Homework #1

MATH4710J — Numerical Methods

Instructor: Manuel CHARLEMAGNE

Student: Turingw1 (ID: 000000)

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Problem 1.

1. Prove that $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ have the same number of elements.
2. Prove that [0, 1] has as many elements as $\mathbb{R}$.
3. Prove that [0, 1] has more elements than $\mathbb{N}$. Hint: understand Cantor's diagonal argument.

Solution.

1. **$\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ have the same number of elements**
   We want to show that the sets of natural numbers $\mathbb{N}$, integers $\mathbb{Z}$, and rational numbers $\mathbb{Q}$ are all countably infinite, i.e., there exists a bijection between each pair.

   Step 1: $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality.

   Define a function $f : \mathbb{N} \to \mathbb{Z}$ by

   $$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

   This function lists the integers in the order $0, 1, -1, 2, -2, 3, -3, \dots$ as n increases. It is a bijection, so $|\mathbb{N}| = |\mathbb{Z}|$. **Step 2: $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality.**

   Every rational number can be written as p/q with $p \in \mathbb{Z}$, $q \in \mathbb{N}$, and $\gcd(p, q) = 1$. Arrange all such pairs (p, q) in a grid and enumerate them diagonally (Cantor's diagonal argument), skipping duplicates and zero denominators. This process lists all rationals in a sequence, showing that $\mathbb{Q}$ is countable.

   Therefore, there exist bijections $\mathbb{N} \leftrightarrow \mathbb{Z} \leftrightarrow \mathbb{Q}$, so all three sets have the same number of elements.

2. **[0, 1] has as many elements as $\mathbb{R}$**
   To show that the interval [0, 1] has the same cardinality as $\mathbb{R}$, we can construct a bijection between these two sets. Consider the function $f : (0, 1) \to \mathbb{R}$ defined by

   $$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right)$$

   This function maps the open interval (0, 1) onto the entire real line $\mathbb{R}$. To include the endpoints 0 and 1, we can extend this function by defining:

   $$f(0) = -\infty, \quad f(1) = +\infty$$

   This shows that there is a bijection between [0, 1] and $\mathbb{R}$, hence they have the same cardinality.

3. **[0, 1] has more elements than $\mathbb{N}$**
To show that the interval [0, 1] has more elements than $\mathbb{N}$, we can use Cantor's diagonal argument. Assume, for the sake of contradiction, that there is a bijection between $\mathbb{N}$ and [0, 1]. This means we can list all numbers in [0, 1] as a sequence:

$$x_1, x_2, x_3, \ldots$$

where each x_i is represented in its decimal expansion. Now, we construct a new number y in [0, 1] by changing the i-th digit of x_i to a different digit (for example, if the i-th digit is 5, change it to 6; if it's 9, change it to 0, etc.). This new number y differs from each x_i in at least one decimal place, meaning y cannot be in the list. This contradiction shows that there is no bijection between $\mathbb{N}$ and [0, 1], hence [0, 1] has more elements than $\mathbb{N}$.

□

Problem 2.

1. Prove the Cauchy-Schwarz inequality (1.20|1.39) over the complex numbers.
2. Show that a distance is always positive.

Solution.

1. **Cauchy-Schwarz Inequality:**

$$\left|\langle x, y\rangle\right| \leq \|x\|\,\|y\|, \qquad x, y \in V,$$

Proof:

$$\begin{aligned} 0 &\leq \langle x - \lambda y,\ x - \lambda y\rangle && (1)\\ &= \langle x, x\rangle - \overline{\lambda}\langle y, x\rangle - \lambda\langle x, y\rangle + |\lambda|^2\langle y, y\rangle. && (2)\end{aligned}$$

Take $\lambda = \dfrac{\langle y, x\rangle}{\langle y, y\rangle}$ (when $y \neq 0$), we have

$$0 \leq \langle x, x\rangle - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle},$$

from which it follows that

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\,\langle y, y\rangle,$$

and hence

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|.$$

which is the Cauchy-Schwarz inequality.

2. **Distance is Always Positive:**
A distance function $d : X \times X \to \mathbb{R}$ on a set X must satisfy the following properties for all $x, y, z \in X$:
1. $d(x, y) = 0$ if and only if $x = y$.
2. Symmetry: $d(x, y) = d(y, x)$.
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$.
Proof

$$\forall x, y \in X, d(x, y) + d(y, x) = 2d(x, y) \geq d(x, x) = 0$$

□

Problem 3.

1. Let f be a linear map from a vector space V_1 into a vector space V_2. Show that the dimension of V_1 is the sum of the dimensions of the kernel and of the image of f. This result is called the **rank-nullity theorem**.

2. Prove that the composition of two linear maps is a linear map.

3. Prove that the inverse of a linear map is a linear map.

Solution.

1. Let $f : V_1 \to V_2$ be a linear map. We want to show that

$$\dim(V_1) = \dim(\ker(f)) + \dim(\operatorname{im}(f)).$$

Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for $\ker(f)$. We can extend this basis to a basis for V_1 by adding vectors $\{v_{k+1}, v_{k+2}, \ldots, v_n\}$ such that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V_1. The images of the added vectors under f, $\{f(v_{k+1}), f(v_{k+2}), \ldots, f(v_n)\}$, form a basis for $\operatorname{im}(f)$ (Since $\{f(v_{k+1}), f(v_{k+2}), \ldots, f(v_n)\}$ can't be 0 unless $v_{k+1}, v_{k+2}, \ldots, v_n$ are in $\ker(f)$). Therefore, we have

$$\dim(V_1) = n = k + (n - k) = \dim(\ker(f)) + \dim(\operatorname{im}(f)).$$

2. Let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be linear maps. We want to show that the composition $g \circ f : V_1 \to V_3$ is linear. For any $u, v \in V_1$ and scalar c, we have

$$(g \circ f)(u+v) = g(f(u+v)) = g(f(u)+f(v)) = g(f(u))+g(f(v)) = (g \circ f)(u)+(g \circ f)(v),$$

and

$$(g \circ f)(cu) = g(f(cu)) = g(cf(u)) = cg(f(u)) = c(g \circ f)(u).$$

Thus, $g \circ f$ is linear.

3. Let $f : V_1 \to V_2$ be a bijective linear map. We want to show that the inverse map $f^{-1} : V_2 \to V_1$ is linear. For any $f(u), f(v) \in V_2$ and scalar c, we have

$$f^{-1}(f(u) + f(v)) = f^{-1}(f(u + v)) = u + v = f^{-1}(f(u)) + f^{-1}(f(v)),$$

and

$$f^{-1}(c(f(u))) = f^{-1}(f(cu)) = cu = cf^{-1}(f(u)).$$

And since f is bijective, every element in V_2 can be written as $f(u)$ for some $u \in V_1$. Thus, f^{-1} is linear.

□

Problem 4.
Intuitively a complete space has "no point missing" anywhere. In particular it means that any Cauchy sequence converges inside the space. In this exercise we show that e is not rational while we can find a Cauchy sequence of rationals converging to e.

1. Show that e is irrational.

2. Show that the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n = (1 + \frac{1}{n})^n$ is a Cauchy sequence converging to e.

3. Is $\mathbb{Q}$ complete? Explain.

Solution.

1. To show that e is irrational, we can use a proof by contradiction. Assume that e is rational, i.e., $e = \frac{p}{q}$ for some integers p and q. Consider the series expansion of e:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

Multiplying both sides by $q!$, we get

$$q!e = q!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) = q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots$$

The left side is an integer since $e = \frac{p}{q}$ and $q!e = p(q-1)!$. However, the right side is not an integer because the terms $\frac{q!}{n!}$ for $n > q$ are not integers. This leads to a contradiction, hence e is irrational.

2. Let

$$a_n = \left(1 + \frac{1}{n}\right)^n, \qquad n \geq 1,$$

and define the function

$$f(x) = x \ln\left(1 + \frac{1}{x}\right), \qquad x \geq 1.$$

Notice that for integer n we have $f(n) = \ln a_n$. To show that $\{a_n\}$ is increasing, it suffices to prove that f is increasing on $[1, \infty)$.

Let $u = 1/x > 0$. Then

$$f'(x) = \ln(1+u) + x \cdot \frac{1}{1+u} \cdot \frac{du}{dx}.$$

Since $u = 1/x$, we have $du/dx = -1/x^2 = -u^2$. Substitution yields

$$f'(x) = \ln(1+u) - \frac{u}{1+u}.$$

Define $g(u) = \ln(1+u) - \frac{u}{1+u}$. Then

$$g'(u) = \frac{1}{1+u} - \frac{1}{(1+u)^2} = \frac{u}{(1+u)^2} > 0 \qquad (u > 0),$$

and $g(0) = 0$. Hence $g(u) > 0$ for all $u > 0$, so $f'(x) > 0$ for $x > 0$. Therefore f is increasing, and thus $\ln a_n = f(n)$ is increasing in n, i.e. $\{a_n\}$ is monotone increasing.

Next, we provide an upper bound. Using the inequality $\ln(1+t) \leq t$ valid for $t > -1$, we obtain for $t = 1/n$:

$$\ln a_n = n \ln\left(1 + \frac{1}{n}\right) \leq n \cdot \frac{1}{n} = 1.$$

Therefore $a_n \leq e^1 = e$ for all n. Consequently, the sequence $\{a_n\}$ is monotone increasing and bounded above by e. By the monotone convergence theorem for real numbers, $\{a_n\}$ converges in $\mathbb{R}$.

Finally, in the metric space $(\mathbb{R}, |\cdot|)$, every convergent sequence is Cauchy. Hence $\{a_n\}$ is a Cauchy sequence.

3. The set of rational numbers $\mathbb{Q}$ is not complete. A metric space is said to be complete if every Cauchy sequence in that space converges to a limit that is also within the same space. However, there exist Cauchy sequences of rational numbers that converge to irrational numbers, which are not in $\mathbb{Q}$. For example, the sequence defined by $a_n = (1+\frac{1}{n})^n$ converges to e, which is irrational. Since $e \notin \mathbb{Q}$, this shows that $\mathbb{Q}$ is not complete.
   For simpler example, consider the sequence defined by $b_n = 3.1, 3.14, 3.141, 3.1415, \ldots$, which converges to π. Since π is irrational and not in $\mathbb{Q}$, this is another example of a Cauchy sequence in $\mathbb{Q}$ that does not converge to a limit in $\mathbb{Q}$. Thus, $\mathbb{Q}$ is not a complete metric space.

□

Problem 5.
1. Write the pseudocode for at least one the following strategy to approximate π.
   (a) The polygons method;
   (b) Machin's formula $\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$ and Taylor series;
2. Implement at least one of the previous algorithms in MATLAB.

Solution.

1. a) For the unit circle (radius 1), the side length of an inscribed regular n-gon is

$$c_n = 2\sin\frac{\pi}{n}, \quad \text{so} \quad p_n = nc_n = 2n\sin\frac{\pi}{n}$$

is the inscribed polygon perimeter. The perimeter of the circumscribed regular n-gon is

$$P_n = 2n\tan\frac{\pi}{n}.$$

Thus

$$n\sin\frac{\pi}{n} \le \pi \le n\tan\frac{\pi}{n}.$$

If we double the number of sides, the inscribed side lengths satisfy

$$c_{2n} = 2\sin\frac{\pi}{2n} = \sqrt{2 - 2\sqrt{1 - \frac{c_n^2}{4}}}\,.$$

Consequently

$$p_{2n} = 2n\,c_{2n}.$$

Repeating the doubling step yields arbitrarily tight bounds for π.

Algorithm 1: Polygon method to approximate π

Input : number of sides n (initially 6), number of iterations k
Output: approximation of π

Function `AlgoHw`(n, k)**:**
  $c_n \leftarrow 2\sin(\pi/n)$;
  $p_n \leftarrow n \cdot c_n$;
  for $i \leftarrow 1$ **to** k **do**
    $c_n \leftarrow \sqrt{2 - 2\sqrt{1 - c_n^2/4}}$;
    $p_n \leftarrow 2n \cdot c_n$;
    $n \leftarrow 2n$;
  end for
  return p_n
end

b) Using Machin's formula and Taylor series to approximate π:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

The Taylor series expansion for $\arctan(x)$ is given by:

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

To approximate π, we can truncate the series after a finite number of terms. Here is the pseudocode for the algorithm:

Algorithm 2: Machin's formula to approximate π

Input : number of terms N
Output: approximation of π

Function `AlgoHw`(N)**:**
  $\arctan(\frac{1}{5}) \leftarrow 0$;
  $\arctan(\frac{1}{239}) \leftarrow 0$;
  for $n \leftarrow 0$ **to** $N-1$ **do**
    $\arctan(\frac{1}{5}) \leftarrow \arctan(\frac{1}{5}) + \frac{(-1)^n(1/5)^{2n+1}}{2n+1}$;
    $\arctan(\frac{1}{239}) \leftarrow \arctan(\frac{1}{239}) + \frac{(-1)^n(1/239)^{2n+1}}{2n+1}$;
  end for
  $\pi \leftarrow 4 \cdot (4 \cdot \arctan(\frac{1}{5}) - \arctan(\frac{1}{239}))$;
  return π
end

2. Here is a MATLAB implementation of the polygon method to approximate π:

Listing 1: Polygon Method to Approximate Pi

```
function pi_approx = approximate_pi_polygon(n, k)
    % n: initial number of sides (e.g., 6)
    % k: number of iterations (doublings)
    c_n = 2 * sin(pi / n); % side length of inscribed polygon
    p_n = n * c_n;         % perimeter of inscribed polygon
    for i = 1:k
        c_n = sqrt(2 - 2 * sqrt(1 - c_n^2 / 4));
        p_n = 2 * n * c_n;
```

```
            n = 2 * n;
        end
        pi_approx = p_n;
    end
```

□

Compiled with LaTeX.