ANALYSIS OF THE EBOLA FRACTIONAL-ORDER MODEL WITH CAPUTO-FABRIZIO DERIVATIVE

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December 14, 2022

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Introduction

Ebola Virus

The Filoviridae family member Ebola virus (EBOV) causes the inflammatory, severe, potentially fatal disease known as (EVD) Ebola viral disease (used to be referred to as Ebola hemorrhagic fever), which affects both humans and great apes. The first species of EBOV was discovered near the Ebola river in the Democratic Republic of Congo in the Central African continent in 1976.

Preliminaries

Definition: C-Derivative. *The fractional-order ϕ-derivative with $\phi \in (0, 1]$ of function $f \in H^1[a, b]$ in Caputo's sense is defined as,*

$$ {}^{C}\mathcal{D}_t^{\phi} f(t) = \frac{1}{\Gamma(1-\phi)} \int_a^t f'(s)(t-s)^{-\phi} ds, \quad t > a. \tag{1} $$

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Definition: CF-derivative. *The new fractional-order ϕ-derivative of a function f in Caputo-Fabrizio's sense can be written as,*

$$ {}^{CF}\mathcal{D}_t^{\phi} f(t) = \frac{(2-\phi)M(\phi)}{2(1-\phi)} \int_a^t f'(s) \exp\left[-\frac{\phi(t-s)}{(1-\phi)}\right] ds, \quad t > a \tag{2} $$

where $M(\phi)$ is the normalizing constant function depending upon ϕ.

Preliminaries

Definition: CF-integral. *The fractional integral of ϕ order of a function f in Caputo-Fabrizio's sense is defined as,*

$$ {}^{CF}\mathcal{I}_t^{\phi} f(t) = \frac{2(1-\phi)}{(2-\phi)M(\phi)} f(t) + \frac{2\phi}{(2-\phi)M(\phi)} \int_a^t f(\tau) d\tau, \quad t > a. \tag{3} $$

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Definition: Laplace Transform of CF-derivative. *The Laplace transform of the fractional-order ϕ-derivative of a function f in Caputo-Fabrizio's sense is defined as,*

$$ \mathcal{L}\left\{ {}^{CF}\mathcal{D}_t^{\phi} f(t), s \right\} = \frac{s\mathcal{L}\{f(t), s\} - f(0)}{(s + \phi(1-s))}, \qquad s \geq 0. \tag{4} $$

Model formulation

Compartments

The total population (N) is divided in six compartments, where each compartments of the proposed model denotes a specific stage of illness,

1. Susceptible that are uninfected (S)
2. Exposed to the Ebola infection (E)
3. Infectious from the infection (I)
4. Hospitalized (H)
5. Deceased or critically sick (D)
6. Recovered from infection (R).

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At any instance of time t, the sum of all the compartments is

$$N(t) = S(t) + E(t) + I(t) + H(t) + D(t) + R(t).$$

Diagram

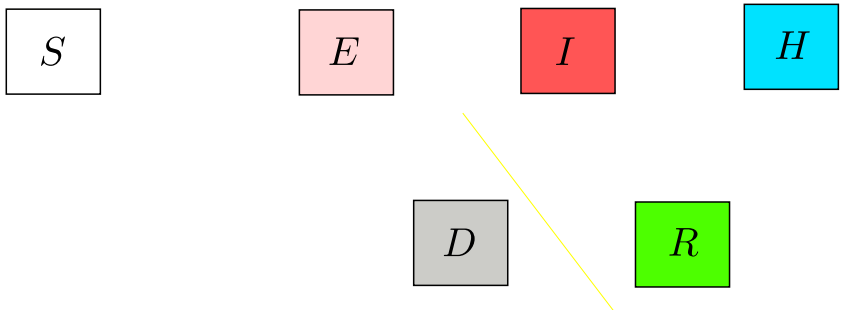

Figure 1: All Compartments.

Diagram

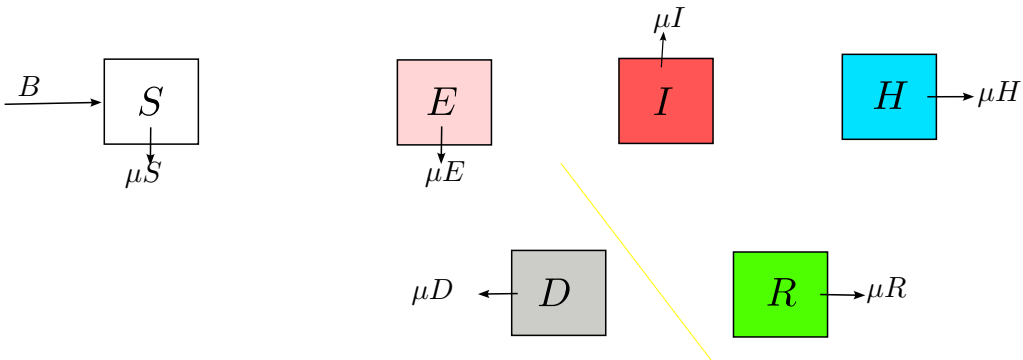

Figure 2: The Flow diagram of the proposed Ebola model.

Diagram

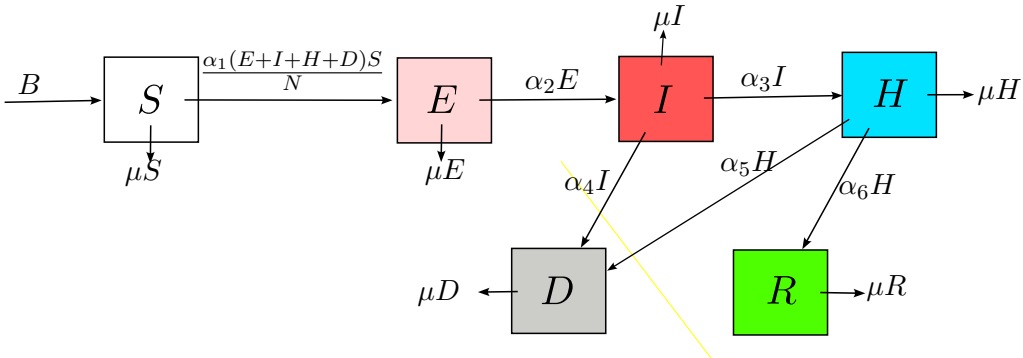

Figure 3: The Flow diagram of the proposed Ebola model.

Governing system

$$
\begin{aligned}
{}^{CF}D_t^{\phi} E(t) &= \frac{\alpha_1 \left(E(t)+I(t)+D(t)+H(t)\right) S(t)}{N(t)} - (\mu+\alpha_2) E(t) \\
{}^{CF}D_t^{\phi} I(t) &= \alpha_2 E(t) - (\alpha_3+\alpha_4+\mu) I(t) \\
{}^{CF}D_t^{\phi} H(t) &= \alpha_3 I(t) - (\alpha_5+\alpha_6+\mu) H(t) \\
{}^{CF}D_t^{\phi} D(t) &= \alpha_4 I(t) + \alpha_5 H(t) - \mu D(t) \\
{}^{CF}D_t^{\phi} R(t) &= \alpha_6 H(t) - \mu R(t) \\
{}^{CF}D_t^{\phi} S(t) &= B - \mu S(t) - \frac{\alpha_1 \left(E(t)+I(t)+D(t)+H(t)\right) S(t)}{N(t)}
\end{aligned}
\tag{5}
$$

It can be represented in vector form as follows, ${}^{CF}\mathcal{D}_t^{\phi}\boldsymbol{\psi}(t) = \mathbf{K}(t, \boldsymbol{\psi}(t))$ with

$$\boldsymbol{\psi}(t) = (E(t), I(t), H(t), D(t), R(t), S(t))' \text{ and } \boldsymbol{\psi}(0) \in \mathbb{R}_+^6 \cup \mathbf{0}.$$

Equilibrium points

Two equillibrium points are calculated which are the steady-state solution of the governing system (5).

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2. The endemic equilibrium point E^1.

Basic Reproduction Number

The next-generation matrix method is used to calculate the the basic reproduction number for this proposed Ebola biological model. It is found to be,

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$$\mathcal{R}_0 = \frac{\begin{array}{r} B(\mu^3 + (\alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\mu^2 + ((\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6)\alpha_2 + \\ (\alpha_5 + \alpha_6)(\alpha_3 + \alpha_4))\mu + ((\alpha_5 + \alpha_6)\alpha_4 + \alpha_3\alpha_5)\alpha_2)\alpha_1 \end{array}}{N(\mu + \alpha_5 + \alpha_6)(\mu + \alpha_3 + \alpha_4)(\mu + \alpha_2)\mu^2} \tag{6}$$

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It denotes the average number of infections spread by an infectious individual in a healthy population.

Analysis of Model

Invariant Positive Region

Lemma 3.1

The proposed fractional-order model for Ebola infection (5)'s feasible domain of solution is

$$\Omega = \left\{ (S, E, I, H, D, R) \in \mathbb{R}_+^6 \mid 0 \leq N \leq \frac{B}{\mu} \right\}$$

positively invariant.

Lipschitz's condition

Lemma 3.2

The kernels $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_4, \mathcal{K}_5,$ and $\mathcal{K}_6$ satisfy the Lipschitz's condition if

$$0 \leq L = \sup\{L_1, L_2, L_3, L_4, L_5, L_6\} < 1$$

where $L_1 = |\alpha_1 s_1 - (\mu + \alpha_2)|, L_2 = |\mu + \alpha_3 + \alpha_4|, L_3 = |\mu + \alpha_5 + \alpha_6|, L_4 = L_5 = \mu, L_6 = |\alpha_1(1 - s_0 - r_0) - \mu|, s_0 = \inf_t \frac{S(t)}{N(t)} \leq \sup_t \frac{S(t)}{N(t)} = s_1,$ and $r_0 = \inf_t \frac{R(t)}{N(t)}$.

Uniqueness of solutions

Theorem 3.1

There exists a unique solution of the proposed fractional system (5) provided if,

$$0 \leq \theta = \frac{2L(1-\phi+\phi T_{\sup})}{M(\phi)(2-\phi)} < 1, \text{ where } L = \sup\{L_1, L_2, L_3, L_4, L_5, L_6\}.$$

UHR-Stability

Definition: UHR-Stability. The fractional-order model ${}^{CF}\mathcal{D}_t^{\phi}\psi(t) = \mathcal{K}(t,\psi(t))$ is generalised Ulam-Hyers-Rassias (UHR) stable in accordance with $\mathcal{Y}(t) \in H^1[J,\mathbb{R}^+]$ if there exists a positive real value ϵ_ϕ (depending upon ϕ) such that for every solution ψ of the following inequality,

$$\left|{}^{CF}\mathcal{D}_t^{\phi}\psi(t) - \mathcal{K}(t,\psi(t))\right| \leq \mathcal{Y}(t),$$

There exists a solution $\tilde{\psi} \in H^1(J,\mathbb{R}^+)$ of the model with the following,

$$|\psi(t) - \tilde{\psi}(t)| \leq \epsilon_\phi \mathcal{Y}(t) \text{ for each } t \in J.$$

UHR-Stability

Theorem 3.2

The fractional-order system (5) governing the proposed Ebola model is generalised UHR-stable in accordance with non-decreasing positive function $\mathcal{Y}$ if

$$0 \leq \theta = \frac{2L(1-\phi+\phi T_{\sup})}{M(\phi)(2-\phi)} < 1.$$

Numerical Scheme

Numerical scheme

The normalisation function is assumed to be $M(\phi) = (2-\phi^2)/(2-\phi)$ which satisfies $M(0) = M(1) = 1$. Considering uniform step-size h along the time axis, and using classical two-step Adams-Bashforth scheme, the following numerical scheme is obtained for the proposed fractional-order system (5).

$$\boldsymbol{\psi}(t_{n+1}) = \boldsymbol{\psi}(t_n) + \frac{2+(3h-2)\phi}{2-\phi^2}\mathbf{K}(t_n, \boldsymbol{\psi}(t_n)) - \frac{2+(h-2)\phi}{2-\phi^2}\mathbf{K}(t_{n-1}, \boldsymbol{\psi}(t_{n-1})) \tag{7}$$

Simulations

The numerical simulation is performed in the MATLAB software.

Table 1: Parameter's value

Parameter	**Value**
B	10
μ	0.1
α_1	0.75
α_2	0.85
α_3	0.425
α_4	0.2
α_5	0.15
α_6	0.25

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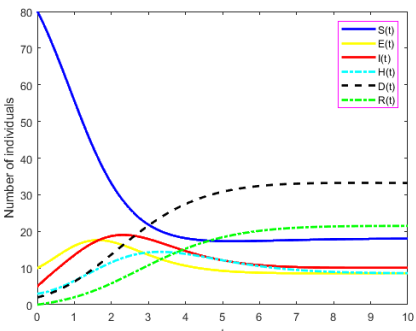
Table 2: Initial condition

Compartment	Value
$S(0)$	80
$E(0)$	10
$I(0)$	5
$H(0)$	3
$D(0)$	2
$R(0)$	0

Simulations

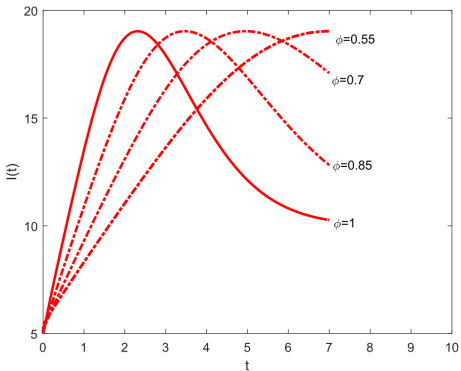
Figure 4: All compartments.



It shows the transmission dynamics of each compartment listed in the proposed model over time. The behaviour is smooth, and it validates the theoretical results.

Simulations

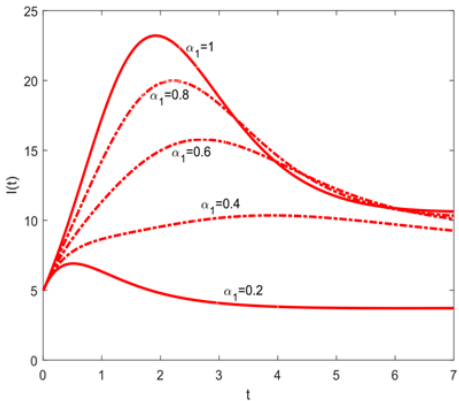
Figure 5: Effect of different values of ϕ on $I(t)$ over time t.



It shows the behaviour for the different values of the fractional order ϕ. For relatively small values of the fractional order ϕ, the number of infectious individuals reaches the exact peak of approximately 19 cases but takes a relatively long time.

Simulations

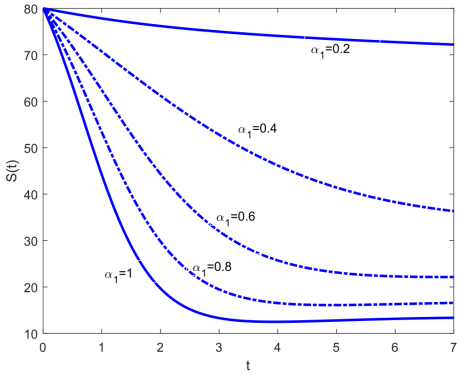
Figure 6: Effect of α_1 on $I(t)$ over time t with integer-order.



It shows the behaviour of the Ebola infectious cases for the different values of the contact rate over time. The relatively more contact rate of susceptible with the pathogen carriers individuals will surge the number of Ebola infectious individuals. The contact rate is the crucial parameter for this model that directly influences the cases of the Ebola. It shows that the most efficient way to control the spread of EBOV infection is to control the contact rate parameter.

Simulations

Figure 7: Effect of the contact rate on susceptible individuals $S(t)$ over time t with integer-order.



Simulations

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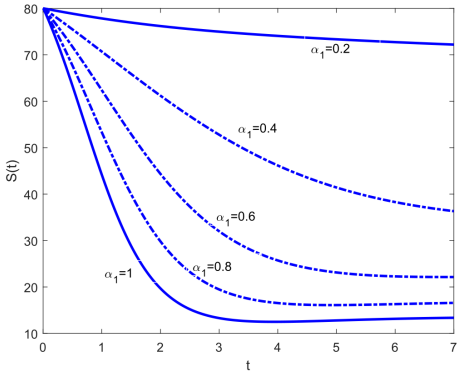
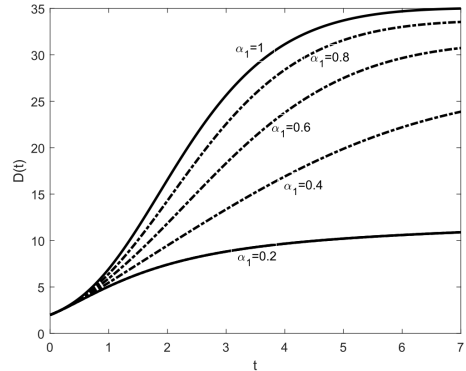


Figure 8: Effect of the contact rate on deceased individuals $D(t)$ over time t with integer-order.



CPU TIME USAGE

Table 3: CPU time usage

Δt	n	Time(s)
0.1	100	0.29 s
0.01	1000	0.34 s
0.001	10^4	0.41 s
0.0001	10^5	1.31 s
0.00001	10^6	2.39 s

The CPU time usage is listed with the different values of step size Δt and iterations n of the mentioned numerical scheme for this proposed model . The table makes it clear that the proposed strategy increases efficiency while taking less time.

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- The generalized UHR-stability condition for the suggested model is found.
- Figures illustrate how the approximate solution of the proposed model differs for integer and fractional orders in numerical simulations.
- The CPU usage time tells that mentioned numerical scheme is efficient and time-saving.

Acknowledgment

The author would like to acknowledge the CSIR-HRDG for the financial assistance.

Thank you

Any Questions ?

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