y_j).$$

Covariance and Correlation ii

- Recall that we obtain the correlation from the covariance by dividing by the square root of the variances.
- Let V be the diagonal matrix whose i-th entry is $\mathrm{Var}(Y_i)$.
    - In other words, V and $\mathrm{Cov}(\mathbf{Y})$ have the same diagonal.
- Then we define the *correlation matrix* as follows:

$$\mathrm{Corr}(\mathbf{Y}) = V^{-1/2}\mathrm{Cov}(\mathbf{Y})V^{-1/2}.$$

- **Exercise**: The (i,j)-th entry of $\mathrm{Corr}(\mathbf{Y})$ is equal to

$$\mathrm{Corr}(Y_i, Y_j).$$

Example i

- Assume that

$$\text{Cov}(\mathbf{Y}) = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{pmatrix}.$$

- Then we know that

$$V = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 25 \end{pmatrix}.$$

Example ii

- Therefore, we can write

$$V^{-1/2} = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.33 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}.$$

- We can now compute the correlation matrix:

Example iii

$$\begin{aligned}\mathrm{Corr}(\mathbf{Y}) &= \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.33 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.33 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0.17 & 0.2 \\ 0.17 & 1 & -0.2 \\ 0.2 & -0.2 & 1 \end{pmatrix}.\end{aligned}$$

Measures of Overall Variability

- In the univariate case, the variance is a scalar measure of spread.
- In the multivariate case, the *covariance* is a matrix.
- No easy way to compare two distributions.
- For this reason, we have other notions of overall variability:

1. **Generalized Variance**: This is defined as the determinant of the covariance matrix.

$$GV(\mathbf{Y}) = \det(\mathrm{Cov}(\mathbf{Y})).$$

2. **Total Variance**: This is defined as the trace of the covariance matrix.

$$TV(\mathbf{Y}) = \mathrm{tr}(\mathrm{Cov}(\mathbf{Y})).$$

Examples i

```r
A <- matrix(c(5, 4, 4, 5), ncol = 2)

results <- eigen(A, symmetric = TRUE,
                 only.values = TRUE)

c("GV" = prod(results$values),
  "TV" = sum(results$values))
```

```
## GV TV
##  9 10
```

Examples ii

```r
# Compare this with the following
B <- matrix(c(5, -4, -4, 5), ncol = 2)

# GV(A) = 9; TV(A) = 10
c("GV" = det(B),
  "TV" = sum(diag(B)))
```

```
## GV TV
##  9 10
```

Measures of Overall Variability (cont'd)

- As we can see, we do lose some information:
  - In matrix B, we saw that the two variables are negatively correlated, and yet we get the same values
- But GV captures *some* information on dependence that TV does not.
  - Compare the following covariance matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}.$$

- *Interpretation*: A small value of the sampled Generalized Variance indicates either small scatter in data points or multicollinearity.

Geometric Interlude i

- A random vector $\mathbf{Y}$ with positive definite covariance matrix Σ can be used to define a distance function on $\mathbb{R}^p$:

$$d(x, y) = \sqrt{(x - y)^T \Sigma^{-1} (x - y)}.$$

- This is called the *Mahalanobis distance* induced by Σ.
- **Exercise**: This indeed satisfies the definition of a distance:
  1. $d(x, y) = d(y, x)$
  2. $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$
  3. $d(x, z) \leq d(x, y) + d(y, z)$

Geometric Interlude ii

- Using this distance, we can construct *hyper-ellipsoids* in $\mathbb{R}^p$ as the set of all points x such that

$$d(x, 0) = 1.$$

- Equivalently:

$$x^T \Sigma^{-1} x = 1.$$

- Since Σ^{-1} is symmetric, we can use the spectral decomposition to rewrite it as:

$$\Sigma^{-1} = \sum_{i=1}^{p} \lambda_i^{-1} v_i v_i^T,$$

where $\lambda_1, \ldots, \lambda_p$ are the eigenvalues of Σ.

Geometric Interlude iii

- We thus get a new parametrization if the hyper-ellipsoid:

$$\sum_{i=1}^{p} \left(\frac{v_i^T x}{\sqrt{\lambda_i}} \right)^2 = 1.$$

- **Theorem**: The volume of this hyper-ellipsoid is equal to

$$\frac{2\pi^{p/2}}{p\Gamma(p/2)} \sqrt{\lambda_1 \cdots \lambda_p}.$$

- In other words, the Generalized Variance is proportional to the square of the volume of the hyper-ellipsoid defined by the covariance matrix.
  - *Note*: the square root of the determinant of a matrix (if it exists) is sometimes called the *Pfaffian*.

Statistical Independence

- The variables $Y_1, \ldots, Y_p$ are said to be *mutually independent* if

$$F(y_1, \ldots, y_p) = F(y_1) \cdots F(y_p).$$

- If $Y_1, \ldots, Y_p$ admit a joint density f (with marginal densities $f_1, \ldots, f_p$), and equivalent condition is

$$f(y_1, \ldots, y_p) = f(y_1) \cdots f(y_p).$$

- **Important property**: If $Y_1, \ldots, Y_p$ are mutually independent, then their joint moments factor:

$$E(Y_1^{n_1} \cdots Y_p^{n_p}) = E(Y_1^{n_1}) \cdots E(Y_p^{n_p}).$$

Linear Combination of Random Variables

- Let $\mathbf{Y} = (Y_1, \ldots, Y_p)$ be a random vector. Let $\mathbf{A}$ be a $q \times p$ matrix, and let $b \in \mathbb{R}^q$.
- Then the random vector $\mathbf{X} := \mathbf{A}\mathbf{Y} + b$ has the following properties:
    - **Expectation**: $E(\mathbf{X}) = \mathbf{A}E(\mathbf{Y}) + b$;
    - **Covariance**: $\mathrm{Cov}(\mathbf{X}) = \mathbf{A}\mathrm{Cov}(\mathbf{Y})\mathbf{A}^T$

Transformation of Random Variables i

- More generally, let $h : \mathbb{R}^p \to \mathbb{R}^p$ be a one-to-one function with inverse $h^{-1} = (h_1^{-1}, \ldots, h_p^{-1})$. Define $\mathbf{X} = h(\mathbf{Y})$.
- Let J be the *Jacobian matrix* of h^{-1}:

$$\begin{pmatrix} \frac{\partial h_1^{-1}}{\partial y_1} & \cdots & \frac{\partial h_1^{-1}}{\partial y_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p^{-1}}{\partial y_1} & \cdots & \frac{\partial h_p^{-1}}{\partial y_p} \end{pmatrix}.$$

- Then the density of $\mathbf{X}$ is given by

$$g(x_1, \ldots, x_p) = f(h_1^{-1}(x_1), \ldots, h_p^{-1}(x_p))|\det(J)|.$$

Transformation of Random Variables ii

- A few comments:
  - *This result is very useful for computing the density of transformations of normal random variables.*
  - If h is a linear transformation $\mathbf{Y} \mapsto A\mathbf{Y}$, then $J = A^{-1}$ (Exercise!).
  - See practice problems for further examples (or go back to your notes from mathematical statistics).

Characteristic function

- We will make use of the **characteristic function** φ_Y of a p-dimensional random vector $\mathbf{Y}$.
- The function $\varphi_Y : \mathbb{R}^p \to \mathbb{C}$ is defined as the expected value
$$\varphi_Y(\mathbf{t}) = E(\exp(i\mathbf{t}^T\mathbf{Y})),$$
where $i^2 = -1$.
- **Note**: The characteristic function of a random variable *always exists*.
- **Example**: The characteristic function of the constant random variable $\mathbf{c}$ is $\varphi(\mathbf{t}) = \exp(i\mathbf{t}^T\mathbf{c})$.

Example I i

- Take the density of a normal distribution:

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

- Using the definition, we get

Example I ii

$$
\begin{aligned}
\varphi(t) &= \int_{-\infty}^{\infty} \exp(itx) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x^2 - 2\mu x + \mu^2 - 2it\sigma^2 x)}{2\sigma^2}\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x^2 - 2(\mu + it\sigma^2)x + \mu^2)}{2\sigma^2}\right) dx.
\end{aligned}
$$

Example I iii

- Let's complete the square:

$$
\begin{aligned}
x^2 - 2(\mu + it\sigma^2)x + \mu^2 &= \Big(x - (\mu + it\sigma^2)\Big)^2 \\
&\quad + \Big(\mu^2 - (\mu + it\sigma^2)^2\Big) \\
&= \Big(x - (\mu + it\sigma^2)\Big)^2 \\
&\quad + \Big(\mu^2 - (\mu^2 + 2it\mu\sigma^2 - (t\sigma^2)^2)\Big) \\
&= \Big(x - (\mu + it\sigma^2)\Big)^2 \\
&\quad + \Big((t\sigma^2)^2 - 2it\mu\sigma^2\Big).
\end{aligned}
$$

Example I iv

- We thus get

$$
\begin{aligned}
&\varphi(t) \\
&= e^{\frac{-\left((t\sigma^2)^2 - 2it\mu\sigma^2\right)}{2\sigma^2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-\left(x - (\mu + it\sigma^2)\right)^2}{2\sigma^2}\right) dx \\
&= \exp\left(-\frac{t^2\sigma^2}{2} + it\mu\right).
\end{aligned}
$$

Example II

- Take the density of a gamma distribution:

$$f(x;\alpha,\beta) = \frac{\beta^\alpha x^{\alpha-1}\exp(-\beta x)}{\Gamma(\alpha)}.$$

- Using the definition, we get

$$\begin{aligned}
\varphi(t) &= \int_0^\infty \exp(itx)\frac{\beta^\alpha x^{\alpha-1}\exp(-\beta x)}{\Gamma(\alpha)}dx \\
&= \frac{(\beta-it)^\alpha}{(\beta-it)^\alpha}\int_0^\infty \frac{\beta^\alpha x^{\alpha-1}\exp(-(\beta-it)x)}{\Gamma(\alpha)}dx \\
&= \frac{\beta^\alpha}{(\beta-it)^\alpha}\int_0^\infty \frac{(\beta-it)^\alpha x^{\alpha-1}\exp(-(\beta-it)x)}{\Gamma(\alpha)}dx \\
&= \left(1-\frac{it}{\beta}\right)^{-\alpha}.
\end{aligned}$$

Properties of the characteristic function i

1. $\varphi_Y(\mathbf{0}) = 1$
2. $|\varphi_Y(\mathbf{t})| \leq 1$ for all $\mathbf{t}$
3. $\varphi_Y(-\mathbf{t}) = \overline{\varphi_Y(\mathbf{t})}$
4. $\varphi_Y(\mathbf{t})$ is uniformly continuous.
5. If $\mathbf{Y} = A\mathbf{X} + b$, then $\varphi_{\mathbf{Y}}(t) = \exp(it^T b)\varphi_{\mathbf{X}}(A^T t)$
6. Two random vectors are equal in distribution if and only if their characteristic functions are equal.
7. The components of $\mathbf{Y} = (Y_1, \ldots, Y_p)$ are mutually independent if and only if $\varphi_Y(\mathbf{t}) = \prod_{i=1}^{p} \varphi_{Y_i}(t_i)$.

Properties of the characteristic function ii

Levy Continuity Theorem
Let $\mathbf{Y}_n$ be a sequence of p-dimensional random vectors, and let φ_n be the characteristic function of $\mathbf{Y}_n$. Then $\mathbf{Y}_n$ converges in distribution to $\mathbf{Y}$ if and only if the sequence φ_n converges pointwise to a function φ that is continuous at the origin. When this is the case, the function φ is the characteristic function of the limiting distribution $\mathbf{Y}$.

Example i

- Let X_n be Poisson with mean n.
  - **Exercise**: The characteristic function of a $Pois(\mu)$ random variable is $\varphi(t) = \exp\left(\mu(e^{it} - 1)\right)$.
- Let $Y_n = \frac{X_n - n}{\sqrt{n}}$ be the standardized random variable.
- **To show**: Y_n converges in a distribution to a standard normal random variable.
- From the properties above, we have

$$\begin{aligned}\varphi_{\mathbf{Y}_n}(t) &= \exp(-itn/\sqrt{n})\varphi_{\mathbf{X}_n}(t/\sqrt{n}) \\ &= \exp\left(n(e^{it/\sqrt{n}} - 1) - itn/\sqrt{n}\right).\end{aligned}$$

Example ii

- We will show that this converges to the characteristic function of the standard normal: $\varphi(t) = \exp(-t^2/2)$.
  - We will use a **change of variables** and the **Taylor expansion** of the exponential distribution around 0.
- First, define $u = it/\sqrt{n}$. We then get $n = -t^2/u^2$ (here we fix t).
  - Note that $u \to 0$ is now equivalent to $n \to \infty$.

Example iii

- Recall the Taylor expansion: as $u \to 0$, we have

$$\exp(u) = 1 + u + \frac{u^2}{2} + o(u^2),$$

where $o(u^2)$ represents a quantity that goes to zero faster than u^2.

Example iv

- We then get

$$
\begin{aligned}
n(e^{it/\sqrt{n}} - 1) - itn/\sqrt{n} &= -\frac{t^2}{u^2}(e^u - 1) + \frac{t^2}{u} \\
&= -\frac{t^2}{u^2}\left(u + \frac{u^2}{2} + o(u^2)\right) + \frac{t^2}{u} \\
&= -\frac{t^2}{u} - \frac{t^2}{2} - \frac{t^2}{u^2}o(u^2) + \frac{t^2}{u} \\
&= -\frac{t^2}{2} - \frac{t^2}{u^2}o(u^2).
\end{aligned}
$$

Example v

- Since the second term goes to zero as $u \to 0$, we can conclude that

$$n(e^{it/\sqrt{n}} - 1) - itn/\sqrt{n} \to \frac{-t^2}{2}, \qquad n \to \infty.$$

- And since the exponential function is continuous everywhere, we get

$$\varphi_{\mathbf{Y}_n}(t) \to \exp\left(\frac{-t^2}{2}\right) \text{ for all } t, \qquad n \to \infty.$$

- The result follows from the Levy Continuity Theorem.

Weak Law of Large Numbers

- We can prove the multivariate (weak) Law of Large Numbers using the Levy Continuity theorem.

WLLN

Let $\mathbf{Y}_n$ be a random sample with characteristic function φ and mean μ. Assume φ is differentiable at the origin. Then $\frac{1}{n}\sum_{k=1}^{n}\mathbf{Y}_k \to \mu$ in probability as $n \to \infty$.

Proof (WLLN) i

- First, note that since φ is differentiable at the origin, we have $\varphi'(0) = i\mu$.
- We can look at the Taylor expansion of φ around 0:

$$\varphi(\mathbf{t}) = 1 + \mathbf{t}^T\varphi'(0) + o(\mathbf{t}) = 1 + i\mathbf{t}^T\mu + o(\mathbf{t}).$$

- Now note that the characteristic function of $\frac{1}{n}\sum_{k=1}^n \mathbf{Y}_k$ is given by

Proof (WLLN) ii

$$
\begin{aligned}
\varphi_n(\mathbf{t}) &= E\left(\exp\left(i\mathbf{t}^T \frac{1}{n}\sum_{k=1}^{n}\mathbf{Y}_k\right)\right) \\
&= E\left(\prod_{k=1}^{n}\exp\left(i\left(\frac{\mathbf{t}}{n}\right)^T\mathbf{Y}_i\right)\right) \\
&= \prod_{k=1}^{n} E\left(\exp\left(i\left(\frac{\mathbf{t}}{n}\right)^T\mathbf{Y}_i\right)\right) \\
&= \varphi\left(\frac{\mathbf{t}}{n}\right)^n.
\end{aligned}
$$

Proof (WLLN) iii

- Using the Taylor expansion of φ, we get

$$\begin{aligned}\varphi_n(\mathbf{t}) &= \varphi\left(\frac{\mathbf{t}}{n}\right)^n \\ &= \left(1 + i\left(\frac{\mathbf{t}}{n}\right)^T \mu + o\left(\frac{1}{n}\right)\right)^n.\end{aligned}$$

- The left-hand side converges to the exponential distribution:

$$\varphi_n(\mathbf{t}) \to \exp(i\mathbf{t}^T\mu).$$

- But this is simply the characteristic function of the constant random variable μ. $\square$

Cramer-Wold Theorem

Two random vectors $\mathbf{X}$ and $\mathbf{Y}$ are equal in distribution if and only if the linear combinations $\mathbf{t}^T\mathbf{X}$ and $\mathbf{t}^T\mathbf{Y}$ are equal in distribution for all vectors $\mathbf{t} \in \mathbb{R}^p$.

Proof
Let $\varphi_{\mathbf{X}}, \varphi_{\mathbf{Y}}$ be the characteristic functions of $\mathbf{X}$ and $\mathbf{Y}$, respectively. Let $s \in \mathbb{R}$. Using the definition, we can see that

$$\varphi_{\mathbf{t}^T\mathbf{X}}(s) = E(\exp(is(\mathbf{t}^T\mathbf{X}))) = E(\exp(i(s\mathbf{t})^T\mathbf{X})) = \varphi_{\mathbf{X}}(s\mathbf{t}).$$

The result follows from the uniqueness of characteristic functions. $\square$

Multivariate Slutsky's Theorem i

Let $\mathbf{X}_n$ be a sequence of q-dimensional random vectors that converge in distribution to $\mathbf{X}$, and let $\mathbf{Y}_n$ be a sequence of p-dimensional random vectors that converge in distribution to a constant vector $\mathbf{c} \in \mathbb{R}^p$. Then for any continuous function $f : \mathbb{R}^{p+q} \to \mathbb{R}^k$, we have

$$f(\mathbf{X}_n, \mathbf{Y}_n) \to f(\mathbf{X}, \mathbf{c}) \quad \text{in distribution.}$$

- Common examples of f include:
  - $f(\mathbf{X}, \mathbf{Y}) = \mathbf{X} + \mathbf{Y}$
  - $f(\mathbf{X}, \mathbf{Y}) = \mathbf{X}^T\mathbf{Y}$ when $p = q$.

Multivariate Slutsky's Theorem ii

- Note that both $\mathbf{X}_n$ or $\mathbf{Y}_n$ could be *matrices*:
    - This follows from the correspondence between the space of $n \times p$ matrices and $\mathbb{R}^{np}$ given by stacking the columns of a matrix into a single column vector.
    - For example, if A_n are $r \times q$ matrices converging to A, then we could conclude

$$A_n \mathbf{X}_n \to A\mathbf{X}.$$

Proof (Slutsky) i

- By the Continuous mapping theorem, it is sufficient to show that

$$(\mathbf{X}_n, \mathbf{Y}_n) \to (\mathbf{X}, c) \quad \text{in distribution.}$$

- For any $\mathbf{u} \in \mathbb{R}^q, \mathbf{v} \in \mathbb{R}^p$, the Cramer-Wold theorem implies

$$\begin{aligned} \mathbf{u}^T \mathbf{X}_n &\to \mathbf{u}^T \mathbf{X} \\ \mathbf{v}^T \mathbf{Y}_n &\to \mathbf{v}^T \mathbf{c}. \end{aligned}$$

Proof (Slutsky) ii

- From the univariate Slutsky's theorem, we get

$$\mathbf{u}^T\mathbf{X}_n + \mathbf{v}^T\mathbf{Y}_n \to \mathbf{u}^T\mathbf{X} + \mathbf{v}^T\mathbf{c}.$$

- If we let $\mathbf{w} = (\mathbf{u}, \mathbf{v})$, we have just shown that, for all $\mathbf{w} \in \mathbb{R}^{q+p}$, we have

$$\mathbf{w}^T(\mathbf{X}_n, \mathbf{Y}_n) \to \mathbf{w}^T(\mathbf{X}, c).$$

- Using once more the Cramer-Wold theorem, we can conclude the proof of this theorem. $\square$

Sample Statistics i

- Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be a random sample from a p-dimensional distribution with mean μ and covariance matrix Σ.
- **Sample mean**: We define the sample mean $\bar{\mathbf{Y}}_n$ as follows:

$$\bar{\mathbf{Y}}_n = \frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i.$$

- *Properties*:
  - $E(\bar{\mathbf{Y}}_n) = \mu$ (i.e. $\bar{\mathbf{Y}}_n$ is an unbiased estimator of μ);
  - $\mathrm{Cov}(\bar{\mathbf{Y}}_n) = \frac{1}{n}\Sigma$.
  - From WLLN: $\bar{\mathbf{Y}}_n \to \mu$ in probability.

Sample Statistics ii

- **Sample covariance**: We define the sample covariance S_n as follows:

$$\mathbf{S}_n = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{Y}_i - \bar{\mathbf{Y}}_n)(\mathbf{Y}_i - \bar{\mathbf{Y}}_n)^T.$$

- *Properties*:
  - $E(\mathbf{S}_n) = \frac{n-1}{n}\Sigma$ (i.e. $\mathbf{S}_n$ is a biased estimator of Σ);
  - If we define $\tilde{\mathbf{S}}_n$ with n instead of $n-1$ in the denominator above, then $E(\tilde{\mathbf{S}}_n) = \Sigma$ (i.e. $\tilde{\mathbf{S}}_n$ is an unbiased estimator of Σ).

Multivariate Central Limit Theorem

Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be a random sample from a p-dimensional distribution with mean μ and covariance matrix Σ. Then

$$\sqrt{n}\left(\bar{\mathbf{Y}}_n - \mu\right) \to N_p(0, \Sigma).$$

Proof
This follows from the Cramer-Wold theorem and the univariate CLT (**Exercise**).

Example i

- Let $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ be a random sample from a p-dimensional distribution with mean μ and covariance matrix Σ.
  - **Exercise**: $E(\mathbf{Y}_n \mathbf{Y}_n^T) = \Sigma + \mu\mu^T$.
- Using Slutsky's theorem and the WLLN, we will show that $\mathbf{S}_n \to \Sigma$.
- By the WLLN, we have that

$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i \mathbf{Y}_i^T \to \Sigma + \mu\mu^T.$$

Example ii

- We then have that

$$\begin{aligned}\tilde{\mathbf{S}}_n &= \frac{1}{n}\sum_{i=1}^{n}(\mathbf{Y}_i - \bar{\mathbf{Y}}_n)(\mathbf{Y}_i - \bar{\mathbf{Y}}_n)^T \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(\mathbf{Y}_i\mathbf{Y}_i^T - \bar{\mathbf{Y}}_n\mathbf{Y}_i^T - \mathbf{Y}_i\bar{\mathbf{Y}}_n^T + \bar{\mathbf{Y}}_n\bar{\mathbf{Y}}_n^T\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\mathbf{Y}_i\mathbf{Y}_i^T - \bar{\mathbf{Y}}_n\bar{\mathbf{Y}}_n^T - \bar{\mathbf{Y}}_n\bar{\mathbf{Y}}_n^T + \bar{\mathbf{Y}}_n\bar{\mathbf{Y}}_n^T \\ &= \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{Y}_i\mathbf{Y}_i^T\right) - \bar{\mathbf{Y}}_n\bar{\mathbf{Y}}_n^T \to \Sigma \qquad \text{(Slutsky).}\end{aligned}$$

- But since $\tilde{\mathbf{S}}_n = \frac{n-1}{n}\mathbf{S}_n$, we also have $\mathbf{S}_n \to \Sigma$. $\square$

Multivariate Delta Method i

Let $\mathbf{Y}_n$ be a sequence of p-dimensional random vectors such that

$$\sqrt{n}\,(\mathbf{Y}_n - \mathbf{c}) \to \mathbf{Z} \quad \text{in distribution,}$$

where $\mathbf{c} \in \mathbb{R}^p$. Furthermore, assume $g : \mathbb{R}^p \to \mathbb{R}^q$ is differentiable at $\mathbf{c}$ with derivative $\nabla g(\mathbf{c})$. Then

$$\sqrt{n}\,(g(\mathbf{Y}_n) - g(\mathbf{c})) \to \nabla g(\mathbf{c})\mathbf{Z} \quad \text{in distribution.}$$

Multivariate Delta Method ii

In other words, we can derive useful approximations: if $\mathbf{Y}_n$ is a random sample with mean $\mathbf{c}$ and covariance matrix Σ:

- $E(g(\mathbf{Y}_n)) \approx g(\mathbf{c})$;
- $\mathrm{Var}(g(\mathbf{Y}_n)) \approx \nabla g(\mathbf{c})\Sigma\nabla g(\mathbf{c})^T$.

Example

- By the Central Limit Theorem, we have

$$\sqrt{n}\left(\bar{\mathbf{Y}}_n - \mu\right) \to N_p(0, \Sigma).$$

- From the Delta method, we get

$$\sqrt{n}\left(g(\bar{\mathbf{Y}}_n) - g(\mu)\right) \to N_p(0, \nabla g(\mu)\Sigma\nabla g(\mu)^T).$$

- For example, if $\mathbf{Y}_n > 0$, then we have

$$\sqrt{n}\left(\log(\bar{\mathbf{Y}}_n) - \log(\mu)\right) \to N_p(0, \tilde{\mu}\Sigma\tilde{\mu}^T),$$

  where $\log$ is applied entrywise, and $\tilde{\mu} = (\mu_1^{-1}, \ldots, \mu_p^{-1})$.