Test for sphericity

Max Turgeon

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We assume $\mathbf{Y}_1, \ldots, \mathbf{Y}_n \sim N_p(\mu, \Sigma)$ with Σ positive definite. Write $V = n\hat{\Sigma}$, where $(\bar{\mathbf{Y}}, \hat{\Sigma})$ is the (unrestricted) MLE for the multivariate normal distribution.

Sphericity means the different components of $\mathbf{Y}$ are **uncorrelated** and have the **same variance**. In other words, we are looking at the following null hypothesis:

$$H_0 : \Sigma = \sigma^2 I_p, \qquad \sigma^2 > 0.$$

Likelihood Ratio Test

We have

$$\begin{aligned}
L(\hat{\mathbf{Y}}, \sigma^2 I_p) &= (2\pi)^{-np/2} |\sigma^2 I_p|^{-n/2} \exp\left(-\frac{1}{2}\mathrm{tr}((\sigma^2 I_p)^{-1} V)\right) \\
&= (2\pi\sigma^2)^{-np/2} \exp\left(-\frac{1}{2\sigma^2}\mathrm{tr}(V)\right).
\end{aligned}$$

Taking the derivative of the logarithm and setting it equal to zero, we find that $L(\hat{\mathbf{Y}}, \sigma^2 I_p)$ is maximised when

$$\widehat{\sigma^2} = \frac{\mathrm{tr} V}{np}.$$

We then get

$$\begin{aligned}
L(\hat{\mathbf{Y}}, \widehat{\sigma^2} I_p) &= (2\pi\widehat{\sigma^2})^{-np/2} \exp\left(-\frac{1}{2\widehat{\sigma^2}}\mathrm{tr}(V)\right) \\
&= (2\pi)^{-np/2} \left(\frac{\mathrm{tr} V}{np}\right)^{-np/2} \exp\left(-\frac{np}{2}\right).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\Lambda &= \frac{(2\pi)^{-np/2} \left(\frac{\mathrm{tr} V}{np}\right)^{-np/2} \exp\left(-\frac{np}{2}\right)}{\exp(-np/2)(2\pi)^{-np/2}|\hat{\Sigma}|^{-n/2}} \\
&= \frac{\left(\frac{\mathrm{tr} V}{np}\right)^{-np/2}}{|n^{-1}V|^{-n/2}} \\
&= \left(\frac{|V|}{(\mathrm{tr} V/p)^p}\right)^{n/2}.
\end{aligned}$$

We can also rewrite this as follows: let $l_1 \geq \cdots \geq l_p$ be the eigenvalues of V. We have

$$\begin{aligned}
\Lambda^{2/n} &= \frac{|V|}{(\mathrm{tr} V/p)^p} \\
&= \frac{\prod_{j=1}^p l_j}{(\frac{1}{p}\sum_{j=1}^p l_j)^p} \\
&= \left(\frac{\prod_{j=1}^p l_j^{1/p}}{\frac{1}{p}\sum_{j=1}^p l_j}\right)^p.
\end{aligned}$$

In other words, the modified LRT $\tilde{\Lambda} = \Lambda^{2/n}$ is the ratio of the geometric to the arithmetic mean of the eigenvalues of V (all raised to the power p).

Note that under H_0, there is only one free parameter, namely σ^2. Therefore the aymptotic theory of likelihood ratio tests implies that

$$-2\log\Lambda \to \chi^2\left(\frac{1}{2}p(p+1) - 1\right).$$

We will provide a better approximation using asymptotic expansions.

First, we need to compute the moments of Λ. We start with the following lemma:

Lemma

Under the null hypothesis, the random variables $\mathrm{tr}V$ and $\frac{\det V}{(\mathrm{tr}V)^p}$ are independent.

Proof

Recall that $V \sim W_p(n-1, \sigma^2 I_p)$, and so its distribution only depends on σ^2. Hence, the distribution of $\frac{\det V}{(\mathrm{tr}V)^p}$ does *not* depend on σ^2, and therefore it is an ancillary statistic.

Now, given that $\widehat{\sigma^2} = \frac{\mathrm{tr}V}{np}$ and given that the multivariate normal forms an exponential family, we know that $(\bar{\mathbf{Y}}, \mathrm{tr}V)$ is a minimal sufficient and complete statistic. Therefore, we can conclude by using Basu's theorem. $\square$

Now, going back to $\tilde{\Lambda}$, note that we have

$$\tilde{\Lambda}\left(\frac{1}{p}\mathrm{tr}V\right)^p = |V|.$$

Using our lemma above, for any h, we can write

$$\begin{aligned} E|V|^h &= E\left(\tilde{\Lambda}\left(\frac{1}{p}\mathrm{tr}V\right)^p\right)^h \\ &= E\tilde{\Lambda}^h E\left(\frac{1}{p}\mathrm{tr}V\right)^{ph}. \end{aligned}$$

In other words, we have

$$E\left(\tilde{\Lambda}^h\right) = \frac{E\left(|V|^h\right)}{E\left(\left(\frac{1}{p}\mathrm{tr}V\right)^{ph}\right)}.$$

Recall the following two results:

1. If $W \sim W_p(m, I_p)$, then $\mathrm{tr}W \sim \chi^2(mp)$.
2. If $W \sim W_p(m, \Sigma)$, then $|W| \sim |\Sigma|\prod_{j=1}^p \chi^2(m-p+j)$.

Therefore, we can get all the moments of $\tilde{\Lambda}$ from the moments of chi-squared distributions.

Proposition

Let $X \sim \chi^2(d)$. Then for $h > -\frac{1}{2}d$, we have

$$E(X^h) = 2^h\frac{\Gamma\left(\frac{1}{2}d + h\right)}{\Gamma\left(\frac{1}{2}d\right)}.$$

Putting all this together, we get the following theorem:

Theorem

The moments of the modified LRT statistic are given by

$$E\left(\tilde{\Lambda}^h\right) = p^{ph}\frac{\Gamma\left(\frac{1}{2}(n-1)p\right)}{\Gamma\left(\frac{1}{2}(n-1)p+ph\right)}\frac{\Gamma_p\left(\frac{1}{2}(n-1)+h\right)}{\Gamma_p\left(\frac{1}{2}(n-1)\right)}.$$

Proof

This follows from our discussion above, the moments of the chi-squared distribution, and the fact that $V \sim \sigma^2 W_p(n-1)$. $\square$

Asymptotic expansion

In a 1949 *Biometrika* paper, George Box studied the distribution theory of a very general class of likelihood ratio tests. It can be applied any time the moments of the likelihood ratio Λ (or some power $W = \Lambda^d$ thereof) have the following expression:

$$E\left(W^h\right) = K\left(\frac{\prod_{j=1}^b y_j^{y_j}}{\prod_{k=1}^a x_k^{x_k}}\right)^h \frac{\prod_{k=1}^a \Gamma\left(x_k(1+h)+\zeta_k\right)}{\prod_{j=1}^b \Gamma\left(y_j(1+h)+\eta_j\right)}, \tag{1}$$

such that

$$\sum_{j=1}^b y_j = \sum_{k=1}^a x_k$$

and K is a constant such that $E\left(\tilde{\Lambda}^0\right) = 1$.

In the context of the test for sphericity, we can take $W = \Lambda^{m/n}$ with $m = n-1$, and we get

$$\begin{aligned}
E\left(W^h\right) &= E\left(\Lambda^{mh/n}\right)\\
&= E\left(\tilde{\Lambda}^{mh/2}\right)\\
&= p^{pmh/2}\frac{\Gamma\left(\frac{1}{2}mp\right)}{\Gamma\left(\frac{1}{2}mp+\frac{1}{2}pmh\right)}\frac{\Gamma_p\left(\frac{1}{2}m+\frac{1}{2}mh\right)}{\Gamma_p\left(\frac{1}{2}m\right)}\\
&= \left(\frac{\Gamma\left(\frac{1}{2}mp\right)}{\Gamma_p\left(\frac{1}{2}m\right)}\right)p^{pmh/2}\frac{\Gamma_p\left(\frac{1}{2}m(1+h)\right)}{\Gamma\left(\frac{1}{2}mp(1+h)\right)}\\
&= \left(\pi^{p(p-1)/4}\frac{\Gamma\left(\frac{1}{2}mp\right)}{\Gamma_p\left(\frac{1}{2}m\right)}\right)\left(p^{pm/2}\right)^h\frac{\prod_{k=1}^p \Gamma_p\left(\frac{1}{2}m(1+h)-\frac{1}{2}(k-1)\right)}{\Gamma\left(\frac{1}{2}mp(1+h)\right)}.
\end{aligned}$$

This is consistent with the general form above, if we take

$$\begin{aligned}
a &= p, \quad x_k = \tfrac{1}{2}m, \quad \zeta_k = -\tfrac{1}{2}(k-1),\\
b &= 1, \quad y_1 = \tfrac{1}{2}mp, \quad \eta_1 = 0,
\end{aligned}$$

and

$$K = \pi^{p(p-1)/4}\frac{\Gamma\left(\frac{1}{2}mp\right)}{\Gamma_p\left(\frac{1}{2}m\right)}.$$