MAT409 - Complex Analysis - II	**Semester II**

Lecture 1
Argument Principle
13^{th} February, 2023

Kapil Chaudhary **kapilchaudhary[at]university-domain**

Prerequisites: You must know in details about the followings

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Prerequisites: You must know in details about the followings

- Residue Theorem
- Zeros and Singularities of a function
- Cauchy Integral Formula

1. Winding number

The winding number η of a contour γ about a point z_0 is defined by:

$$\eta(\gamma, z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - z_0}$$

It indicates how many times the γ curve circles z_0 in the anticlockwise direction. A positive winding number is given to anticlockwise winding and a negative winding number to clockwise winding. The winding number is sometimes called as index as well by some authors, so don't be confused.

$$\text{Ind}_\gamma(z_0) = \eta(\gamma, z_0)$$

See 2 for problems.

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Proposition 1 *Let γ and σ be two closed rectifiable*[1] *curve having same initial points then*

- $\eta(\gamma, a) = -\eta(-\gamma, a)$ *for every* $a \notin \{\gamma\}$,
- $\eta(\gamma + \sigma, a) = \eta(\gamma, a) + \eta(\sigma, a)$ *for every* $a \notin \{\gamma\} \cup \{\sigma\}$.

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Proof: Left to reader. ■

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Proof: Left to reader. ■

Definition 2 *Let G be an open set then γ is homologous to zero ($\gamma \approx 0$) if $\eta(\gamma, w) = 0$ for each $w \in \mathbb{C} - G$.*

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2. Argument Principle

The argument principle (or principle of the argument) is a consequence of the residue theorem. It connects the winding number of a curve with the number of zeros and poles inside the curve. This is useful for applications (mathematical and otherwise) where we want to know the location of zeros and poles.

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Theorem 3 *Let $\Omega \subset \mathbb{C}$ be open. Let $\gamma \approx 0$ be a closed rectifiable curve inside Ω such that interior of the γ belongs to Ω. Let f be a meromorphic function such that f has no zeros and poles on γ. Then*

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz &= \begin{pmatrix}\text{number of zeros of } f \text{ inside interior} \\ \text{of } \gamma \text{ counted with multiplicities}\end{pmatrix} - \begin{pmatrix}\text{number of poles of f inside interior} \\ \text{of } \gamma \text{ counted with multiplicities}\end{pmatrix}. \\ &= Z_{f,\gamma} - P_{f,\gamma}\end{aligned}$$

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Ouline of proof: What are the singularities of the function f'/f?

- zeros of f.
- poles of f as function f is not defined at pole.
- Any other? No! (Why?)

Calculate the residue of the function f'/f at singularities (zeroes and poles of f) and use **residue theorem**.

Proof: All the singularities of the function f'/f are either zeroes of function f or poles of the function f. Let z_0 be the zero of order n of $f(z)$. Then, for every z in the neighbourhood of z_0, we have

$$f(z) = (z - z_0)^n g(z)$$

where $g(z_0) \neq 0$ and $g(z)$ is holomorphic in neighbourhood of z_0.

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{n(z-z_0)^{n-1}g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)} \\ &= \frac{n}{z-z_0} + \frac{g'(z)}{g(z)}.\end{aligned}$$

So, z_0 is a simple pole of f'/f and $\operatorname{res}_{z=z_0} \frac{f'(z)}{f(z)} = n$. Let z_1 lebe the pole of order m of $f(z)$. Then, for every z in the neighbourhood of z_1, we have

$$f(z) = (z-z_1)^{-m} h(z)$$

where $h(z_1) \neq 0$ and $h(z)$ is holomorphic in neighbourhood of z_1.

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{-m(z-z_1)^{-m-1}h(z) + (z-z_1)^{-m} h'(z)}{(z-z_1)^{-m} h(z)} \\ &= \frac{-m}{z-z_1} + \frac{h'(z)}{h(z)}.\end{aligned}$$

So, z_1 is a simple pole of f'/f and $\operatorname{res}_{z=z_1} \frac{f'(z)}{f(z)} = -m$. Using **residue theorem**,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz &= \text{sum of the residue of the function } \frac{f'}{f} \\ &= \begin{pmatrix}\text{number of zeros of } f \text{ inside interior} \\ \text{of } \gamma \text{ counted with multiplicities}\end{pmatrix} - \begin{pmatrix}\text{number of poles of f inside interior} \\ \text{of } \gamma \text{ counted with multiplicities}\end{pmatrix}.\end{aligned}$$

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Remark 1 *Argument principle implies that* $I := \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz$ *is an integer. Further, if we deform continuously*[2] *the* γ *and/or* $f(z)$, *value of* I *does not change as long as they are well defined.*

[2] A continuous deformation of a region is where we can shrink or twist the region but without tearing. Gluing is also not allowed.

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Question: Why the above theorem is called Argument principle?

Hint: Read introduction part of argument principle from book - *Complex Variables and Applications* by *Brown and Churchill*.

Question: Is this theorem somehow related to the winding number or index of the contour γ?

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Theorem 4 (Argument's Principle) *Let* $\Omega \subset \mathbb{C}$ *be open. Let* $\gamma \approx 0$ *be a closed simple curve oriented in counterclockwise direction and* f *be a meromorphic function inside interior of* γ *(means* f *is analytic inside* γ *except for finitely many points, also* f *should not have any zero or singularity on* γ*). Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = Z_{f,\gamma} - P_{f,\gamma} = \eta(f\circ\gamma, 0)$$

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Outline of proof: This proof is easy, just use the transformation $w = f(z)$.

Proof: Let us use the transformation $w = f(z)$ With this change of variables the contour $z = \gamma(t)$

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becomes $w = f(z) = f(\gamma(t)) = f \circ \gamma(t)$ and $dw = f'(z)dz$. so,

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz &= \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dw}{w}\\ &= \frac{1}{2\pi i}\int_{f\circ\gamma}\frac{dw}{w-0}\\ &= \eta(f\circ\gamma, 0)\end{aligned}$$

Now, using Theorem 3

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \eta(f\circ\gamma, 0) = Z_{f,\gamma} - P_{f,\gamma}$$

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Corollary 5 *Assume that $f\circ\gamma$ does not go through -1 (i.e. there is no zeroes of $1 + f(z)$ on γ). Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{1+f(z)}dz = \eta(f\circ\gamma, -1) = Z_{f+1,\gamma} - P_{f,\gamma}$$

Proof: Applying the argument principle on function $1 + f$,

$$\frac{1}{2\pi i}\int_\gamma \frac{(1+f)'(z)}{1+f(z)}dz = \eta((1+f)\circ\gamma, 0) = Z_{1+f,\gamma} - P_{1+f,\gamma}$$

Note that: $(1 + f)'(z) = f'(z)$, $P_{1+f,\gamma} = P_{f,\gamma}$ (why?) and function $1 + f$ winds about 0 if and only if f winds about -1. ■

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Corollary 6 *Consider f be a polynomial of degree n then for sufficiently large positive number R,*

$$\frac{1}{2\pi i}\oint_{|z|=R}\frac{f'(z)}{f(z)}dz = n.$$

Outline of proof: Given that f is a polynomial of degree n. Using fundamental theorem of algebra, we can assume that $r_1, r_2, r_3, \cdots r_n$ (need not to be distinct) are zeroes of f counted multiplicity wise. i.e.

$$f(z) = C(z-r_1)(z-r_2)(z-r_3)\cdots(z-r_n)$$

Now, consider the counterclockwise oriented circle $|z| = R$ with sufficiently large radius R (R can be chosen in such a way that the open disc $|z| < R$ contains all the roots $r_1, r_2, r_3, \cdots r_n$ of f)

$$R = 1 + \max\{|r_1|, |r_2|, \cdots |r_n|\}$$

Now, use the argument principle.

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Now, use the argument principle.

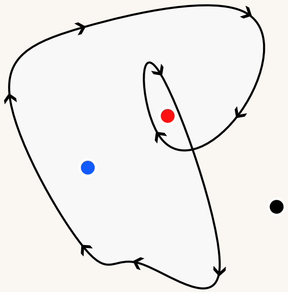
Theorem 7 (Extended Argument Principle) *Let f be a meromorphic function with zeros $z_1, z_2, \cdots z_n$ and poles $p_1, p_2, \cdots p_m$ counted multiplicities wise in the region G. If g is analytic in region G and γ be a closed rectifiable curve such that $\gamma \approx 0$ and it does not pass through any of z_i or p_j. Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}g(z)dz = \sum_{i=1}^{n} g(z_i)\eta(\gamma, z_i) - \sum_{j=1}^{m} g(p_j)\eta(\gamma, p_j)$$

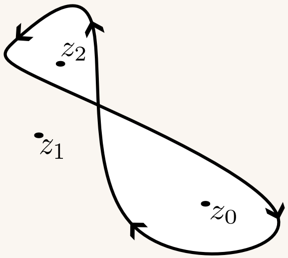
Proof: Exempted. ■

Problems

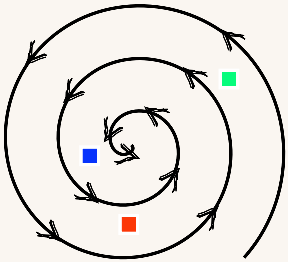
1. Find the winding number of the following curves γ about points mentioned.



(a) about red, blue and black point



(b) about z_0, z_1 and z_1



(c) about red, blue and green point

2. Let $\gamma(t) = it$ with $-\infty < t < \infty$ (the y-axis). Let $f(z) = 1/(z+1)$. Describe the curve $f \circ \gamma(t)$.
3. Let $f(z) = z^2 + z$. Find the winding number of $f \circ \gamma$ about 0 using **argument principle**.

(a) $\gamma_1(t) = 2e^{it}$, $0 \leq t \leq 2\pi$

(b) $\gamma_2(t) = e^{it}$, $0 \leq t \leq 2\pi$

(c) $\gamma_3(t) = \frac{1}{2}e^{it}$, $0 \leq t \leq 2\pi$

4. Find the value of the following integrals using **argument principle**. Further, verify the answers using **residue theorem**.

   (a) $\displaystyle\int_{|z|=2} \frac{3z^2-2z+1}{(z^2+1)(z-1)} dz$

   (b) $\displaystyle\int_{|z|=2} \frac{z^2}{z-1} dz$

5. Let $f(z) = \dfrac{(z-2)^2 z^3}{(z+5)^2(z+i)^3(z-1)^4}$. Compute the value of $\displaystyle\int_{|z|=3} \frac{f'(z)}{f(z)} dz$.

6. Let p be a polynomial of degree 2 such that $p(0) = 2024$, and the following condition holds:

$$\oint_{|z|=2} z\frac{p'(z)}{p(z)} dz = 0, \quad \oint_{|z|=2} z^2\frac{p'(z)}{p(z)} dz = -4.$$

   Then, what is the coefficient of z^2 in polynomial p?

3. Rouche's Theorem

Theorem 8 *Let γ be a simple closed curve. Assume f, h be the analytic function on and inside γ except for finitely many poles (no zeroes and poles of f, h on γ) such that $|h| < |f|$ everywhere on γ. Then*

$$\eta((f+h)\circ\gamma, 0) = \eta(f\circ\gamma, 0)$$

$$Z_{f+h} - P_{f+h} = Z_f - P_f$$

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$$\eta((f+h)\circ\gamma, 0) = \eta(f\circ\gamma, 0)$$
$$Z_{f+h} - P_{f+h} = Z_f - P_f$$

Outline of Proof: We will verify the hypothesis of argument principle on $f, h, f+h, (f+h)/f$.

Now, $|h| < |f| \implies \left|\frac{h}{f}\right| < 1 \implies \left(\frac{h}{f}\right)\circ\gamma$ lies strictly inside unit circle.

This means, $\left(1+\frac{h}{f}\right)\circ\gamma$ lies strictly inside unit circle centred at 1.

0 doesn't belongs to the interior of $\left(1+\frac{h}{f}\right)\circ\gamma$. Consequently, $\eta\left(\left(1+\frac{h}{f}\right)\circ\gamma, 0\right) = 0$.

Let $g = 1+\frac{h}{f} = \frac{f+h}{f}$, so $\int_\gamma \frac{g'}{g} = 0$ (Using argument principle, as $\eta(g\circ\gamma, 0) = 0$)

Now,

$$g' = \left(1+\frac{h}{f}\right)' = \frac{fh' - hf'}{f^2} \implies \frac{g'}{g} = \frac{fh' - hf'}{f(f+h)} = \frac{(f+h)'}{f+h} - \frac{f'}{f}$$

Hence,

$$0 = \int_\gamma \frac{g'}{g} = \int_\gamma \frac{(f+h)'}{f+h} - \int_\gamma \frac{f'}{f} = 2\pi i\left[\eta((f+h)\circ\gamma, 0) - \eta(f\circ\gamma, 0)\right]$$

$$\implies \begin{aligned} \eta((f+h)\circ\gamma,0) = \eta(f\circ\gamma,0) \\ Z_{f+h} - P_{f+h} = Z_f - P_f \end{aligned}$$

Proof: Write yourself using given outline. ■

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Corollary 9 (Special Case of Rouche's Theorem) *Let C be a circle in an open set $\Omega \subset \mathbb{C}$. Further, assume that f_1, f_2 be two holomorphic function (no poles) on Ω s.t. $|f_1(z)| > |f_2(z)| > 0$ for each $z \in C$. Then, $f_1 + f_2$ and f_1 have same number of zeroes inside the interior of C.*

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Proof: Since, functions f_1, f_2 are analytic on Ω so is $f_1 + f_2$. i.e. $P_{f_1+f_2} = P_{f_1} = 0$. Using above theorem,

$$Z_{f_1+f_2} = Z_{f_1}$$

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Example. Show that all the zeroes of $z^5 + 3z + 1$ lies inside $|z| = 2$.

$$\implies \begin{gathered} \eta((f+h)\circ\gamma, 0) = \eta(f\circ\gamma, 0) \\ Z_{f+h} - P_{f+h} = Z_f - P_f \end{gathered}$$

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Example. Show that all the zeroes of $z^5 + 3z + 1$ lies inside $|z| = 2$.
Solution. Assume two holomorphic functions $f_1 = z^5, f_2 = 3z + 1$ and let $C = \{z \in \mathbb{C} \mid |z| = 2\}$. Now, $|z^5| > |3z + 1|$ for each $z \in C$. So, it satisfies the hypothesis of the Corollary 9. Thus, $z^5 + 3z + 1$ have same number of zeroes as z^5 inside circle C. But all the zeros of z^5 are inside the circle C. Consequently, all the zeroes of $z^5 + 3z + 1$ lies inside $|z| = 2$.

$$\implies \begin{aligned} \eta((f+h)\circ\gamma,0) = \eta(f\circ\gamma,0) \\ Z_{f+h} - P_{f+h} = Z_f - P_f \end{aligned}$$

Proof: Write yourself using given outline. ∎

Corollary 9 (Special Case of Rouche's Theorem) *Let C be a circle in an open set $\Omega \subset \mathbb{C}$. Further, assume that f_1, f_2 be two holomorphic function (no poles) on Ω s.t. $|f_1(z)| > |f_2(z)| > 0$ for each $z \in C$. Then, $f_1 + f_2$ and f_1 have same number of zeroes inside the interior of C.*

Proof: Since, functions f_1, f_2 are analytic on Ω so is $f_1 + f_2$. i.e. $P_{f_1+f_2} = P_{f_1} = 0$. Using above theorem,

$$Z_{f_1+f_2} = Z_{f_1}$$

∎

Example. Show that all the zeroes of $z^5 + 3z + 1$ lies inside $|z| = 2$.
Solution. Assume two holomorphic functions $f_1 = z^5, f_2 = 3z + 1$ and let $C = \{z \in \mathbb{C} \mid |z| = 2\}$. Now, $|z^5| > |3z+1|$ for each $z \in C$. So, it satisfies the hypothesis of the Corollary 9. Thus, $z^5 + 3z + 1$ have same number of zeroes as z^5 inside circle C. But all the zeros of z^5 are inside the circle C. Consequently, all the zeroes of $z^5 + 3z + 1$ lies inside $|z| = 2$.

Problems

1. Use Rouche's theorem to prove that all the zeroes of the following equation lies in the annulus $0.5 \leq |z| < 1.25$.
$$z^6 + (1+i)z + 1 = 0$$
2. Find the number of roots of $h(z) = 6z^4 + z^3 - 2z^2 + z - 1 = 0$ inside unit disc $|z| \leq 1$.
3. **(True/False)** Suppose f is a holomorphic function on and inside simple closed curve γ such that that it has n zeroes inside γ then f' has $n-1$ zeroes inside γ. **Hint:** $f(z) = e^z - 1$ inside $|z| = 3\pi$.
4. Prove that the equation $z = 2 - e^{-z}$ has exactly one root in the right half-plane $\{Re(z) > 0\}$.
5. Use Rouche's theorem to count the number of zeroes for $f(z) = z^2 - 4 + 3e^{-z}$ on the right half plane $\{Re(z) > 0\}$.
6. Show that $f(z) = z + 3 + 2e^z$ has one root in the left half-plane $\{Re(z) < 0\}$.
7. Use Rouche's Theorem to prove the fundamental theorem of algebra.

4. Mobius Transformation

Definition 10 *A Mobius (Fractional linear or Bilinear) transformation is a function of the form*

$$T : z \to \frac{az+b}{cz+d}, \quad \text{where } a, b, c, d \in \mathbb{C} \mid ad - bc \neq 0.$$

Note: *whenever* $ad - bc = 0 \implies T$ *is a constant function (How?).*

Remark 2 *The inverse of a Mobius transformation is again a Mobius transformation.*

$$T^{-1} : z \to \frac{dz-b}{-cz+a}$$

Remark 3 *Each Fractional linear transformation is the composition of the following four elementary maps:*

- *(Translation)* $z \to z + z_0,\ z_0 \in \mathbb{C}$
- *(Dilation or Scaling)* $z \to \lambda z\,,\ \lambda \in \mathbb{R}^+$
- *(Rotation)* $z \to ze^{i\theta}, \theta \in \mathbb{R}$
- *(Inversion)* $z \to 1/z$

Whenever $c = 0 \implies T(z) = \frac{a}{d}z + \frac{b}{d}$ is the composition of dilation, rotation, followed by a translation.

Whenever $c \neq 0$

$$\begin{aligned} T(z) = \frac{az+b}{cz+d} &= \frac{az}{c(z+d/c)} + \frac{b}{c(z+d/c)} \\ &= \frac{a}{c} - \frac{ad}{c^2(z+d/c)} + \frac{b}{c(z+d/c)} \\ &= \frac{a}{c} + \frac{bc-ad}{c^2}\left(\frac{1}{z+d/c}\right) \end{aligned}$$

is the composition of translation, inversion, dilation, rotation followed by a translation.

Example. Write down the Mobius transformation $T : z \to \dfrac{z+i}{z-i}$ as composition of translation, scaling, rotation, and inversion. **Ans:** $1 + 2e^{i\pi/2}\left(\dfrac{1}{z-i}\right)$

Definition 11 *A point z is said to be fixed point of transformation T if it satisfies $T(z) = z$.*

Remark 4 *Any Mobius transformation (except the identity transformation) can have at most 2 fixed points in $\mathbb{C}_\infty$.'*

$$T(z) = \frac{az+b}{cz+d} = z \implies cz^2 + (d-a)z - b = 0.$$

Example. Find the fixed points of the Mobius transformation $T : z \to \dfrac{3z-1}{z+5}$. **Ans:** $z = -1$.

Theorem 12 *A Mobius transformation is completely determined by it's action on three distinct points in $\mathbb{C}_\infty$.*

Proof: Suppose to contrary, there are two distinct Mobius transformation S and T which (both) maps three distinct points a, b, c to α, β, γ respectively. i.e.

$$S(a) = T(a) = \alpha$$
$$S(b) = T(b) = \beta$$
$$S(c) = T(c) = \gamma.$$

Now, consider the Mobius map $T^{-1} \circ S$ having three distinct fixed points a, b, c. So, it must be an identity transformation. ie. $T^{-1} \circ S = \text{Id} \implies S = T$. ■

Theorem 13 *A Mobius transformation maps* <u>*circles and lines*</u> *to* <u>*circles and lines*</u> *complex plane* $\mathbb{C}$.

Proof: Let S be a circle and L be a line in complex plane $\mathbb{C}$. Any Mobius transform consisting of rotation, translation and scaling maps circle S to a circle S' and line L to line L'.

Now, consider a circle $|z - z_0| = r$ and $\mathtt{I} : z \to w = 1/z$ be the inversion map.

$$\begin{aligned} |z - z_0|^2 &= r^2 \\ \implies (z - z_0)\overline{(z - z_0)} &= r^2 \\ \implies |z|^2 + |z_0|^2 - 2\text{Re}(\overline{z}z_0) - r^2 &= 0 \\ \implies \frac{1}{|w|^2} + |z_0|^2 - 2\frac{\text{Re}\,(wz_0)}{|w|^2} - r^2 &= 0 \end{aligned}$$

Whenever $|z_0| = r$, we get $2\text{Re}\,(wz_0) = 1$. Assuming $w = u + iv, z_0 = x_0 + iy_0 \implies 2(ux_0 - vy_0) = 1$ is a line in complex plane.

Otherwise, we get $1 - 2\text{Re}(wz_0) + |w|^2\left(|z_0|^2 - r^2\right) = 0 \implies |w|^2 + \dfrac{1 - 2\text{Re}(wz_0)}{|z_0|^2 - r^2} = 0$ which can be further written as

$$\left|w - \frac{\overline{z_0}}{|z_0|^2 - r^2}\right|^2 - \frac{r^2}{\left(|z_0|^2 - r^2\right)^2} = 0$$

is an equation of circle in $\mathbb{C}$.
Consider a line $L : 2\text{Re}(z\overline{z_0}) = a$ for some $a \in \mathbb{R}$ then $w = 1/z$ gives $2\text{Re}(wz_0) = a|w|^2$.
Whenever $a = 0 \implies 2\text{Re}(wz_0) = 0$ which is a line in complex plane through origin.
Whenever $a \neq 0$, Equation of the line $2\text{Re}(wz_0) = 0$ becomes

$$\begin{aligned}
&wz_0 + \overline{wz_0} - a|w|^2 = 0\\
\implies &|w|^2 - \frac{(wz_0 + \overline{wz_0})}{a} + \frac{|z_0|^2}{a^2} - \frac{|z_0|^2}{a^2} = 0\\
\implies &\left|w - \frac{\overline{z_0}}{a}\right|^2 = \frac{|z_0|^2}{a^2}.
\end{aligned}$$

is a circle in complex plane. Hence, all the elementary maps of a mobius transformation maps lines and circle to lines and circles in complex plane $\mathbb{C}$. ■

Definition 14 *The Conformal maps are the functions that preserves angle between curves.*

Problems

- Find a Mobius transformation with two fixed points, namely i and $1 - i$.
- Find the inverse of Mobius transformation $T : z \to \dfrac{z + i}{2z + 3i}$.
-

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Report any dead link to me at **kapilchaudhary@gujaratuniversity.ac.in**