MATH471 — Introduction to Numerical Methods

Homework 1

Ailin & Manuel — GC (Fall 2025)

Reminders

- Write in a neat and legible handwriting or use LaTeX
- Clearly explain the reasoning process
- Write in a complete style (subject, verb, and object)
- Be critical on your results

Ex. 1 — *Cardinality*

1. Prove that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ have the same number of elements.
2. Prove that $[0, 1]$ has as many elements as $\mathbb{R}$.
3. Prove that $[0, 1]$ has more elements than $\mathbb{N}$. *Hint:* understand Cantor's diagonal argument.

Ex. 2 — *Slides*

1. Prove the Cauchy-Schwarz inequality (1.20|1.39) over the complex numbers.
2. Show that a distance is always positive.

Ex. 3 — *Linear algebra*

1. Let f be a linear map from a vector space V_1 into a vector space V_2. Show that the dimension of V_1 is the sum of the dimensions of the kernel and of the image of f. This result is called the rank nullity theorem.
2. Prove that the composition of two linear maps is a linear map.
3. Prove that the inverse of a linear map is a linear map.

Ex. 4 — *Convergence of rationals to irrationals*

Intuitively a *complete space* has "no point missing" anywhere. In particular it means that any Cauchy sequence converges inside the space. In this exercise we show that e is not rational while we can find a Cauchy sequence of rationals converging to e.

1. Show that e is irrational.
2. Show that the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = \left(1 + \frac{1}{n}\right)^n$ is a Cauchy sequence converging to e.
3. Is $\mathbb{Q}$ complete? Explain.

Ex. 5 — *Pi*

1. Write the pseudocode for at least one the following strategy to approximate π.
   a) The polygons method;
   b) Machin's formula $\frac{\pi}{4} = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239}$ and Taylor series;
2. Implement at least one of the previous algorithms in MATLAB.