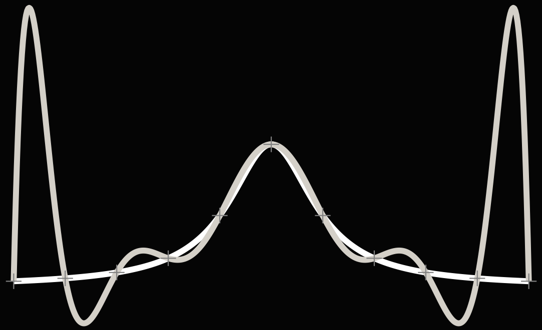
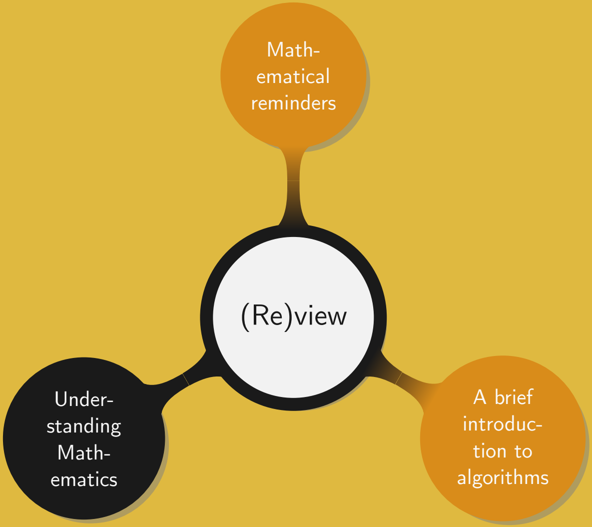


Introduction to Numerical Methods

1. (Re)view

Ailin & Manuel – Fall 2025

Chapter organisation



Mathematics is easy

The universe cannot be read until we have learnt its language and become familiar with the characters in which it is written. It is written in mathematical language, the letters being the triangles, circles, and other geometrical figures without which it would be humanly impossible to comprehend a single word.

Galileo Galilei

Applied and Pure Mathematics

Mathematics splits into two main types:

- Applied: develop mathematical methods to answer the needs from other fields of science such as physics, engineering, or finance.
- Pure: study of abstract concepts that are totally disconnected from the real world.

Relation between pure and applied Mathematics:

- Applied Mathematics highly relies on pure Mathematics to model the real world
- Pure Mathematics is often the result of real world abstractions

Abstraction process:

1. Take a real world problem
2. Phrase it into mathematical terms
3. Eliminate any “physical dependence”
4. Widen the application by generalizing the concept

Goal: use numerical approximations to solve mathematical analysis problems

Applications: physics, engineering, biology, weather prediction, finance

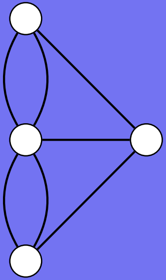
Remark. Numerical analysis is not to be confused with numerical methods: the latter focuses on the study of methods to solve a given problem, while the goal of the former is to develop them. As a result numerical analysis requires a better understanding of the underlying mathematics.

Goal: classify objects with respect to their behavior under smooth deformation

Applications: study the inherent structure of an object with respect to properties such as convergence, connectedness, and continuity

Example.

- Euler's seven bridges of Königsberg
- Is it possible to walk across each bridge exactly once?
- Non-affecting factors: distance between bridges, shape of the "islands"
- Affecting factor: number of "islands" and bridges, as well as their arrangement



Goal: classify sets with respect to how their elements behave under some given operations

Applications: number theory, analysis, geometry, theoretical physics

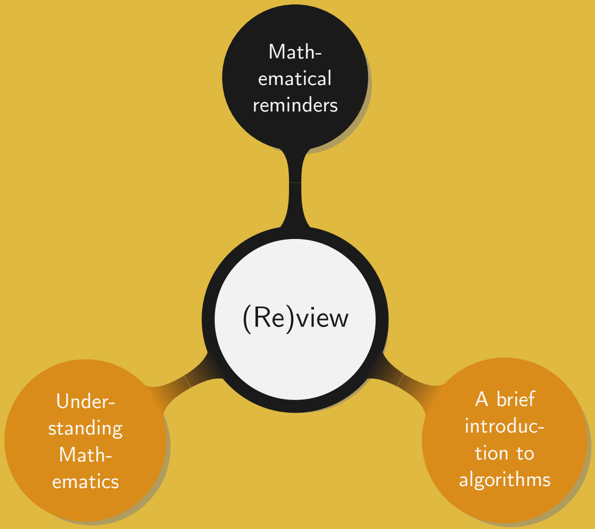
Remark. Although from a historical point of view algebra started with the study of objects with the goal of discovering useful properties, this approach changed during the 20-th century. Nowadays sets are studied from an abstract and formal perspective without taking into account any application.

Goal: find a systematic way to measure the size of any set

Applications: probability theory, analysis

Remark. The idea consists in defining a function called *measure* which maps subsets of a set to positive real values. The main issue is how to keep the size consistent, i.e. given a countable number of disjoint subsets the measure of their union must be equal to the sum of their measures.

Chapter organisation



Definition (Group)

A *group* is a pair $(G, +)$ consisting of a set G and a *group operation* $+\colon G \times G \to G$ that verifies the following properties:

1. *Associativity:* $a + (b + c) = (a + b) + c$ for all $a, b, c \in G$;
2. *Existence of a unit element:* there exists an element $e_+ \in G$ such that for all $a \in G$, $a + e_+ = e_+ + a = a$;
3. *Existence of an inverse:* for any $a \in G$ there exists an element $a^{-1} \in G$ such that $a + a^{-1} = a^{-1} + a = e_+$;

A group is said to be *abelian* if it also verifies

4. *Commutativity:* $a + b = b + a$ for all $a, b \in G$;

Examples.

- $(\mathbb{Z}, +)$ is an abelian group but $(\mathbb{Z}, \times)$ is not.
- Let $\mathcal{F}(X, \mathbb{R})$ be the set of the functions from a set X into $\mathbb{R}$. Then $(\mathcal{F}(X, \mathbb{R}), +)$ is a group but $(\mathcal{F}(X, \mathbb{R}), \circ)$ is not.

Definition (Ring)

A *ring* is a triple $(R, +, \cdot)$ consisting of a set R and two *binary operations* $(+)$ and $(\times)$ defined from $R \times R$ to R , and such that

1. $(R, +)$ is an abelian group;
2. *Multiplicative unit:* there exists an element $e_\times \in R$ such that for all $a \in R$
$$a \times e_\times = e_\times \times a = a;$$
3. *Associativity:* for any $a, b, c \in R$, $a \times (b \times c) = (a \times b) \times c$;
4. *Distributivity:* for any $a, b, c \in R$,
$$a \times (b + c) = (a \times b) + (a \times c), (b + c) \times a = (b \times a) + (c \times a);$$

A ring is said to *commutative* if it also verifies

5. *Commutativity:* $a \times b = b \times a$ for all $a, b \in R$;

Definition (Field)

Let $(\mathbb{K}, +, \times)$ be a commutative ring with unit element of addition 0 and unit element of multiplication 1. Then $\mathbb{K}$ is a *field* if

i. $0 \neq 1$

ii. For every $a \in \mathbb{K}^*$ there exists an element a^{-1} such that

$$a \times a^{-1} = 1.$$

Remark. Another way of writing this definition is to say that $(\mathbb{K}, +, \times)$ is a field if $(\mathbb{K}, +)$ and $(\mathbb{K} \setminus \{0\}, \times)$ are abelian groups, $0 \neq 1$, and $(\times)$ distributes over $(+)$.

Definition (Vector space)

A *vector space* $(V, +, \cdot)$ over a field $(\mathbb{K}, +, \times)$ is a set on which the $(+)$ and $(\cdot)$ operations are defined and closed. Moreover, for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\alpha, \beta \in \mathbb{K}$ the following conditions hold.

i. $(V, +)$ is an abelian group;
ii. *Compatibility of scalar and field multiplications:*

$$\alpha \cdot (\beta \cdot \mathbf{v}) = (\alpha \times \beta) \cdot \mathbf{v};$$

iii. *Distributivity of scalar multiplication for vector addition:*

$$\alpha \cdot (\mathbf{u} + \mathbf{v}) = \alpha \cdot \mathbf{u} + \alpha \cdot \mathbf{v};$$

iv. *Distributivity of scalar multiplication for field addition:*

$$(\alpha + \beta) \cdot \mathbf{v} = \alpha \cdot \mathbf{v} + \beta \cdot \mathbf{v};$$

v. *Scalar multiplication identity:* for all $\mathbf{v} \in V, 1_F \cdot \mathbf{v} = \mathbf{v}$;

Elements of V are called *vectors* and the ones of F *scalar*.

For the sake of simplicity we do not denote vectors using bold fonts, for instance we now write u instead of $\mathbf{u}$.

Definitions

1. Let V be a vector space. A subset of V whose elements are linearly independent and span V is called a *basis* of V.
2. If V is spanned by a finite number of vectors then V is said to have a *finite dimension*. The *dimension* of V is defined as the cardinal of the basis.
3. Let V_1 and V_2 be two vector spaces over a field $\mathbb{K}$. A function f defined from V_1 into V_2 is a *linear map* or *linear map* if:
   - For any $x, y \in V_1$, $f(x+y) = f(x) + f(y)$;
   - For any $x \in V_1$ and $a \in \mathbb{K}$, $f(ax) = af(x)$;
4. A bijective linear map is called an *isomorphism*.

Definition

Let E and F be two vector spaces, and f be a linear map from E to F. The *kernel* of f, noted $\ker f$, is the set of all the elements x from E such that $f(x) = 0$.

Lemma

For two vector spaces E and F of similar dimension, a linear map f from E into F is bijective if and only if $\ker f = \{0\}$.

Proof. If f is bijective then clearly $\ker f = \{0\}$.

Since $\ker f = \{0\}$, f is at least injective. Moreover as E and F have similar dimension, a basis from E is mapped by f into a basis of F. Hence f is bijective. $\square$

Let $\mathbb{K}$ be $\mathbb{R}$ the field of the reals, or $\mathbb{C}$ the field of complexes.

Definitions

Let V be a vector space over a field $\mathbb{K}$.

1. The map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ is an *inner product*, if for $u, v, w \in V$ and $a \in \mathbb{K}$:
   1. *Conjugate symmetry:* $\langle u, v \rangle = \overline{\langle v, u \rangle}$;
   2. *Linearity*: $\langle au, v \rangle = a\langle u, v \rangle$ and $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$;
   3. *Positivity:* $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$;

   A vector space with an inner product is an *inner product space*.

2. The map $\| \cdot \| : V \to \mathbb{R}_+$ is a *norm*, if for any $u, v \in V$ and $a \in \mathbb{K}$:
   1. $\|u\| = 0 \Leftrightarrow u = 0$;
   2. $\|au\| = |a|\|u\|$;
   3. *Triangle inequality:* $\|u + v\| \leq \|u\| + \|v\|$;

   A vector space with an norm is an *normed vector space*.

We now show that inner product spaces have a naturally defined norm, inducing a normed vector space. In this course we restrict our attention to the simpler case $\mathbb{K} = \mathbb{R}$.

Proposition

Let E be an inner-product space over $\mathbb{R}$. Then E is a normed vector space and for $u \in E$ the norm is defined by $|u| = \sqrt{\langle u, u \rangle}$.

Proof. We simply verify definition 1.18.

(i) Clearly $|u| = 0$ if and only if $u = 0$.

(ii) For $\lambda \in \mathbb{R}$, $|\lambda u| = \sqrt{\langle \lambda u, \lambda u \rangle} = |\lambda|\sqrt{\langle u, u \rangle} = |\lambda||u|$.

(iii) We have

$$\begin{aligned} |u+v|^2 = \langle u+v, u+v \rangle &= |u|^2 + |v|^2 + 2\langle u, v \rangle \\ &\leq |u|^2 + |v|^2 + 2|u||v| = (|u| + |v|)^2 \end{aligned} \tag{1.1}$$

Hence from equation (**??**) we obtain $|u+v| \leq |u| + |v|$. $\square$

Since $|\cdot|$ is a norm we denote it $\|\cdot\|$.

Theorem (Cauchy-Schwarz inequality)

Let E be an inner product space over $\mathbb{R}$ and $u, v \in E$. Then

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|.$$

Proof. Let $\lambda \in \mathbb{R}$. Using the notation from the theorem we have

$$0 \leq |u - \lambda v|^2 = \langle u, u\rangle - 2\lambda\langle u, v\rangle + \lambda^2\langle v, v\rangle.$$

Viewing the equation as a quadratic polynomial, its discriminant $\langle u, v\rangle^2 - \langle u, u\rangle\langle v, v\rangle$ cannot be positive since it is over $\mathbb{R}$ and we do not want $|u - \lambda v|^2$ to be negative.

Hence we obtain

$$|\langle u, v\rangle| \leq \sqrt{\langle u, u\rangle} \cdot \sqrt{\langle v, v\rangle} = \|u\| \cdot \|v\|.$$

$\square$

Remarks.

- Although the previous proof of the Cauchy-Schwarz inequality is only valid when $\mathbb{K} = \mathbb{R}$, the result can be proven when $\mathbb{K} = \mathbb{C}$ at the cost of a slightly more complex strategy.
- The equality only happens when the vectors u and v are linearly dependent.

Proposition

Let E be an inner-product space over $\mathbb{R}$ and $u, v \in E$.

1. Parallelogram law: $\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$;
2. $\|u+v\|^2 - \|u-v\|^2 = 4\langle u, v\rangle$;

Proof. The proofs is trivial as we have

$$\begin{aligned}\|u+v\|^2+\|u-v\|^2 &= \|u\|^2+\|v\|^2+2\langle u,v\rangle+\|u\|^2+\|v\|^2-2\langle u,v\rangle \\ &= 2(\|u\|^2+\|v\|^2),\end{aligned}$$

and

$$\begin{aligned}\|u+v\|^2-\|u-v\|^2 &= \|u\|^2+\|v\|^2+2\langle u,v\rangle-\|u\|^2-\|v\|^2+2\langle u,v\rangle \\ &= 4\langle u,v\rangle.\end{aligned}$$

$\square$

Remark. Most normed vector spaces over $\mathbb{R}$ and $\mathbb{C}$ do not have an inner-product. In other words, the converse of proposition 1.19 is false. However if for a norm the parallelogram law is verified then this norm arises from an inner-product.

Definition

For two normed vector spaces E and F, we denote by $\mathrm{L}(E,F)$ the $\mathbb{K}$-vector space of the linear maps from E to F. We define the supremum norm of a linear map u as

$$\|u\| = \sup_{x \in E\setminus\{0\}} \frac{\|u(x)\|}{\|x\|}.$$

An element of $\mathrm{L}(E,F)$ is said to be *bounded* if it has finite norm. All bounded linear maps form a subspace $\mathcal{L}(E,F)$ of $\mathrm{L}(E,F)$.

Remark. Note that E are F are two vectors spaces which can be functions spaces themselves, i.e. their vectors are functions. This means that u is a linear map which is applied to a function. Differentiation and integration are common examples of such u.

Proposition

Let E and F be two vector spaces over $\mathbb{K}$ and $u \in \mathcal{L}(E, F)$. If $(x_n)_{n\in\mathbb{N}}$ is a sequence converging to l, then $\big(u(x_n)\big)_n$ converges to $u(l)$.

Proof. Let n be a positive integer. Then

$$\|u(x_n) - u(l)\| = \|u(x_n - l)\| \leq \|u\|\|x_n - l\|,$$

i.e. if $(x_n)_n$ converges, then so does $\big(u(x_n)\big)_n$. □

Remark. This result is only valid for $u \in \mathcal{L}(E, F)$, i.e. when u has finite norm. If u had infinite norm then we would not be able to bound $\|u(x_n) - u(l)\|$. In particular to find a counter example it is necessary to think of an infinite dimensional space, for instance by looking at a sequence of converging functions.[1]

[1] Refer to h1.6 for an example of discontinuous liner map.

Definitions

1. Let X be a set. A *distance* or *metric* on X is a function $d : X \times X \to \mathbb{R}_+$ such that for all $x, y, x \in X$:
   1. $d(x, y) = 0$ if and only if $x = y$;
   2. $d(x, y) = d(y, x)$;
   3. $d(x, z) \leq d(x, y) + d(y, z)$ (triangular inequality);

   A set endowed with a distance is called a *metric space*.
2. Let X and Y be two metric spaces endowed with distances d and d', respectively. A function $f : X \to Y$ is *continuous* at a, if for all $\varepsilon \in \mathbb{R}_+^*$, there exists $\eta \in \mathbb{R}_+^*$, such that $d(a, x) < \eta$ implies $d'(f(a), f(x)) < \varepsilon$.
3. Let X and Y be two metric spaces. A function $f : X \to Y$ is *continuous* if it is continuous at any point of X.

We now introduce more refined notions of continuity.

Definitions

Let X and Y be two metric spaces endowed with metrics d and d', respectively, and $f : X \to Y$.

1. The function f is *uniformly continuous* if for all $\varepsilon \in \mathbb{R}_+^*$, there exists $\eta \in \mathbb{R}_+^*$, such that $d(x, y) < \eta$ implies $d'(f(x), f(y)) < \varepsilon$, for $x, y \in X$.
2. The function f is *Lipschitz continuous* if there exists k such that for any $x, y \in X$, $d'(f(x), f(y)) \leq kd(x, y)$.
3. Let X be an interval of $\mathbb{R}$. The function f is *absolutely continuous* if for all $\varepsilon \in \mathbb{R}_+^*$, there exists $\eta \in \mathbb{R}_+^*$, such that whenever $\{[x_i, y_i] : i = 1, \cdots, n\}$ is a finite collection of mutually disjoint subintervals of X, $\sum_{i=0}^n |y_i - x_i| < \eta$ implies $\sum_{i=0}^n d'(f(x_i), f(y_i)) < \varepsilon$.

Remark.

1. A Lipschitz continuous function is uniformly continuous.
2. A uniformly continuous function is continuous.
3. A continuous function is rarely uniformly continuous.
4. An absolutely continuous function on a finite union of closed real intervals is uniformly continuous.
5. A Lipschitz continuous function on a closed real interval is absolutely continuous.

To derive a useful result on the continuity of linear maps over a normed vector space we introduce the notion of ball.

Definition

Let (X, d) be a metric space, $a \in X$ and $r \in \mathbb{R}_+^*$. The set of all the points at a distance strictly less than r from a is called an *open ball*, or simply a *ball*, of radius r and is denoted $B_d(a, r)$.

Theorem

Let E and F be two normed vector spaces over $\mathbb{K}$ and u be a linear map. The following conditions are equivalent.

- (i) $u \in \mathcal{L}(E, F)$;
- (ii) u is uniformly continuous;
- (iii) u is continuous;
- (iv) u is continuous at any $x \in E$;
- (v) u is bounded over any bounded subset of E;
- (vi) There exists $a \in E$ and $r \in \mathbb{R}_+^*$ such that u is bounded over $B(a, r)$;

Proof. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) and (v) $\Rightarrow$ (vi) are trivial.

(i) $\Rightarrow$ (ii). If $u \in \mathcal{L}(E, F)$, then by linearity, for $x, y \in E$ we have

$$\|u(x) - u(y)\| = \|u(x - y)\| \leq \|u\|\|x - y\|.$$

So u is $\|u\|$-Lipschitz continuous, and in particular uniformly continuous.

(iv) $\Rightarrow$ (v). If u is continuous at a then there exists $r > 0$ such that $\|u(x) - u(a)\| < 1$ when $\|x - a\| < r$. Thus for $x \in B(a, r)$ we have $\|u(x)\| \leq 1 + \|u(a)\|$, meaning that u is bounded over $B(a, r)$. But as any bounded subset of E is contained in a ball, this shows that u is bounded over any bounded subset of E.

(vi) $\Rightarrow$ (i). Without loss of generality we prove the result for $a = 0$. Let $x \in E$ and $y \in B(0, r)$ such that $\|y\|x = \|x\|y$. Then we apply u and by linearity we get $\|y\| \cdot \|u(x)\| = \|x\| \cdot \|u(y)\|$.

Since y is in $B(0, r)$ we know the existence of m such that $\|u(y)\| \leq m$ and $\|y\| < r$. Setting $M = \max(m, r)$, we get $\|u(x)\| \leq \frac{M}{r}\|x\|$, meaning that $\|u(x)\|$ is bounded. As expected, this shows that $\|u\| = \sup \frac{\|u(x)\|}{\|x\|}$ is bounded. $\square$

We now turn our attention to the notion of differentiability. Let E and F be two vector spaces of finite dimension over $\mathbb{R}$ and U be a non-empty open from E. We then define f as a function from U into F.

Lemma

If for $u \in \mathrm{L}(E, F)$, $\lim_{x \to 0} \frac{u(x)}{\|x\|} = 0$, then u is the null function.

Proof. By definition of the limit when x tends to 0, for any $\varepsilon > 0$, there exists $\eta > 0$ such that $\|u(x)\| \leq \varepsilon \|x\|$, if $\|x\| \leq \eta$. To prove that u is null, we need to show that this remains true for any $y \in E \setminus \{0\}$. We define $z = \frac{\eta}{\|y\|} y$. Then by the linearity of u

$$\|u(z)\| = \eta \frac{\|u(y)\|}{\|y\|}.$$

But as $\|z\| = \eta$, we can write $\|u(z)\| \leq \varepsilon \|z\| = \varepsilon \eta$. Hence, for any $y \neq 0$ from E, $\|u\| = \frac{\|u(y)\|}{\|y\|} \leq \varepsilon$. $\square$

Proposition

There exists at most one linear map $\ell \in \mathrm{L}(E, F)$ such that, for $a \in U \subset E$,

$$\lim_{x \to 0} \frac{1}{\|x\|} \left[f(a+x) - f(a) - \ell(x) \right] = 0.$$

Proof. Let $\ell_1, \ell_2 \in \mathrm{L}(E, F)$ matching the condition from the proposition. Then

$$\lim_{x \to 0} \frac{(\ell_1 - \ell_2)(x)}{\|x\|} = 0.$$

From lemma 1.30 it means ℓ_1 and ℓ_2 are equal. $\square$

Remark. From this result we see that ℓ is tangent to f at a.

Definitions

1. When it exists, the linear map ℓ is called the *differential of* f *at* a. Then the function f is said to be *differentiable* at a. The differential of f at a is denoted $\mathrm{d}f(a)$.
2. If for all $a \in U \subset E$, $\mathrm{d}f(a)$ exists then we define the application
$$\begin{aligned} \mathrm{d}f : U &\longrightarrow \mathrm{L}(E, F) \\ x &\longmapsto \mathrm{d}f(x). \end{aligned}$$
This application is called the *differential of* f.
3. When f is *n times differentiable* on $U \subset E$, $n \in \mathbb{N}^*$, the differential of order n of f is n-linear and given by
$$\begin{aligned} \mathrm{d}^n f : U &\longrightarrow \mathrm{L}_n(E, F) = \mathrm{L}(E, \mathrm{L}_{n-1}(E, F)) \\ x &\longmapsto \mathrm{d}^n f(x). \end{aligned}$$

Definitions

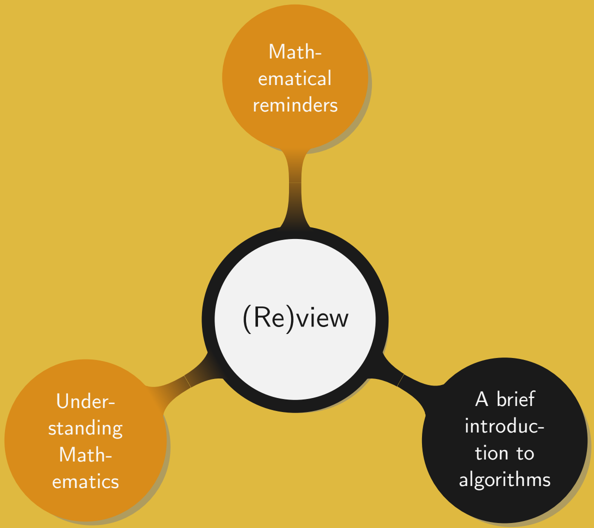
Let f be a function n times differentiable over $U \subset E$.

1. The function f is said to be in the differentiability class C^n if $\mathrm{d}^n f$ is continuous over U.
2. The function f is C^∞ if it is C^n for all $n \in \mathbb{N}$.

Remark.

- i. We will often write $f \in C^n[a, b]$ to say that f is defined over $[a, b] \subset \mathbb{R}$, and is in the differentiability class C^n.
- ii. The set of all the $C^n[a, b]$ functions is a vector space.
- iii. A continuous function is sometimes said to be C^0.

In the next section we focus on the less abstract notion of algorithm. In particular we will see how they should be designed and presented in the assignments and exams.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Algorithm: Recipe telling the computer how to solve a problem.

Example. Detail an algorithm to prepare a jam sandwich.

Actions: `cut`, `listen`, `spread`, `sleep`, `read`, `take`, `eat`, `dip`

Objects: `knife`, `guitar`, `bread`, `honey`, `jam jar`, `sword`

Algorithm. (*Sandwich making*)

Input : 1 bread, 1 jamjar, 1 knife

Output: 1 jam sandwich

take the knife and cut 2 slices of bread;

dip the knife into the jamjar;

spread the jam on the bread, using the knife;

assemble the 2 slices together, jam on the inside;

An algorithm systematically solves a *well-defined* problem:

- The *Input* is clearly expressed
- The *Output* solves the problem
- The *Algorithm* provides a precise step-by-step procedure starting from the Input and leading to the Output

Algorithms can be described using one of the three following ways:

- English
- Pseudocode
- Programming language

Algorithm. (*Insertion Sort*)

Input : $a_1, \ldots, a_n$, n unsorted elements
Output: the $a_i, 1 \le i \le n$, in increasing order

for $j \leftarrow 2$ **to** n **do**
  $i \leftarrow 1$;
  while $a_j > a_i$ **do** $i \leftarrow i + 1$;
  $m \leftarrow a_j$;
  for $k \leftarrow 0$ **to** $j - i - 1$ **do** $a_{j-k} \leftarrow a_{j-k-1}$;
  $a_i \leftarrow m$
end for
return $(a_1, \ldots, a_n)$

Setup: a robot arm solders chips on a board in n contact points

Goal: minimize the time to attach a chip to the board

Assumptions:

- The arm moves at constant speed
- Once a chip has been attached another one is soldered

Defining the Input and Output:

- Input: a set S of n points in the plane
- Output: the shortest path visiting all the points in S

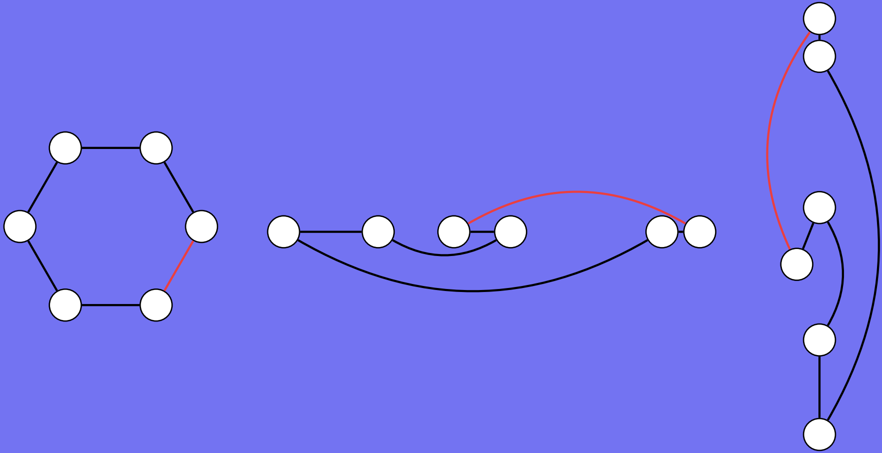
Algorithm. (*Nearest neighbor*)

Input : a set $S = \{s_0, \cdots, s_{n-1}\}$ of n points in the plane
Output: the shortest cycle visiting all the points in S

```
p_0 ← s_0;
for i ← 1 to n − 1 do
    p_i ← closest unvisited neighbor to p_{i−1};
    Visit p_i;
end for
return ⟨p_0, ..., p_{n−1}⟩
```

A first failure

How does the nearest neighbor algorithm (1.39) perform in the following three cases?



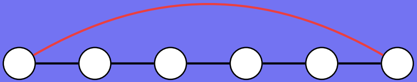
Algorithm. (*Closest pair*)

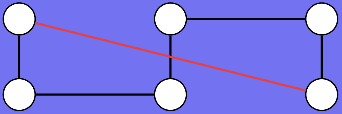
Input : a set S of n points in the plane

Output: the shortest cycle visiting all the points in S

for $i \leftarrow 1$ **to** $n-1$ **do**
    $d \leftarrow \infty$;
    foreach *pair of end points* $\langle s, t\rangle$ *from distinct vertex chains* **do**
        if $dist(s, t) \leq d$ **then**
            $s_m \leftarrow s$; $t_m \leftarrow t$; $d \leftarrow dist(s, t)$;
        end if
    end foreach
    Connect s_m and t_m by an edge;
end for
return *all the points starting from one of the two end points*

Applying the closest pair algorithm (1.41) on the following vertices arrangement yields the two graphs:





Possible strategy to ensure the most optimal path:

- Enumerate all the possible paths
- Select the one that minimizes the total length

Drawback: for only 20 vertices $20! = 2432902008176640000$ paths have to be explored...

A difference:

- Algorithm: always output a correct result
- Heuristic: idea serving as a guide to solve a problem with no guarantee of always providing a good solution

Correctness and efficiency:

- An algorithm working on a set of input does not imply it will work on all instances
- Efficient algorithm totally solving a problem might not exist

Common traps when defining the Input and Output:

- Are they precise enough?
- Can all the Input be easily and efficiently generated?
- Could there be any confusion on the expected Output?

Example. For an Output, what does it mean to “find the best route”?

The shortest in distance, the fastest in time, or the one minimizing the number of turns?

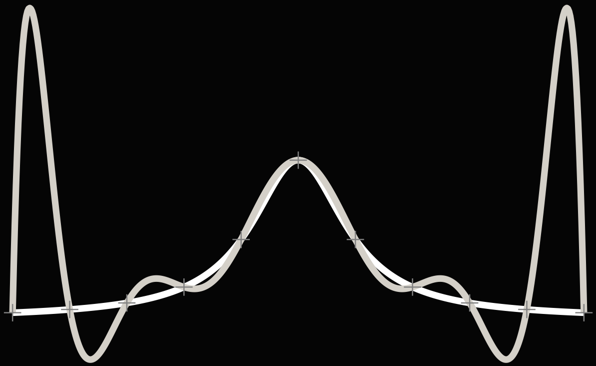
Conclusion: where to start (Input) and where to go (Ouput) must be expressed in simple, precise, and clear terms.

Finding good counter-examples:

- Seek simplicity: make it clear why the algorithm fails
- Think small: algorithms failing for large Input often fail for smaller one
- Test the extremes: study special cases, e.g. inputs equal, tiny, huge...
- Think exhaustively: test whether all the possible cases are covered by the algorithm
- Track weaknesses: check if the underlying idea behind the algorithm has any “unexpected” impact on the output

Key points

- How hard mathematics is?
- What are topology, algebra, and measure theory?
- For E and F two normed vector spaces, what is $\mathcal{L}(E, F)$?
- How to write pseudocode in the assignments and exams
- How to ensure the correctness of an algorithm



Thank you!