Problem Set 1–STAT 7200

1. Let A be a $p \times q$ matrix, and let B be a $q \times p$ matrix.
   (a) Prove that $\det(I + AB) = \det(I + BA)$.
   (b) Prove that AB and BA have the same nonzero eigenvalues.

2. Let $\mathbf{x} \in \mathbb{R}^p$, and let A be a $p \times p$ symmetric matrix. Prove that

$$\frac{\partial \mathbf{x}^T A \mathbf{x}}{\partial \mathbf{x}} = 2A\mathbf{x}.$$

3. Let (X, Y) be uniformly distributed on the unit disk $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$. Let $R = \sqrt{X^2 + Y^2}$. Find the distribution and the density function of R.

4. Let $X, Y \sim U(0, 1)$ be independent. Find the density functions of $U = X - Y$ and $V = X/Y$.

5. Find the characteristic function of the binomial, Poisson, and chi-square distributions.

6. Let $X_1, \ldots, X_n \sim Exp(\beta)$. Find the characteristic function of X_i. Use the result to prove that $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \beta)$.

7. Reduce the multivariate Central Limit Theorem to the univariate CLT, i.e. assuming the univariate result, prove the multivariate one.

8. Let $(X_{11}, X_{21}), \ldots, (X_{1n}, X_{2n})$ be a random sample with mean $\mu = (\mu_1, \mu_2)$ and variance Σ. Let

$$\bar{X}_1 = \frac{1}{n} \sum_{i=1}^{n} X_{1i}, \qquad \bar{X}_2 = \frac{1}{n} \sum_{i=1}^{n} X_{2i}.$$

Define $Y_n = \bar{X}_1 / \bar{X}_2$. Find the limiting distribution of the sequence Y_n.