Wishart Distribution

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STAT 7200–Multivariate Statistics

Objectives

- Understand the distribution of covariance matrices
- Understand the distribution of the MLEs for the multivariate normal distribution
- Understand the distribution of *functionals* of covariance matrices
- Visualize covariance matrices and their distribution

Before we begin... i

- In this section, we will discuss *random matrices*
  - Therefore, we will talk about distributions, derivatives and integrals over *sets of matrices*
- It can be useful to identify the space $M_{n,p}(\mathbb{R})$ of $n \times p$ matrices with $\mathbb{R}^{np}$.
  - We can define the function $\mathrm{vec} : M_{n,p}(\mathbb{R}) \to \mathbb{R}^{np}$ that takes a matrix M and maps it to the np-dimensional vector given by concatenating the columns of M into a single vector.

$$\mathrm{vec}\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = (1, 2, 3, 4).$$

Before we begin... ii

- Another important observation: structural constraints (e.g. symmetry, positive definiteness) reduce the number of "free" entries in a matrix and therefore the dimension of the subspace.
    - E.g. If A is a symmetric $p \times p$ matrix, there are only $\frac{1}{2}p(p+1)$ independent entries: the entries on the diagonal, and the off-diagonal entries above the diagonal (or below).

Wishart distribution i

- Let S be a random, positive semidefinite matrix of dimension $p \times p$.
    - We say S follows a *standard Wishart distribution* $W_p(m)$ if we can write

$$S = \sum_{i=1}^m \mathbf{Z}_i \mathbf{Z}_i^T, \quad \mathbf{Z}_i \sim N_p(0, I_p) \text{ indep..}$$

    - We say S follows a *Wishart distribution* $W_p(m, \Sigma)$ with scale matrix Σ if we can write

$$S = \sum_{i=1}^m \mathbf{Y}_i \mathbf{Y}_i^T, \quad \mathbf{Y}_i \sim N_p(0, \Sigma) \text{ indep..}$$

Wishart distribution ii

- We say S follows a *non-central Wishart distribution* $W_p(m, \Sigma; \Delta)$ with scale matrix Σ and non-centrality parameter Δ if we can write

$$S = \sum_{i=1}^m \mathbf{Y}_i\mathbf{Y}_i^T, \quad \mathbf{Y}_i \sim N_p(\mu_i, \Sigma) \text{ indep.}, \quad \Delta = \sum_{i=1}^m \mu_i\mu_i^T.$$

Example i

- Let $S \sim W_p(m)$ be Wishart distributed, with scale matrix $\Sigma = I_p$.
- We can therefore write $S = \sum_{i=1}^m \mathbf{Z}_i \mathbf{Z}_i^T$, with $\mathbf{Z}_i \sim N_p(0, I_p)$.

Example ii

- Using the properties of the trace, we have

$$\begin{aligned}\operatorname{tr}(S) &= \operatorname{tr}\left(\sum_{i=1}^m \mathbf{Z}_i \mathbf{Z}_i^T\right) \\ &= \sum_{i=1}^m \operatorname{tr}\left(\mathbf{Z}_i \mathbf{Z}_i^T\right) \\ &= \sum_{i=1}^m \operatorname{tr}\left(\mathbf{Z}_i^T \mathbf{Z}_i\right) \\ &= \sum_{i=1}^m \mathbf{Z}_i^T \mathbf{Z}_i.\end{aligned}$$

- Recall that $\mathbf{Z}_i^T \mathbf{Z}_i \sim \chi^2(p)$.

Example iii

- Therefore $\mathrm{tr}(S)$ is the sum of m independent copies of a $\chi^2(p)$, and so we have

$$\mathrm{tr}(S) \sim \chi^2(mp).$$

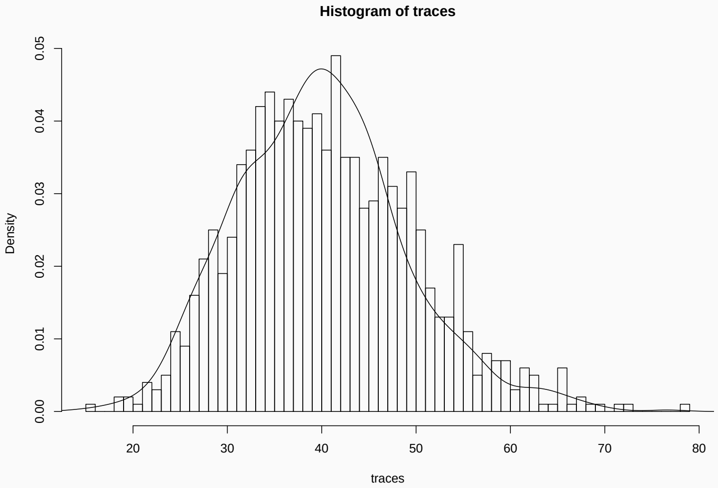
```r
B <- 1000
n <- 10; p <- 4

traces <- replicate(B, {
  Z <- matrix(rnorm(n*p), ncol = p)
  W <- crossprod(Z)
  sum(diag(W))
})
```

Example iv

```r
hist(traces, 50, freq = FALSE)
lines(density(rchisq(B, df = n*p)))
```

Example v



Histogram of traces
Density
0.00
0.01
0.02
0.03
0.04
0.05
20
30
40
50
60
70
80
traces

Non-singular Wishart distribution i

- As defined above, there is no guarantee that a Wishart variate is invertible.
- **To show**: if $S \sim W_p(m, \Sigma)$ with Σ positive definite, S is invertible almost surely whenever $m \geq p$.

Lemma: Let Z be an $n \times n$ random matrix where the entries Z_{ij} are iid $N(0, 1)$. Then $P(\det Z = 0) = 0$.

Proof: We will prove this by induction on n. If $n = 1$, then the result hold since $N(0, 1)$ is absolutely continuous.

Now let $n > 1$ and assume the result holds for $n - 1$. Write

Non-singular Wishart distribution ii

$$Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix},$$

where Z_{22} is $(n-1) \times (n-1)$. Note that by assumption, we have $\det Z_{22} \neq 0$ almost surely. Now, by the Schur determinant formula, we have

$$\begin{aligned} \det Z &= \det Z_{22} \det \left(Z_{11} - Z_{12} Z_{22}^{-1} Z_{21} \right) \\ &= (\det Z_{22}) \left(Z_{11} - Z_{12} Z_{22}^{-1} Z_{21} \right). \end{aligned}$$

Non-singular Wishart distribution iii

We now have

$$\begin{aligned}
P(|Z| = 0) &= P(|Z| = 0, |Z_{22}| \neq 0) + P(|Z| = 0, |Z_{22}| = 0) \\
&= P(|Z| = 0, |Z_{22}| \neq 0) \\
&= P(Z_{11} = Z_{12}Z_{22}^{-1}Z_{21}, |Z_{22}| \neq 0) \\
&= E\left(P(Z_{11} = Z_{12}Z_{22}^{-1}Z_{21}, |Z_{22}| \neq 0 \mid Z_{12}, Z_{22}, Z_{21})\right) \\
&= E(0) \\
&= 0,
\end{aligned}$$

Non-singular Wishart distribution iv

where we used the laws of total probability (Line 1) and total expectation (Line 4). Therefore, the result follows from induction. $\square$

We are now ready to prove the main result: let $S \sim W_p(m, \Sigma)$ with $\det \Sigma \neq 0$, and write $S = \sum_{i=1}^m \mathbf{Y}_i \mathbf{Y}_i^T$, with $\mathbf{Y}_i \sim N_p(0, \Sigma)$. If we let $\mathbb{Y}$ be the $m \times p$ matrix whose i-th row is $\mathbf{Y}_i$. Then

$$S = \sum_{i=1}^m \mathbf{Y}_i \mathbf{Y}_i^T = \mathbb{Y}^T \mathbb{Y}.$$

Non-singular Wishart distribution v

Now note that

$$\mathrm{rank}(S) = \mathrm{rank}(\mathbb{Y}^T\mathbb{Y}) = \mathrm{rank}(\mathbb{Y}).$$

Furthermore, if we write $\Sigma = LL^T$ using the Cholesky decomposition, then we can write

$$\mathbb{Z} = \mathbb{Y}(L^{-1})^T,$$

where the rows $\mathbf{Z}_i$ of $\mathbb{Z}$ are $N_p(0, I_p)$ and $\mathrm{rank}(\mathbb{Z}) = \mathrm{rank}(\mathbb{Y})$.

Finally, we have

Non-singular Wishart distribution vi

$$\begin{aligned}\mathrm{rank}(S) &= \mathrm{rank}(\mathbb{Z})\\ &\geq \mathrm{rank}(\mathbf{Z}_1, \ldots, \mathbf{Z}_p)\\ &= p \quad \text{(a.s.)},\end{aligned}$$

where the last equality follows from our Lemma. Since $\mathrm{rank}(S) = p$ almost surely, it is invertible almost surely. $\square$

Definition
If $S \sim W_p(m, \Sigma)$ with Σ positive definite and $m \geq p$, we say that S follows a *nonsingular* Wishart distribution. Otherwise, we say it follows a *singular* Wishart distribution.

Additional properties i

Let $S \sim W_p(m, \Sigma)$.

- We have $E(S) = m\Sigma$.
- If B is a $q \times p$ matrix, we have

$$BSB^T \sim W_p(m, B\Sigma B^T).$$

- If $T \sim W_p(n, \Sigma)$, then

$$S + T \sim W_p(m + n, \Sigma).$$

Additional properties ii

Now assume we can partition S and Σ as such:

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

with S_{ii} and Σ_{ii} of dimension $p_i \times p_i$. We then have

- $S_{ii} \sim W_{p_i}(m, \Sigma_{ii})$
- If $\Sigma_{12} = 0$, then S_{11} and S_{22} are independent.

Characteristic function i

- The definition of characteristic function can be extended to *random matrices*:
  - Let S be a $p \times p$ random matrix. The characteristic function of S evaluated at a $p \times p$ symmetric matrix T is defined as

$$\varphi_S(T) = E\left(\exp(i\mathrm{tr}(TS))\right).$$

- We will show that if $S \sim W_p(m, \Sigma)$, then

$$\varphi_S(T) = |I_p - 2i\Sigma T|^{-m/2}.$$

- First, we will use the Cholesky decomposition $\Sigma = LL^T$.

Characteristic function ii

- Next, we can write

$$S = L\left(\sum_{j=1}^m \mathbf{Z}_j\mathbf{Z}_j^T\right)L^T,$$

  where $\mathbf{Z}_j \sim N_p(0, I_p)$.
- Now, fix a symmetric matrix T. The matrix L^TTL is also symmetric, and therefore we can compute its spectral decomposition:

$$L^TTL = U\Lambda U^T,$$

  where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_p)$ is diagonal and $UU^T = I_p$.

Characteristic function iii

- We can now write

Characteristic function iv

$$
\begin{aligned}
\operatorname{tr}(TS) &= \operatorname{tr}\left(TL\left(\sum_{j=1}^{m}\mathbf{Z}_j\mathbf{Z}_j^T\right)L^T\right) \\
&= \operatorname{tr}\left(U\Lambda U^T\left(\sum_{j=1}^{m}\mathbf{Z}_j\mathbf{Z}_j^T\right)\right) \\
&= \operatorname{tr}\left(\Lambda U^T\left(\sum_{j=1}^{m}\mathbf{Z}_j\mathbf{Z}_j^T\right)U\right) \\
&= \operatorname{tr}\left(\Lambda\left(\sum_{j=1}^{m}(U^T\mathbf{Z}_j)(U^T\mathbf{Z}_j)^T\right)\right).
\end{aligned}
$$

Characteristic function v

- Two key observations:
  - $U^T \mathbf{Z}_j \sim N_p(0, I_p)$;
  - $\mathrm{tr}\left(\Lambda \mathbf{Z}_j \mathbf{Z}_j^T\right) = \sum_{k=1}^p \lambda_k Z_{jk}^2$.
- Putting all this together, we get

$$
\begin{aligned}
E\left(\exp(i\mathrm{tr}(TS))\right) &= E\left(\exp\left(i\sum_{j=1}^m\sum_{k=1}^p \lambda_k Z_{jk}^2\right)\right)\\
&= \prod_{j=1}^m\prod_{k=1}^p E\left(\exp\left(i\lambda_k Z_{jk}^2\right)\right).
\end{aligned}
$$

Characteristic function vi

- But $Z_{jk}^2 \sim \chi^2(1)$, and so we have

$$\varphi_S(T) = \prod_{j=1}^m \prod_{k=1}^p \varphi_{\chi^2(1)}(\lambda_k).$$

- Recall that $\varphi_{\chi^2(1)}(t) = (1 - 2it)^{-1/2}$, and therefore we have

$$\varphi_S(T) = \prod_{j=1}^m \prod_{k=1}^p (1 - 2i\lambda_k)^{-1/2}.$$

Characteristic function vii

- Since $\prod_{k=1}^{p}(1-2i\lambda_k)^{-1/2} = |I_p - 2i\Lambda|^{-1/2}$, we then have

$$\begin{aligned}
\varphi_S(T) &= \prod_{j=1}^{m} |I_p - 2i\Lambda|^{-1/2} \\
&= |I_p - 2i\Lambda|^{-m/2} \\
&= |I_p - 2iU\Lambda U^T|^{-m/2} \\
&= |I_p - 2iL^T T L|^{-m/2} \\
&= |I_p - 2i\Sigma T|^{-m/2}
\end{aligned}$$

$\square$

Density of Wishart distribution

- Let $S \sim W_p(m, \Sigma)$ with Σ positive definite and $m \geq p$. The density of S is given by

$$f(S) = \frac{1}{2^{pm/2}\Gamma_p(\frac{m}{2})|\Sigma|^{m/2}} \exp\left(-\frac{1}{2}\mathrm{tr}(\Sigma^{-1}S)\right) |S|^{(m-p-1)/2},$$

where

$$\Gamma_p(u) = \pi^{p(p-1)/4} \prod_{i=0}^{p-1} \Gamma\left(u - \frac{i}{2}\right), \quad u > \frac{1}{2}(p-1).$$

- *Proof*: Compute the characteristic function using the expression for the density and check that we obtain the same result as before (**Exercise**).

Sampling distribution of sample covariance

- We are now ready to prove the results we stated a few lectures ago.
- Recall again the univariate case:
  - $\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$;
  - $\bar{X}$ and s^2 are independent.
- In the multivariate case, we want to prove:
  - $(n-1)S_n \sim W_p(n-1, \Sigma)$;
  - $\bar{\mathbf{Y}}$ and S_n are independent.
- We will show that using the **multivariate Cochran theorem**

Cochran theorem

Let $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ be a random sample with $\mathbf{Y}_i \sim N_p(0, \Sigma)$, and write $\mathbb{Y}$ for the $n \times p$ matrix whose i-th row is $\mathbf{Y}_i$. Let A, B be $n \times n$ symmetric matrices, and let C be a $q \times n$ matrix of rank q. Then

1. $\mathbb{Y}^T A \mathbb{Y} \sim W_p(m, \Sigma)$ if and only if $A^2 = A$ and $\mathrm{tr} A = m$.
2. $\mathbb{Y}^T A \mathbb{Y}$ and $\mathbb{Y}^T B \mathbb{Y}$ are independent if and only if $AB = 0$.
3. $\mathbb{Y}^T A \mathbb{Y}$ and $C\mathbb{Y}$ are independent if and only if $CA = 0$.

Application i

- Let $C = \frac{1}{n}\mathbf{1}^T$, where $\mathbf{1}$ is the n-dimensional vector of ones.
- Let $A = I_n - \frac{1}{n}\mathbf{1}\mathbf{1}^T$.
- Then we have

$$\mathbb{Y}^T A \mathbb{Y} = (n-1)S_n, \qquad C\mathbb{Y} = \bar{\mathbf{Y}}^T.$$

- We need to check the conditions of Cochran's theorem:
  - $A^2 = A$;
  - $CA = 0$;
  - $\mathrm{tr} A = n - 1$.

Application ii

- Using Parts 1. and 3. of the theorem, we can conclude that
  - $(n-1)S_n \sim W_p(n-1, \Sigma)$;
  - $\bar{\mathbf{Y}}$ and S_n are independent.

Proof (Cochran theorem) i

Note 1: We typically only use one direction ($\Leftarrow$).

Note 2: We will only prove the first part.

- Since A is symmetric, we can compute its spectral decomposition as usual:

$$A = U \Lambda U^T.$$

- By assuming $A^2 = A$, we are forcing the same condition on the eigenvalues:

$$\Lambda^2 = \Lambda.$$

- But only two real numbers are possible $\lambda_i \in \{0, 1\}$.

Proof (Cochran theorem) ii

- Given that $\mathrm{tr}A = m$, and after perhaps reordering the eigenvalues, we have

$$\lambda_1 = \cdots = \lambda_m = 1, \quad \lambda_{m-1} = \cdots = \lambda_n = 0.$$

- Now, set $\mathbb{Z} = U^T \mathbb{Y}$, and let $\mathbf{Z}_i$ be the i-th row of $\mathbb{Z}$. We have

$$\begin{aligned}
\mathrm{Cov}(\mathbb{Z}) &= E((U^T \mathbb{Y})^T (U^T \mathbb{Y})) \\
&= E(\mathbb{Y}^T U U^T \mathbb{Y}) \\
&= E(\mathbb{Y}^T \mathbb{Y}) \\
&= \mathrm{Cov}(\mathbb{Y}).
\end{aligned}$$

Proof (Cochran theorem) iii

- Therefore, the covariance structures of $\mathbb{Y}$ and $\mathbb{Z}$ are the same:
  - The vectors $\mathbf{Z}_1, \ldots, \mathbf{Z}_n$ are still independent.
  - $\mathbf{Z}_i \sim N_p(0, \Sigma)$.
- We can now write

$$\begin{aligned}
\mathbb{Y}^T A \mathbb{Y} &= \mathbb{Y}^T U \Lambda U^T \mathbb{Y} \\
&= \mathbb{Z}^T \Lambda \mathbb{Z} \\
&= \sum_{i=1}^{m} \mathbf{Z}_i \mathbf{Z}_i^T.
\end{aligned}$$

- Therefore, we conlude that $\mathbb{Y}^T A \mathbb{Y} \sim W_p(m, \Sigma)$. $\square$