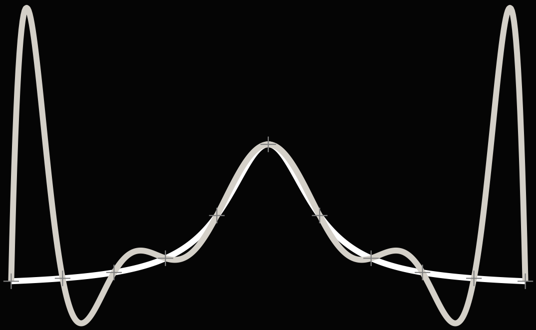
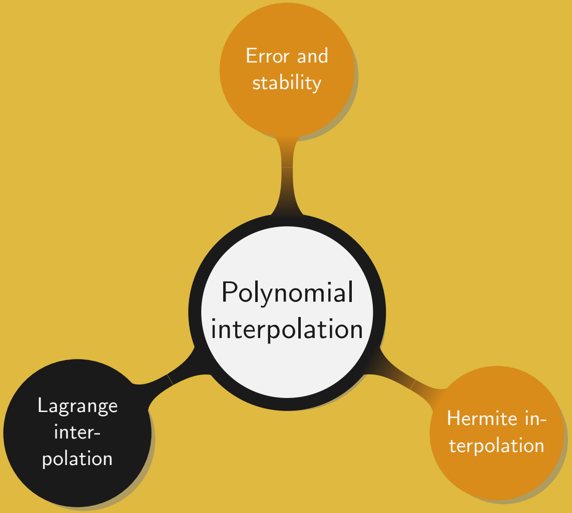


# Introduction to Numerical Methods

## 3. Polynomial interpolation

Ailin & Manuel – Fall 2025

# Chapter organisation



Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and $x_1, \cdots, x_n$ be $n+1$ distinct points from $[a, b]$. From a mathematical point of view, if we want to construct a polynomial $P$ such that $f(x_i) = P(x_i)$, for all $x_i$, $0 \leq i \leq n$, we first need to ensure its existence.

We denote by $\mathbb{R}_n[x]$ the set of the polynomials of $\mathbb{R}[x]$ with degree less than $n$, $n \in \mathbb{N}$.

**Theorem**

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and $x_0, \cdots, x_n$ be $n+1$ distinct points from $[a, b]$. There exists a unique polynomial $P(x) \in \mathbb{R}_n[x]$ such that for all $0 \leq i \leq n$ $P(x_i) = f(x_i)$ .

Proof. By definition $P$ exists if and only if there exist $a_0, \cdots, a_n$ such that for all the $x_i$, $0 \leq i \leq n$, $P(x) = \sum_{k=0}^{n} a_k x^k = f(x)$.

Rewriting the previous equations as a matrix yields

$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}}_{V} \cdot \begin{pmatrix} a_0 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} f(x_0) \\ \vdots \\ f(x_n) \end{pmatrix}.$$

Hence the $a_i$, $0 \leq i \leq n$, exist if and only if $V$ is invertible, i.e. $\det V \neq 0$. This is clearly the case since $V$ is the Vandermonde matrix and $\det V = \prod_{\substack{i,j=0 \\ i<j}}^{n} (x_j - x_i)$. Therefore at least one polynomial $P$, of degree at most $n$, interpolates $f$ over $[a, b]$.

Let $P_1$ and $P_2$ be two polynomials of degree less than $n$ such that $P_1(x_i) = f(x_i) = P_2(x_i)$, for $0 \leq i \leq n$. Then

$$\begin{cases} Q = P_1 - P_2, \ \deg Q \leq n, \\ Q(x_i) = 0, \ 0 \leq i \leq n. \end{cases}$$

Since $Q$ is of degree at most $n$ but has $n+1$ roots, it is identically zero, meaning that $P_1 = P_2$. Hence $P$ is unique. $\square$

Based on the previous proof a straight forward solution to determine $P$ is to invert the matrix $V$. However since this is a long and expensive operation, we instead construct an independent linear system where the $f(x_i)$ are the components of $P$.

In other words we write the polynomial $P$ as

$$P(x) = \sum_{i=0}^{n} f(x_i)\ell_i(x), \text{ where } \ell_i \in \mathbb{R}_n[X] \text{ and } \ell_i(x_j) = \begin{cases} 0 \text{ if } j \neq i \\ 1 \text{ if } j = i \end{cases}.$$

By construction, $\ell_i(x_j) = 0$ if $j \neq i$, meaning that $x_j$ is a root of $\ell_i$ for $j = 0, \cdots, i-1, i+1, \cdots, n$. Therefore $\ell_i$ is divisible by $\prod_{\substack{j=0 \\ j\neq i}}^{n}(x - x_j)$ and there exists $c_i$ such that $\ell_i(x) = c_i \prod_{\substack{j=0 \\ j\neq i}}^{n}(x - x_j)$. Since we also know $\ell_i(x_i) = 1$ we can deduce $c_i = \dfrac{1}{\prod_{\substack{j=0 \\ j\neq i}}^{n}(x_i - x_j)}$. Hence, $\ell_i(x) = \prod_{\substack{j=0 \\ j\neq i}}^{n} \dfrac{x - x_j}{x_i - x_j}$, for $0 \leq i \leq n$.

**Definition** (Lagrange polynomials)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function, and $x_0, \cdots, x_n$ be $n+1$ distinct points from $[a, b]$. The polynomial

$$L_n(f) = \sum_{i=0}^{n} f(x_i)\ell_i(x), \text{ with } \ell_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \quad \text{for } 0 \leq i \leq n,$$

is called the *Lagrange form of the interpolation polynomial* and the $\ell_i(x)$, *Lagrange polynomials*.

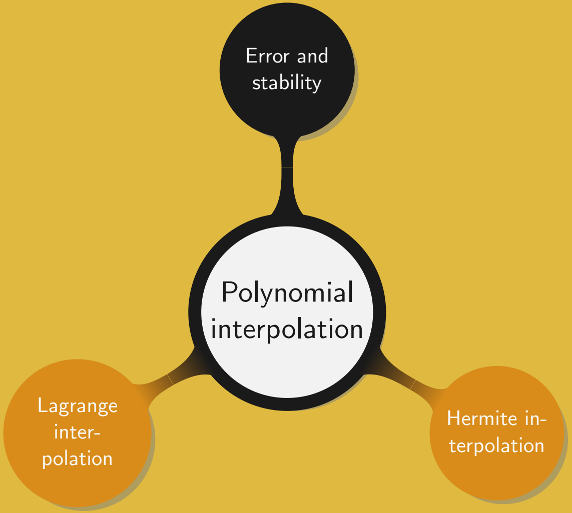
Example. Let $f(t_0), \cdots, f(t_n)$ represent $n+1$ measurements at a regular interval, what is the value of $f$ at $(t_{i+1} - t_i)/2$?

**Algorithm.** (*Lagrange interpolation*)

**Input** : $n+1$ pairs $(x_0, f(x_0)), \cdots, (x_n, f(x_n))$, and $c \in [\min_i x_i, \max_i x_i]$
**Output:** $f(c)$

$res \leftarrow 0$;
**for** $i \leftarrow 0$ **to** $n$ **do**
  $\ell_i \leftarrow f(x_i)$;
  **for** $j \leftarrow 0$ **to** $n$ **do**
    **if** $j \neq i$ **then** $\ell_i \leftarrow \ell_i \frac{c-x_j}{x_i-x_j}$;
  **end for**
  $res \leftarrow res + \ell_i$
**end for**
**return** $res$

## Chapter organisation



From theorem 3.3 we know that a unique polynomial, depending on the $x_i$, interpolates $f$. We now want to study the error of interpolation, i.e. the difference between $f(x)$ and $L_n(f)$ when $x \neq x_i$.

**Theorem**

Let $[a, b]$ be a real interval, $(x_i)_{i=0,\cdots,n}$ be $n+1$ distinct points of $[a, b]$, and $f$ be in $C^{n+1}[a, b]$. If $L_n(f)$ is the Lagrange interpolation polynomial of $f$ at the points $x_i$, then for all $x$ in $[a, b]$ there exists $\xi_x \in [\min(x, \min_{0\leq i\leq n} x_i), \max(x, \max_{0\leq i\leq n} x_i)] \subset [a, b]$ such that

$$Ef(x) = f(x) - L_n(f)(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{j=0}^{n} (x - x_j).$$

Proof. Let $x \in [a, b]$. For the sake of clarity we denote $L_n(f)$ by $P$.

If $x = x_j$, then $f(x_j) - P(x_j) = 0$ while $\dfrac{f^{(n+1)}(\xi_x)}{(n+1)!} \displaystyle\prod_{i=0}^{n}(x_j - x_i)$ is also 0 for any $\xi_x$. So the result holds.

Therefore we now assume $x \notin \{x_0, \cdots, x_n\}$ and define the polynomial $\Pi_n(t) = \prod_{j=0}^{n}(t - x_j)$ for any $t \in [a, b]$. We then construct the polynomial

$$Q(t) = P(t) + \frac{f(x) - P(x)}{\Pi_n(x)} \Pi_n(t) \tag{3.1}$$

and the function $g(t) = f(t) - Q(t)$.

We now investigate some properties of $g$. First note that $g \in C^{n+1}[a, b]$ for $f \in C^{n+1}[a, b]$ and $Q \in C^{\infty}[a, b]$.

Then observe that $g(x_i) = 0$ since $f(x_i) = P(x_i)$ and $\Pi_n(x_i) = 0$.

Moreover as $g(x) = f(x) - P(x) - \frac{f(x)-P(x)}{\Pi_n(x)}\Pi_n(x) = 0$, we can conclude that $g$ has $n+2$ distinct roots in $[a, b]$.

As a result we know that $g \in C^{n+1}[a, b]$ and has $n+2$ distinct roots in $[a, b]$. Therefore Rolle's theorem (2.18) can be applied, and there exists $\xi_x$ in the smallest interval containing the roots of $g$, such that $g^{(n+1)}(\xi_x) = 0$.

Clearly the smallest interval containing the roots of $g = f - Q$ is $[\min(x, \min\limits_{0\leq i\leq n} x_i), \max(x, \max\limits_{0\leq i\leq n} x_i)]$ and

$$f^{(n+1)}(\xi_x) - Q^{(n+1)}(\xi_x) = 0.$$

Then from (**??**), it is easy to get $Q^{(n+1)}(t) = \frac{f(x)-P(x)}{\Pi_n(x)}\Pi_n^{(n+1)}(t)$.

Finally as $\Pi_n^{(n+1)}(t) = (n+1)!$ and $Q^{(n+1)}(t) = \frac{f(x)-P(x)}{\Pi_n(x)}(n+1)!$, we obtain

$$Ef(x) = f(x) - P(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{j=0}^{n} (x - x_j).$$

$\square$

**Corollary**

Using the previous notations and setting $M_n = \max_{t \in [a,b]} \left| f^{(n+1)}(t) \right|$,

$$\left| f(x) - L_n(f) \right| \leq \frac{M_n}{(n+1)!} \left| \Pi_n(x) \right| \leq \frac{M_n}{(n+1)!} (b-a)^{n+1}.$$

As $|\Pi_n(x)| \leq \max_{t\in[a,b]} |\Pi_n(t)|$ the previous corollary (3.13) yields

$$|f(x) - L_n(f)| \leq \frac{M_n}{(n+1)!} \max_{t\in[a,b]} |\Pi_n(t)|.$$

In other words the error decreases as $\max_{t\in[a,b]} |\Pi_n(t)|$ decreases, meaning that the error depends on the choice of the nodes $x_i$. Therefore we now focus on how to choose the $x_i$ in order to minimize $\max_{t\in[a,b]}$.

Observing that $\Pi_n$ is a monic polynomial of degree $n+1$, we want to choose the $x_i$ such that

$$\max_{x\in[a,b]} \left|\prod_{j=0}^{n}(x - x_j)\right| \leq \max_{x\in[a,b]} \left|q(x)\right|, \quad \forall q \in E_{n+1},$$

where $E_{n+1} = \left\{P \in \mathbb{R}_{n+1}[x], P \text{ is monic}\right\}$.

For the sake of simplicity we assume $a = -1$ and $b = 1$ and then apply the bijective change of variable

$$\begin{aligned} \varphi : [-1, 1] &\longrightarrow [a, b] \\ X &\longmapsto \frac{b-a}{2}X + \frac{b+a}{2} \end{aligned} \tag{3.2}$$

to recover the original case.

**Theorem**

For any polynomial $q$ in $E_n$, $\max_{t \in [-1,1]} |q(t)| \geq 2^{1-n}$.

Proof. Assume there exists $r \in E_n$ such that $\max_{t \in [-1,1]} |r(t)| < 2^{1-n}$.

From proposition 2.59 we know that the $n$th Chebyshev polynomial of the first kind has leading coefficient $2^{n-1}$, and extrema $-1$ and $1$ at the points $x_i'$, $0 \leq i \leq n$.

Therefore $\overline{T}_n(x) = 2^{1-n} T_n(x)$ is in $E_n$, and $m(t) = r(t) - \overline{T}_n(t)$ is a polynomial of degree at most $n-1$ over $[-1, 1]$. Moreover at the $x_i'$, $m$ evaluates as $m(x_i') = r(x_i') - (-1)^k 2^{1-n}$.

As $m$ changes sign $n$ times, the intermediate values theorem (2.11) implies that $m$ has $n$ distinct roots. This is impossible since $\deg m < n$. ↯ □

Now note that as $\overline{T}_n(x) = 2^{1-n} T_n(x)$, we have

$$\max_{x \in [-1,1]} |\overline{T}_n(x)| = 2^{1-n} \max_{x \in [-1,1]} |T_n(x)| = 2^{1-n}.$$

Together with theorem 3.15 it means that choosing the $x_i$ as the roots of $T_n$ minimizes the error. In this case it is given by

$$\left|f(x) - L_n(f)\right| \leq \frac{1}{(n+1)!} \cdot \max_{x\in[-1,1]} \left|\prod_{k=0}^{n}(x - x_k)\right| \cdot \max_{x\in[-1,1]} \left|f^{(n+1)}(x)\right|.$$

But as $\prod_{k=0}^{n}(x - x_k)$ is exactly $\overline{T}_{n+1}(x)$ we get from corollary 3.13

$$\left|f(x) - L_n(f)\right| \leq \frac{M_n}{2^n(n+1)!}.$$

All the previous discussion being only valid over $[-1, 1]$ we now use the map $\varphi(X)$ (**??**) in order to determine the best nodes over any interval $[a, b]$ of $\mathbb{R}$.

From proposition 2.59 the $(n+1)$th Chebyshev polynomial of the first kind has roots $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$, $0 \leq k \leq n$. Hence after applying the map, the roots are given by

$$x_k = \frac{b+a}{2} + \frac{b-a}{2}\cos\left(\frac{2k+1}{2n+2}\pi\right), \quad \forall 0 \leq k \leq n.$$

The only task left is the estimation of $\max_{x\in[a,b]}\left|\prod_{k=0}^{n}(x-x_k)\right|$. First we rewrite

$$\begin{aligned}\left|\prod_{k=0}^{n}(x-x_k)\right| &= \left|\prod_{k=0}^{n}\left(\frac{b+a}{2}+\frac{b-a}{2}x-x_k\right)\right| \\ &= \left|\prod_{k=0}^{n}\left(\frac{b-a}{2}x-\frac{b-a}{2}\cos\left(\frac{2k+1}{2n+2}\pi\right)\right)\right|.\end{aligned}$$

Then factoring by $(b-a)/2$ we obtain

$$\left|\prod_{k=0}^{n}(x-x_k)\right| = \left(\frac{b-a}{2}\right)^{n+1}\left|\prod_{k=0}^{n}\left(x-\cos\left(\frac{2k+1}{2n+2}\pi\right)\right)\right|.$$

Finally the max over $[a,b]$ is given by

$$\begin{aligned}\max_{x\in[a,b]}\left|\prod_{k=0}^{n}(x-x_k)\right| &= \left(\frac{b-a}{2}\right)^{n+1}\max_{x\in[-1,1]}\left|\prod_{k=0}^{n}(x-x_k)\right| \\ &= \left(\frac{b-a}{2}\right)^{n+1}\frac{1}{2^n} \\ &= 2\left(\frac{b-a}{4}\right)^{n+1}.\end{aligned}$$

Hence the error in the Lagrange interpolation of $f \in C^{n+1}[a, b]$ is minimized when choosing the nodes as the roots of the $(n+1)$th Chebyshev polynomials of the first kind. In this case the error is given by

$$\left|f(x) - L_n(f)\right| \leq \frac{2M_n}{(n+1)!}\left(\frac{b-a}{4}\right)^{n+1}.$$

We now turn our attention to the “quality” of the interpolation as the number of nodes increases.

**Definition** (Stability of a numerical method)

In general, a numerical method is said to *stable* if a small perturbation on the inputs induces a small variation on the result.

In the case of Lagrange interpolation let $\widetilde{f}$ be the perturbed value of a continuous function $f$, and define $\varepsilon = \widetilde{f} - f$.

The Lagrange form of the interpolation polynomial (def. 3.7) of $f$ and $\widetilde{f}$ is given by $L_n(f) = \sum_{i=0}^{n} f(x_i)\ell_i(x)$ and $L_n(\widetilde{f}) = \sum_{i=0}^{n} \widetilde{f}(x_i)\ell_i(x)$, respectively.

Noting that

$$\begin{aligned} L_n(\widetilde{f}) - L_n(f) &= \sum_{i=0}^{n} \widetilde{f}(x_i)\ell_i(x) - \sum_{i=0}^{n} f(x_i)\ell_i(x) \\ &= \sum_{i=0}^{n} \left(\widetilde{f}(x_i) - f(x_i)\right) \ell_i(x) \\ &= L_n(\widetilde{f} - f), \end{aligned}$$

we get

$$\left|L_n(\varepsilon)\right| \leq \max \left|\widetilde{f}(x_i) - f(x_i)\right| \cdot \max_{x\in[a,b]} \sum_{i=0}^{n} \left|\ell_i(x)\right| . \tag{3.3}$$

This means that a perturbation on the inputs is "amplified" by a factor $\max_{x\in[a,b]} \sum_{i=0}^{n} \left|\ell_i(x)\right|$ on the output.

**Definition**

In Lagrange interpolation the number $\Lambda_n = \max_{x\in[a,b]} \sum_{i=0}^{n} \left|\ell_i(x)\right|$ is called the *Lebesgue constant*.

As the number $n$ of nodes increases in Lagrange interpolation, $\Lambda_n$ increases such that it tends to infinity. Since Lebesgue constant only depends on the choice of the nodes it means the more nodes the less stable.

It can be shown that any set of interpolation nodes leads to $\Lambda_n > \dfrac{2}{\pi}\log(n+1)+\dfrac{1}{2}$. In particular the Chebyshev nodes are nearly optimal, $\Lambda_n$ growing at a rate $\frac{2}{\pi}\log n + \frac{2}{\pi}\left(\gamma + \log\frac{8}{\pi}\right) + o(1)$ with $\gamma$ the Euler constant given by $\gamma = \int_0^{\infty} \frac{1}{\lfloor x\rfloor} - \frac{1}{x}\,\mathrm{d}x$.[1].

These results being more advanced and complex to prove we now focus our attention on the case of equidistant nodes.

[1] A slightly weaker result will be proven in homework 5

**Proposition**

When applying Lagrange interpolation with equidistant nodes, $\Lambda_n$ becomes larger than $\dfrac{2^n}{4n^2}$ as $n$ tends to infinity.

Proof. First observe that the result holds when $n = 1$ as $\Lambda_1 = 1$.

Let $k \geq 2$. Since the nodes are equidistant $x_k = x_0 + kh$ for all $1 < k \leq n$ and some constant $h$. Similarly for any $x \in [a, b]$ there exists $s$ such that $x = x_0 + sh$. Then

$$\prod_{\substack{j=0\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j} = \prod_{\substack{j=0\\ j\neq i}}^{n} (s - j) \Big/ \prod_{\substack{j=0\\ j\neq i}}^{n} (i - j)$$

First we rewrite the denominator as $\left|\prod_{\substack{j=0\\ j\neq i}}^{n} (i - j)\right| = i!(n - i)!$.

The goal being to minimize $\max_{x\in[a,b]} \sum_{i=0}^{n} |\ell_i(x)|$ looking at the behaviour of the numerator for any point that is not a node will yield a result. Note however that this result will not be optimal, i.e. it might be far below the actual maximum.

For the sake of simplicity we now assume $s = 1/2$. In that case

$$\Lambda_n \geq \sum_{i=0}^{n} \left| \ell_i \left( x_0 + \frac{h}{2} \right) \right| = \sum_{i=0}^{n} \frac{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdots (i + \frac{3}{2}) \cdot (i + \frac{1}{2}) \cdots (n - \frac{1}{2})}{i!(n-i)!}.$$

As $k \geq 2$ it is easy to see that $k - 1/2 \geq k - 1$ and get

$$\left| \ell_i \left( x_0 + \frac{h}{2} \right) \right| \geq \frac{1}{4n(i-1)} \cdot \frac{1 \cdot 2 \cdots (n-1)n}{i!(n-i)!}.$$

Rearranging the terms and noting that $i-1 \leq n$ results in

$$\left|\ell_i(x_0+\frac{1}{2})\right| \geq \frac{n!}{4n^2 i!(n-i)!}.$$

Hence we finally obtain the expected result

$$\Lambda_n \geq \left(\sum_{i=0}^{n} \binom{n}{i}\right) \frac{1}{4n^2} = \frac{2^n}{4n^2}.$$

$\square$

As we know how $\Lambda_n$ behaves we now want to more precisely investigate how it affects $L_n$ viewed as a function.

**Theorem**

Let $E = C[a, b]$ for $[a, b]$ a real interval and $F = \mathbb{R}_n[x]$. Then $L_n$ is in $\mathcal{L}(E, F)$ and $\|L_n\| = \Lambda_n$.

Proof. It is simple to see that $L_n$ is a linear application as

$$\begin{aligned}
L_n(af + bg)(x) &= \sum_{i=0}^{n} \left(\ell_i(x)(af + bg)(x_i)\right) \\
&= \sum_{i=0}^{n} \left(\ell_i(x)(af(x_i) + bg(x_i))\right) \\
&= a\sum_{i=0}^{n} \left(\ell_i(x)f(x_i)\right) + b\sum_{i=0}^{n} \left(\ell_i(x)g(x_i)\right) \\
&= aL_n(f)(x) + bL_n(g)(x)
\end{aligned}$$

Moreover as $L_n$ is a continuous function over the real interval $[a, b]$ the extreme value theorem (2.14) states that $L_n$ attains its maximal value. So $L_n$ belongs to $\mathcal{L}(E, F)$.

Over $\mathcal{L}(E, F)$ we have $\|L_n\| = \sup\limits_{\substack{f \in [a,b] \\ f \neq 0}} \dfrac{\|L_n(f)\|_\infty}{\|f\|_\infty}$ (definition 1.23). But based on equation (**??**), we also have $\|L_n(f)\|_\infty \leq \|f\|_\infty \Lambda_n$, and we can conclude that

$$\|L_n\| \leq \Lambda_n. \tag{3.4}$$

The only thing left to prove is that $\|L_n\| \geq \Lambda_n$. Let $\xi \in [a, b]$ be such that

$$\Lambda_n = \sum_{i=0}^{n} |\ell_i(\xi)| = \max_{x \in [a,b]} \sum_{i=0}^{n} |\ell_i(x)|.$$

Defining $\text{sign}(x)$ over $\mathbb{R}$ by

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases},$$

we can rewrite $\Lambda_n$ as

$$\Lambda_n = \sum_{i=0}^{n} \ell_i(\xi) \cdot \text{sign}(\ell_i(\xi)). \tag{3.5}$$

In order to prove the final part of the result we now construct a continuous, piecewise affine function such that $\|g\|_\infty = 1$, and $g(x_i) = \text{sign}(\ell_i(\xi))$, $i = 1, \cdots, n$.

Let $\sigma : \{0, \cdots, n\} \to \{0, \cdots, n\}$ such that for all $0 \leq i \leq n-1$, $x_{\sigma(i)} < x_{\sigma(i+1)}$. Then for each interval $[x_{\sigma(i)}, x_{\sigma(i+1)}]$ there exists a polynomial $P_i$ of degree at most one such that $P_i(x_{\sigma(i)}) = \operatorname{sign}(\ell_{\sigma(i)}(\xi))$ and $P_i(x_{\sigma(i+1)}) = \operatorname{sign}(\ell_{\sigma(i+1)}(\xi))$.

"Connecting" all the "pieces" together we construct the function

$$
\begin{aligned}
g : [a, b] &\longrightarrow \mathbb{R} \\
x &\longmapsto \begin{cases} \operatorname{sign}(\ell_{\sigma(0)}(\xi)) & a \leq x \leq x_{\sigma(0)} \\ P_i(x) & x_{\sigma(i)} \leq x \leq x_{\sigma(i+1)}, \quad 0 \leq i \leq n-1 \\ \operatorname{sign}(\ell_{\sigma(n)}(\xi)) & x_{\sigma(n)} \leq x \leq b \end{cases}
\end{aligned}
$$

By construction it is clear that $g$ is continuous, $\|g\|_\infty = 1$, and $g(x_i) = \operatorname{sign}(\ell_i(\xi))$.

Therefore equation (**??**) can be applied to $g$

$$\Lambda_n = \sum_{i=0}^{n} \ell_i(\xi) g(x_i) = L_n(g)(\xi).$$

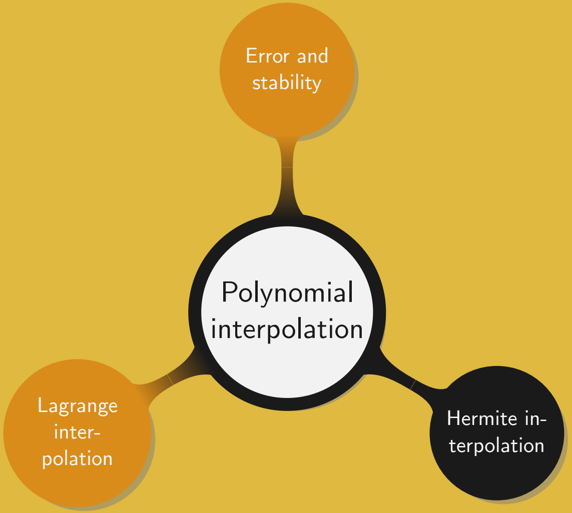
In term on norms this means $\Lambda_n \leq \|L_n(g)\|_\infty \leq \|L_n\| \cdot \|g\|_\infty$. But as $\|g\|_\infty = 1$ we have proven that $\Lambda_n \leq \|L_n\|$.

Together with equation (**??**) this completes the proof.

□

**Corollary**

The Lagrange interpolation method with equidistant nodes becomes unstable as the number of nodes increases.


Error and stability
Polynomial interpolation
Lagrange inter- polation
Hermite in- terpolation

Let $f$ be a continuous function defined from a real interval $[a, b]$ into $\mathbb{R}$ and $x_0, \cdots, x_k$ be $k+1$ distinct points from $[a, b]$.

Assuming there is a finite family of integers $\alpha_0, \cdots, \alpha_k$ such that for all $i$, $f^{(l)}(x_i)$ exists for $0 \leq l \leq \alpha_i$, we want to find a polynomial $P$ verifying

$$P^{(l)}(x_i) = f^{(l)}(x_i), \quad \text{for } 0 \leq l \leq \alpha_i, \text{ and } i = 0, \cdots, k.$$

From a more informal perspective we are given some values from an unknown function as well as some values of its derivatives. The goal is then to find a polynomial that interpolates this function at all the points and their defined derivatives.

This type of interpolation is called *Hermite interpolation*.

**Theorem**

Let $x_0, \cdots, x_k$ be $k+1$ distinct points from $[a, b]$, and $f$ be a continuous function from $[a, b]$ into $\mathbb{R}$ such that there is a finite sequence of integers $\alpha_0, \cdots, \alpha_k$ for which $f^{(l)}(x_i)$ exists, $0 \leq l \leq \alpha_i$, and $i \in \{0, \cdots, k\}$.

Setting $n = k + \sum_{i=0}^{k} \alpha_i$, there exists a unique polynomial $P$ of degree at most $n$, called *Hermite interpolation polynomial*, such that

$$P^{(l)}(x_i) = f^{(l)}(x_i), \quad \text{for } 0 \leq l \leq \alpha_i, \ i = 0, \cdots, k.$$

Proof. We define the application $L$ by

$$\begin{aligned} L : \mathbb{R}_n[x] &\longrightarrow \mathbb{R}^{n+1} \\ P &\longmapsto \left( P(x_0), P'(x_0), \cdots, P^{(\alpha_0)}(x_0), \cdots, P(x_k), \cdots, P^{(\alpha_k)}(x_k) \right). \end{aligned}$$

Since $L$ is clearly a linear application different from 0, proving that it is bijective will directly yield the existence and unicity of $P$.

Let $P$ be in the kernel of $L$. Then $L(P) = 0$, i.e. $P^{(l)}(x_j) = 0$, for all $0 \leq l \leq \alpha_j$, $0 \leq j \leq k$. In other words $x_j$ is a root of order $\alpha_j + 1$ of $P$. Hence $P$ is divisible by $\prod_{j=0}^{k}(x - x_j)^{\alpha_j+1}$ and there exists a polynomial $q$ such that

$$P(x) = q(x) \prod_{j=0}^{k} (x - x_j)^{\alpha_j+1}.$$

If $q$ is not identically equal to 0 then

$$\begin{aligned} \deg P &= \deg q + \deg \prod_{j=0}^{k} (x - x_j)^{\alpha_j+1} = \deg q + \sum_{j=0}^{k} (\alpha_j + 1) \\ &= \deg q + k + 1 + \sum_{j=0}^{k} \alpha_j = \deg q + n + 1. \end{aligned}$$

We just proved that $P$ is of degree larger than $n+1$, which is not since it belongs to $\mathbb{R}_n[x]$. $\lightning$

Hence $q$ is identically 0, and $\ker L = \{0\}$. So by lemma 1.17, $L$ is a bijection. $\square$

Remark. The previous theorem does not provide any information on how to construct $P$. As a simple overview the idea consists in setting $P^{(l)}(x_i)$ to $f^{(l)}(x_i)$, for $0 \leq l \leq \alpha_i$, $i = 0, \cdots, k$. Then express all these elements in the canonical basis $(e_1, \cdots, e_n)$ of $\mathbb{R}^{n+1}$ and determine $L^{-1}(e_j)$ for all $j \in \{0, \cdots, n\}$. Finally write $P$ in term of $L^{-1}(e_j)$, for all $j \in \{0, \cdots, n\}$.

**Theorem**

Let $x_0, \cdots, x_k$ be $k+1$ distinct points from $[a,b]$, and $f : [a,b] \to \mathbb{R}$ be in $C^{n+1}[a,b]$, such that there is a finite sequence of integers $\alpha_0, \cdots, \alpha_k$ for which $f^{(l)}(x_i)$ exists, $0 \leq l \leq \alpha_i$, and $i \in \{0, \cdots, k\}$.

If $P$ is the Hermite interpolation polynomial of $f$ at $x_0, \cdots, x_k$, then for all $x \in [a,b]$ there exists $\xi_x \in \left[\min(x, \min_i x_i), \max(x, \max_i x_i)\right]$ such that

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{k} (x - x_i)^{\alpha_i + 1}.$$

Proof. Since the formula is clearly true when $x = x_i$, we assume $x \neq x_i$, $0 \leq i \leq k$.

We now proceed following a similar strategy as in the case of Lagrange interpolation.

Calling $\Pi(t)$ the product $\prod_{i=0}^{k}(t-x_i)^{\alpha_i+1}$ we want to prove the existence of $\xi_x$ such that

$$(n-1)!\frac{f(x)-P(x)}{\Pi(x)} - f^{(n+1)}(\xi_x) = 0,$$

where $P$ is Hermite's polynomial.

Let $Q(t) = \frac{f(x)-P(x)}{\Pi(x)}\Pi(t) - f(t) + P(t)$. Then $Q$ is a function with $n+2$ zeros: all the $x_i$ counted with their multiplicity, and $x$.

Therefore, by Rolle's theorem (2.18) there exists $\xi_x$ in the smallest interval containing the roots of $Q$, such that $Q^{(n+1)}(\xi_x) = 0$.

Moreover as $P \in \mathbb{R}_n[x]$ we know that $P^{(n+1)} = 0$ and obtain

$$Q^{(n+1)}(t) = \frac{f(x) - P(x)}{\Pi(x)} \Pi^{(n+1)}(t) - f^{(n+1)}(t).$$

Finally, noting that $\Pi$ is a monic polynomial from $\mathbb{R}_{n+1}[x]$ yields

$$\frac{f(x) - P(x)}{\Pi(x)} (n+1)! = f^{(n+1)}(\xi_x).$$

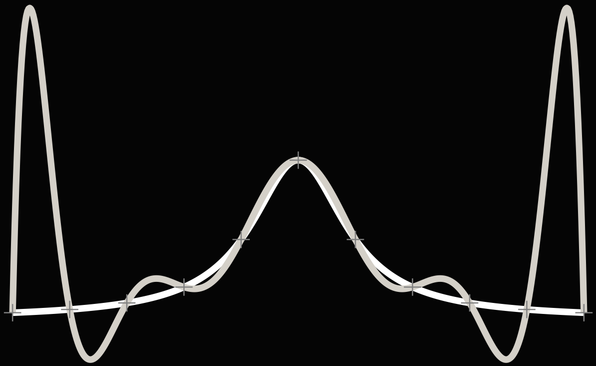
Hence

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{i=0}^{k} (x - x_i)^{\alpha_i + 1}.$$

$\square$

# Key points

- Explain Lagrange interpolation
- What is a good choice of nodes for Lagrange interpolation?
- What is the stability of a numerical method?
- What does the Lebesgue constant measure?
- Explain Hermite interpolation



# Thank you!