Analysis of Algorithms 1 (Fall 2011)
Istanbul Technical University Computer Eng. Dept.

Chapter 13
Red-Black Trees

Last updated: December 03, 2009

Purpose

Review Binary Search Trees

Introduce Red-Black Trees

Review 2-3 and 2-3-4 trees

Rotations and other operations in RB Trees

Outline

Binary Search Tree (BST) review

Red and Black Trees

2-3 and 2-3-4 trees

Operations on Red and Black Trees

Tree

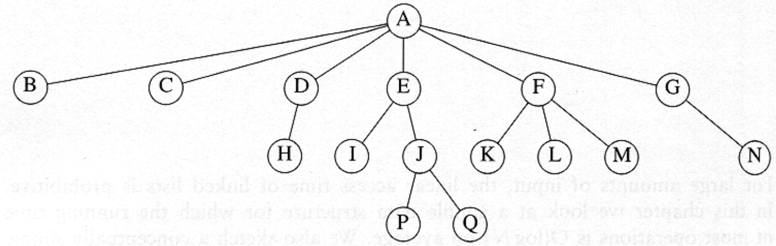
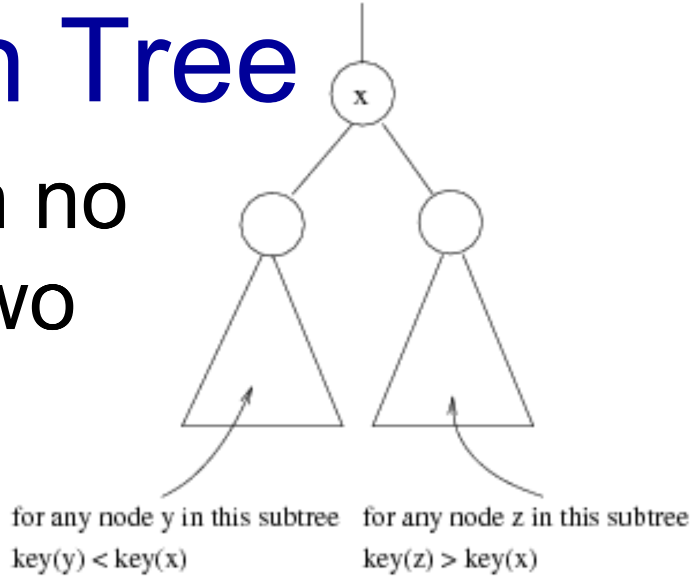


Figure 4.2 A tree

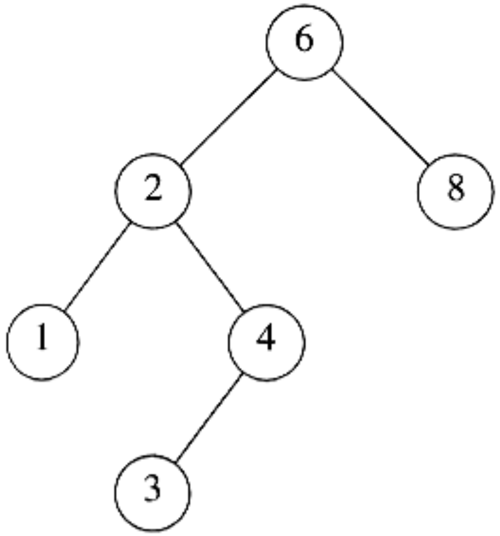
- ***Child* and *parent***
  - Every node except the root has one parent
  - A node can have an arbitrary number of children
- ***Leaves:*** Nodes with no children
- ***Sibling:*** nodes with same parent
- ***Path Length*:** number of edges on the path
- ***Depth* of a node:** length of the unique path from the root to that node. The depth of a tree is equal to the depth of the deepest leaf
- ***Height* of a node**: length of the longest path from that node to a leaf, all leaves are at height 0, The height of a tree is equal to the height of the root
- *Ancestor* and *descendant*

Binary Search Tree

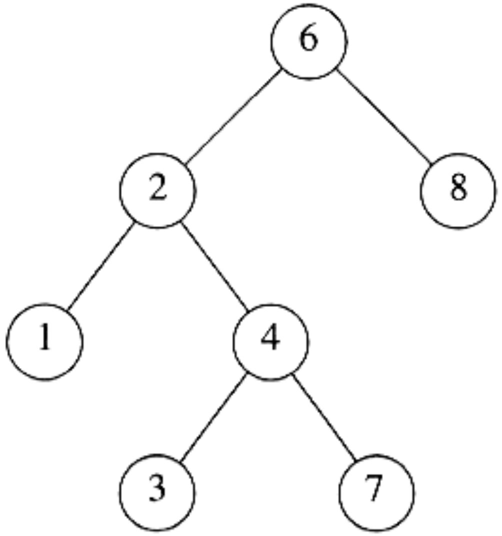


- **Binary tree:** A tree in which no node can have more than two children
- **Binary search tree:**
- Stores keys in the nodes in a way so that searching, insertion and deletion can be done efficiently.
- For every node X, all the keys in its left subtree are smaller than the key value in X, and all the keys in its right subtree are larger than the key value in X

Binary Search Tree

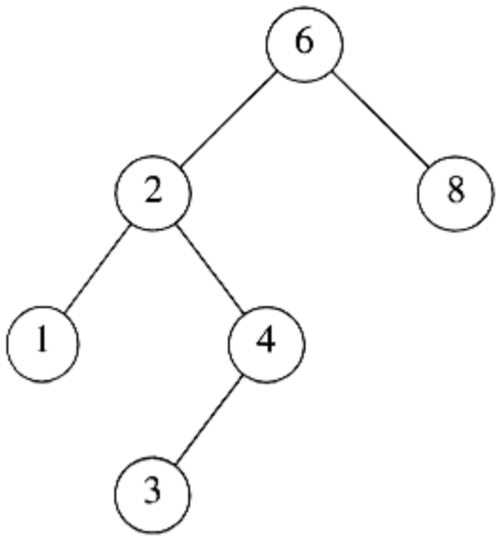


A binary search tree



Not a binary search tree
WHY?

Binary Search Tree



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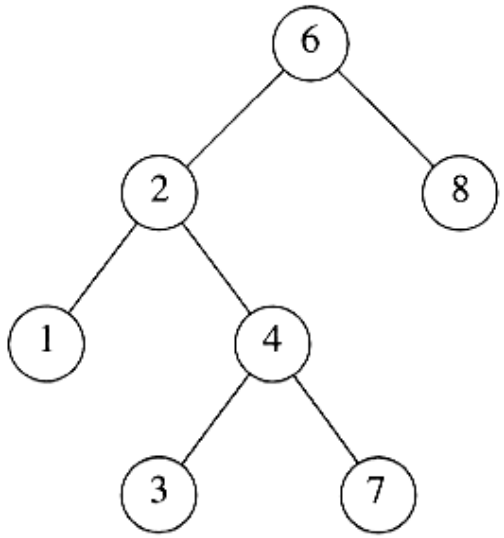


Not a binary search tree
Because 7>6

Binary Search Tree Operations

- **Tree Traversal:** Used to print out the data in a tree in a certain order
- **Pre-order traversal (node-left-right)**
  - Print the data at the root
  - Recursively print out all data in the left subtree
  - Recursively print out all data in the right subtree
- See also, **post-order: left-right-node** and **in-order:left-node-right** traversal.

Binary-search-tree sort

$T \leftarrow \varnothing$ ▷ Create an empty BST

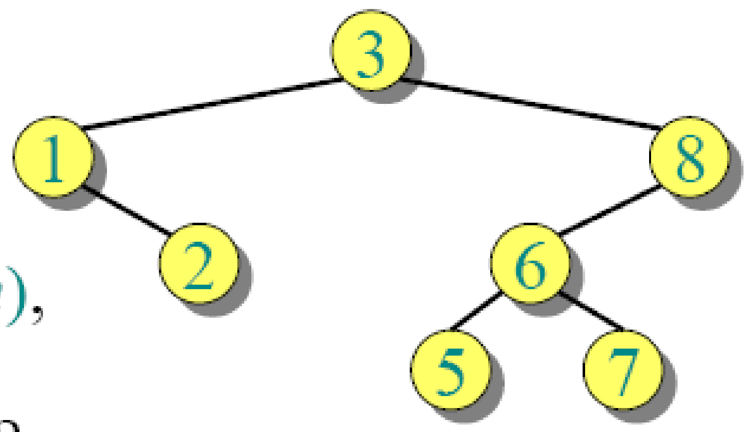
for $i = 1$ to n

do TREE-INSERT $(T, A[i])$

Perform an inorder tree walk (traversal) of T.

Example:

$A = [3\ 1\ 8\ 2\ 6\ 7\ 5]$



Tree-walk time $= O(n)$, but how long does it take to build the BST?

Node depth

The depth of a node = the number of comparisons made during TREE-INSERT. Assuming all input permutations are equally likely, we have

Average node depth

$$= \frac{1}{n} E\left[\sum_{i=1}^{n} (\#\text{comparisons to insert node } i)\right]$$

$$= \frac{1}{n} O(n \lg n) \qquad \text{(quicksort analysis)}$$

$$= O(\lg n) \; .$$

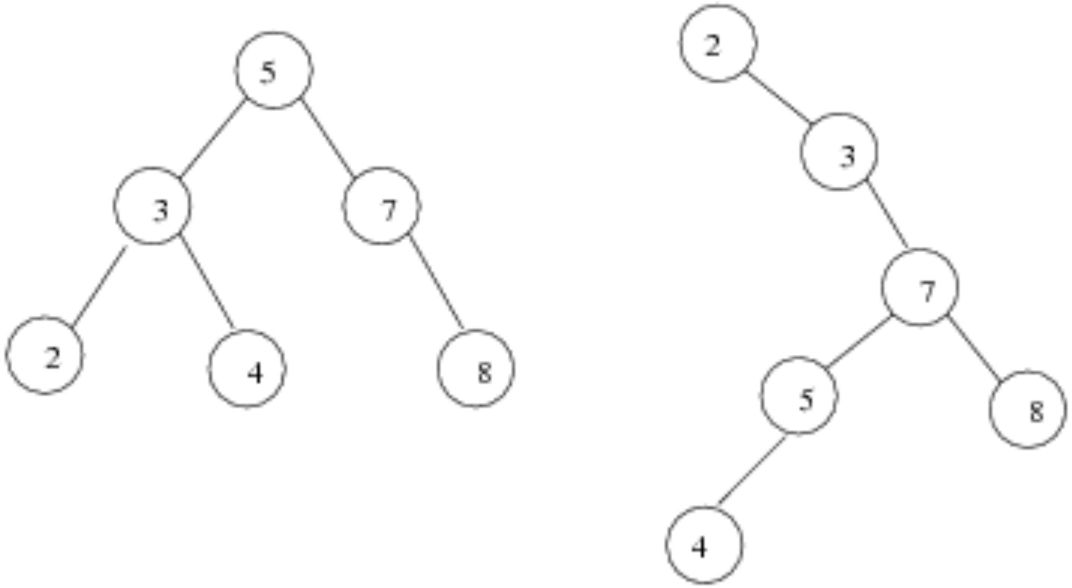
Height of a randomly built binary search tree

Outline of the analysis:

- Use ***Jensen's inequality***, which says that $f(E[X]) \leq E[f(X)]$ for any convex function f and random variable X.
- Analyze the ***exponential height*** of a randomly built BST on n nodes, which is the random variable $Yn = 2^{Xn}$, where Xn is the random variable denoting the height of the BST.
- Prove that $2^{E[Xn]} \leq E[2^{Xn}] = E[Yn] = O(n^3)$, and hence that $E[Xn] = O(\lg n)$.

Binary Search Tree Operations

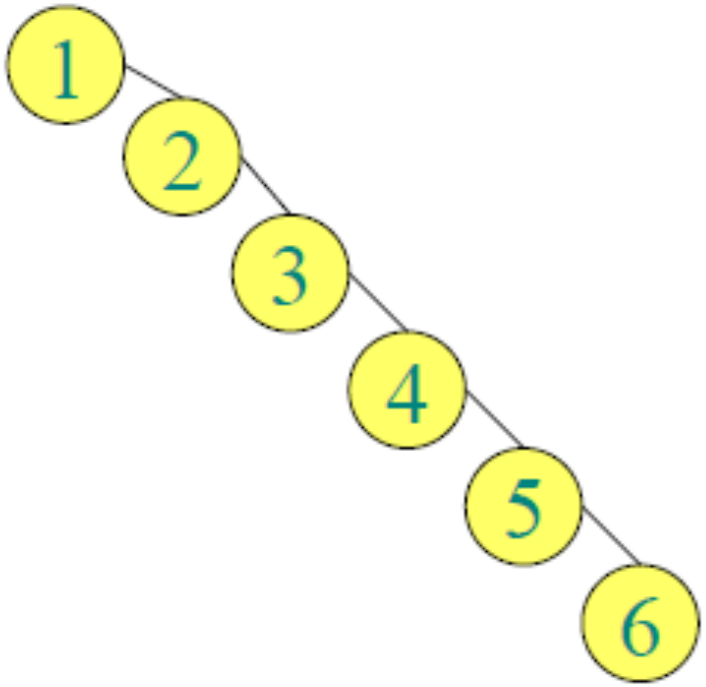
- Linear access time of linked lists is prohibitive
- Does there exist any simple data structure for which the running time of most operations (search, insert, delete) is O(log N)?



Two binary search trees representing the same set.

Average depth of a node is O(log N); maximum depth of a node is O(N)

Balanced search trees, or how to avoid **this** even in the worst case



Balanced search tree

Balanced search tree: A search-tree data structure for which a height of $O(\lg n)$ is guaranteed when implementing a dynamic set of n items.

Examples:

- AVL trees
- 2-3 trees
- 2-3-4 trees
- B-trees
- Red-black trees

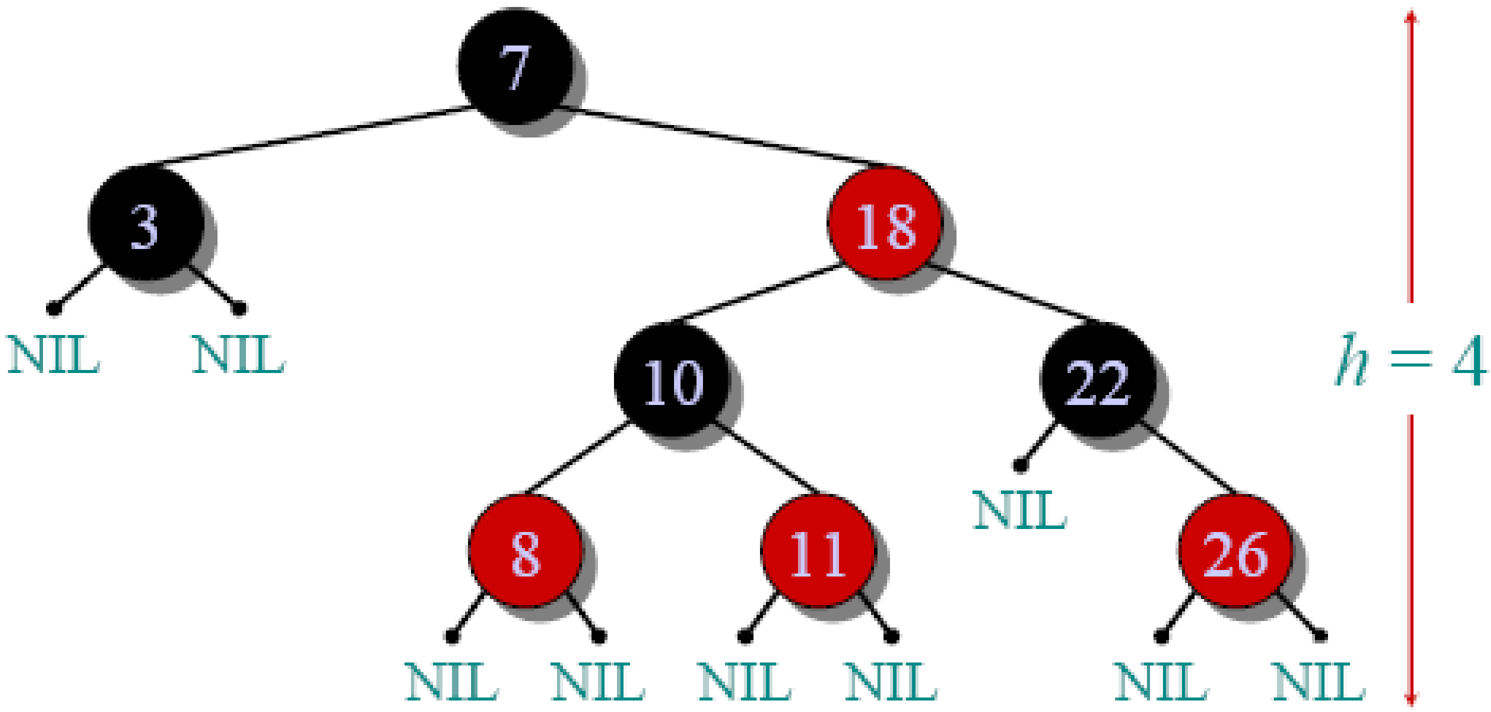
Red-black trees

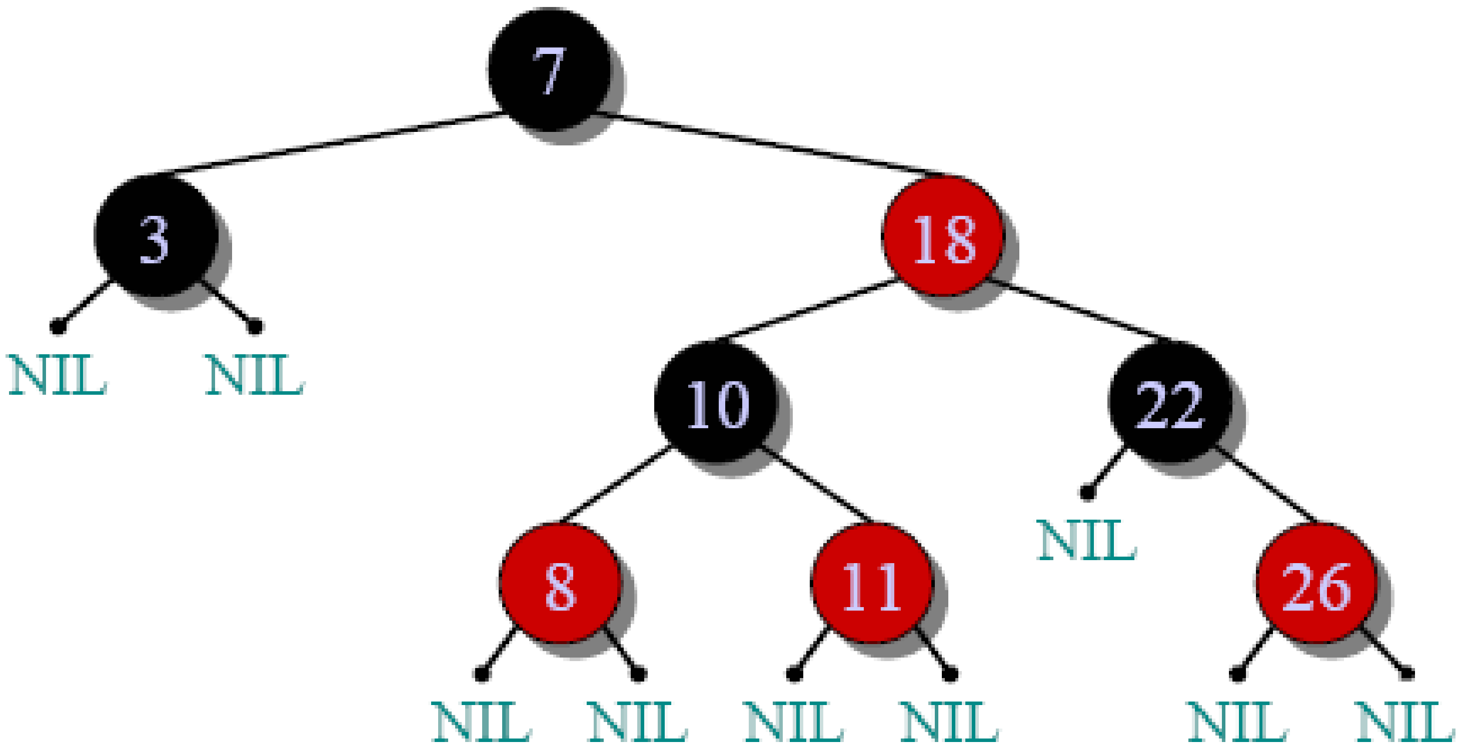
BSTs (Binary Search Tree) with an extra one-bit color field in each node.

Red-black properties:

1. Every node is either red or black.
2. The root and leaves (NIL's) are black.
3. If a node is red, then its parent is black.
4. All simple paths from any node x to a descendant leaf have the same number of black nodes = black-height(x).

Red-black tree example





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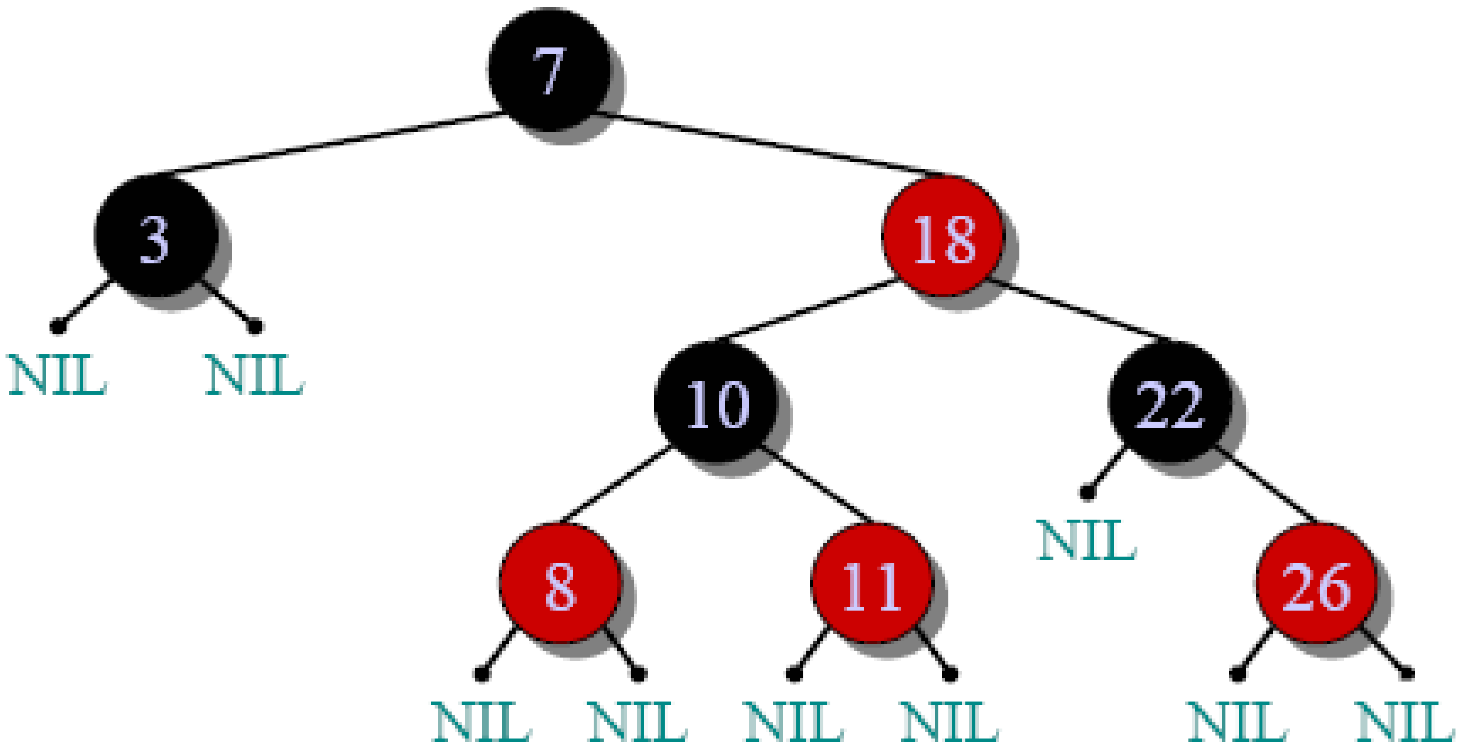
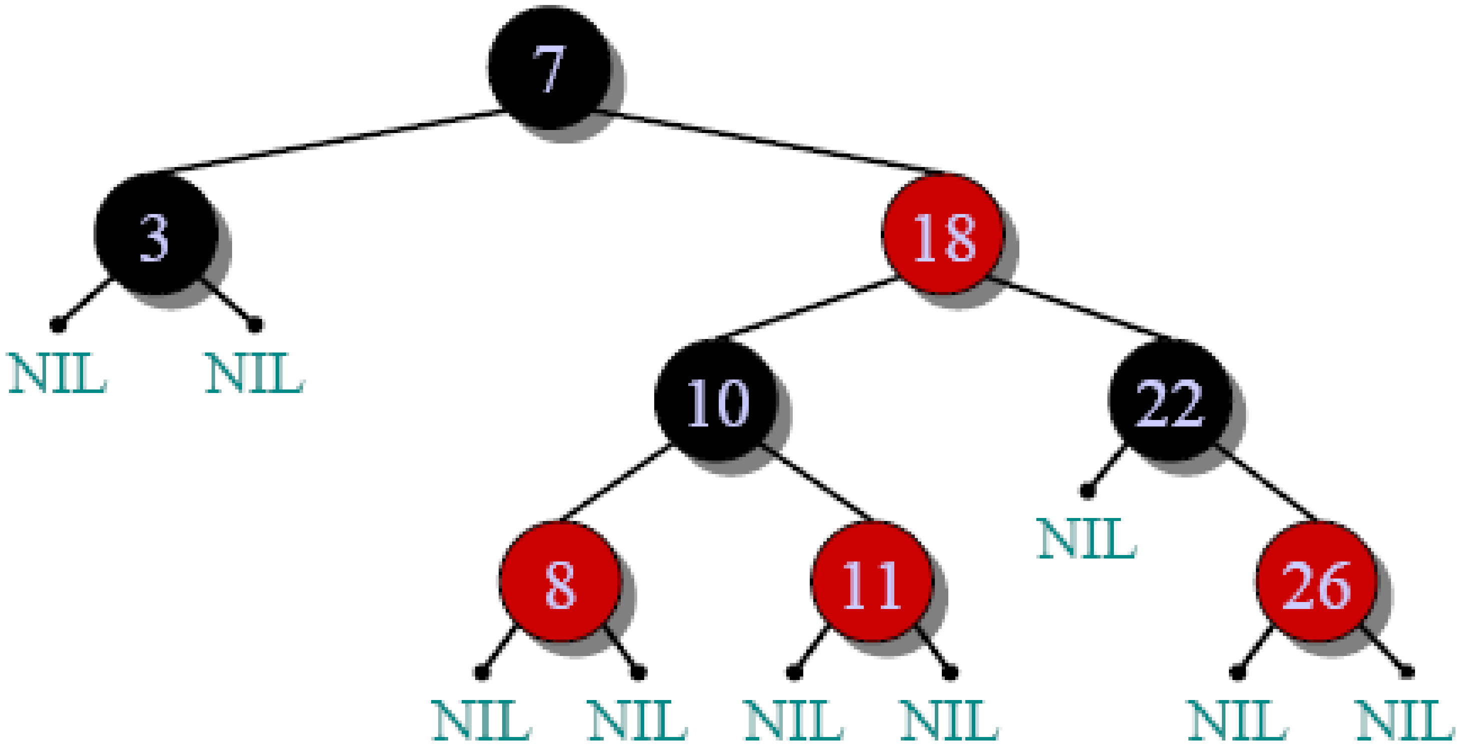


2. The root and leaves (NIL's) are black.



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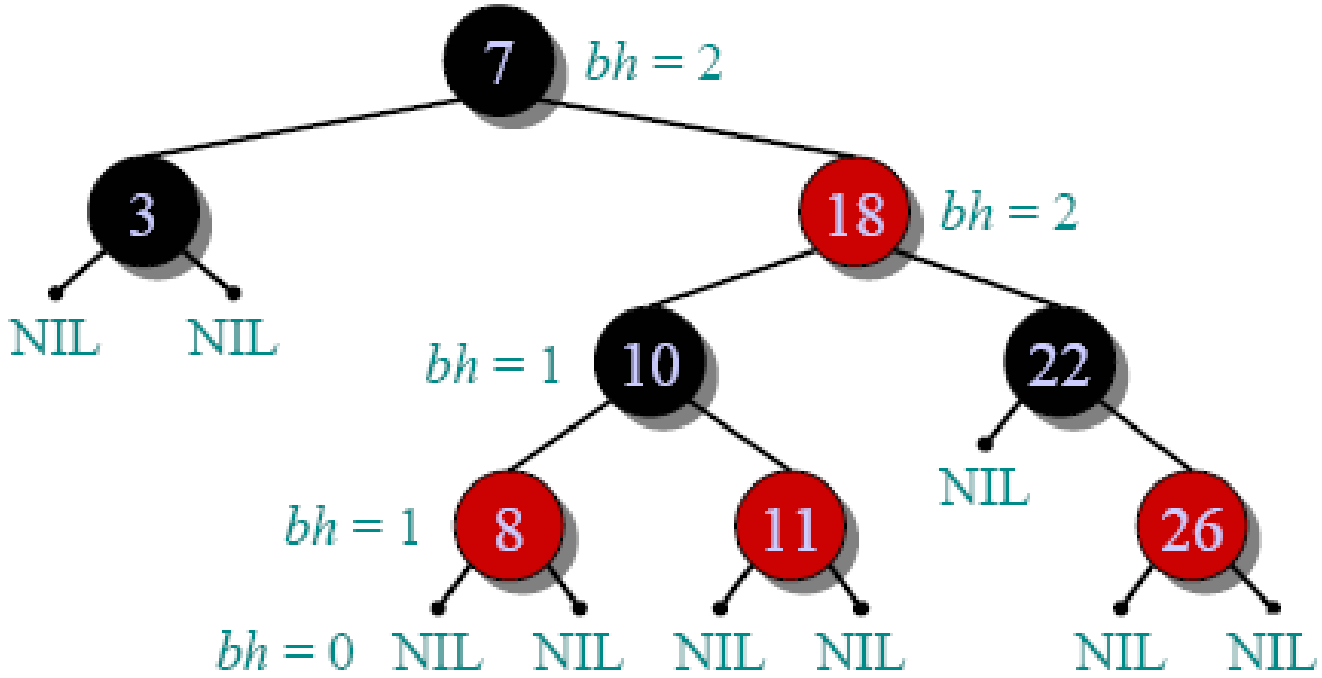
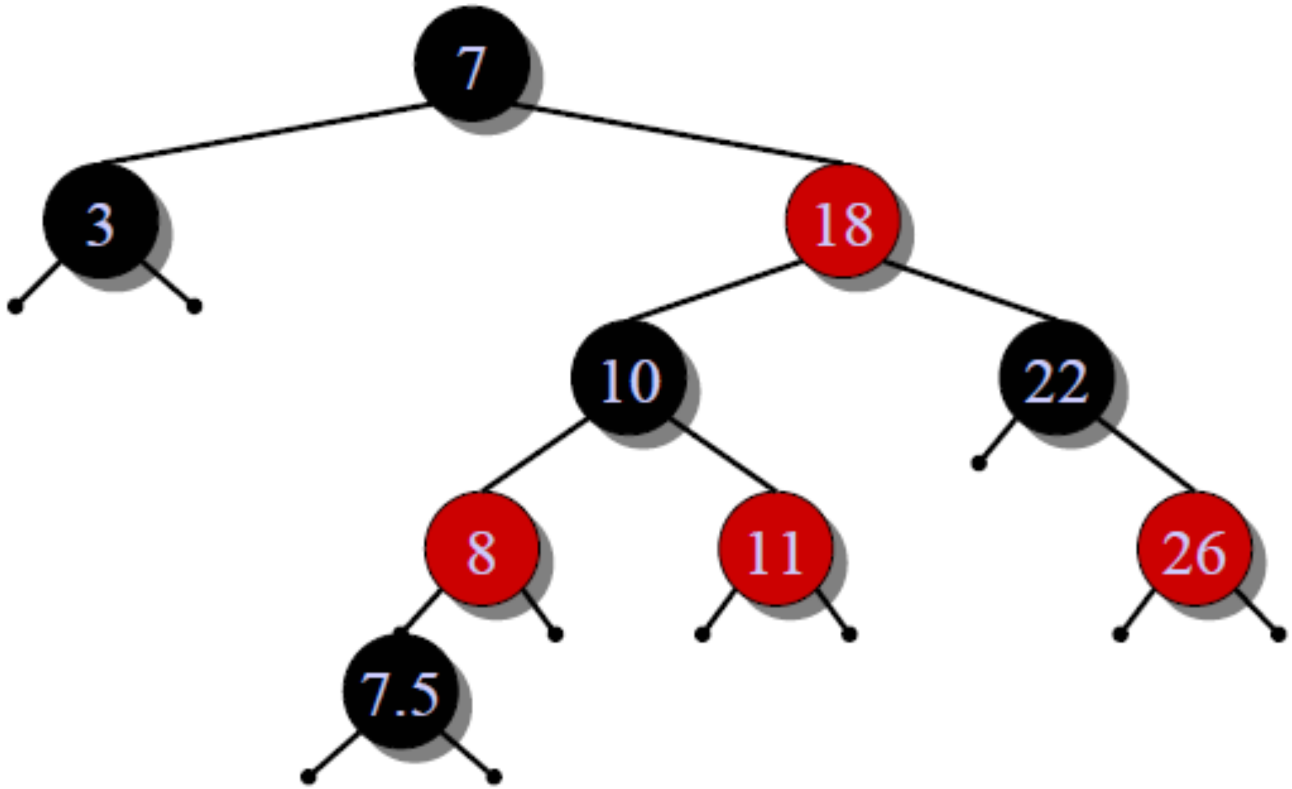


4. All simple paths from any node *x* to a descendant leaf have the same number of black nodes = *black-height*(*x*).

What properties would we like to prove about red-black trees ?

They always have **O(log n) height**
There is an **O(log n)** time **insertion** procedure which preserves the red-black properties

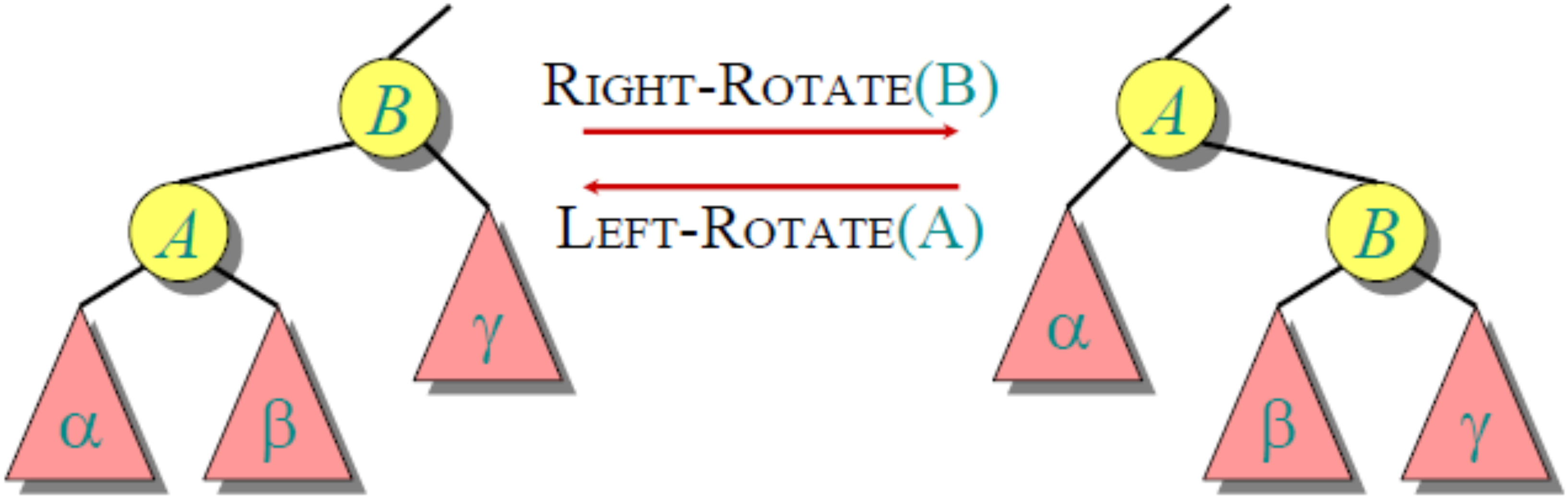
• Is it true that, after we add a new element to a tree, we can always recolor the tree to keep it red-black ?



- Is it true that, after we add a new element to a tree, we can always recolor the tree to keep it red-black ?

NO. After insertions, sometimes we need to juggle nodes around

Rotations

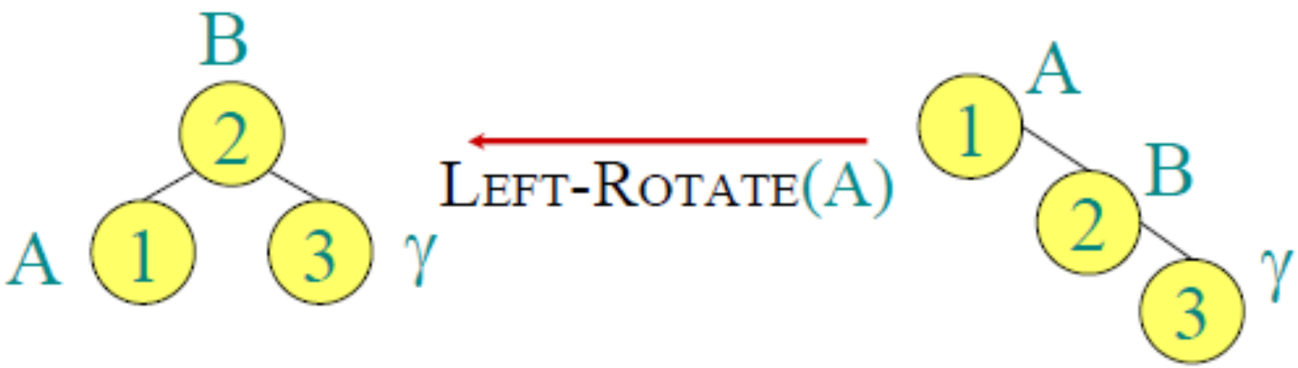


Rotations maintain the inorder ordering of keys:

- $a \in \alpha, b \in \beta, c \in \gamma \Rightarrow a \leq A \leq b \leq B \leq c$.

A rotation can be performed in $O(1)$ time.

Rotations can reduce height



Red-black tree wrap-up

- Can show how
  - O(log n) re-colorings
  - 1 rotation

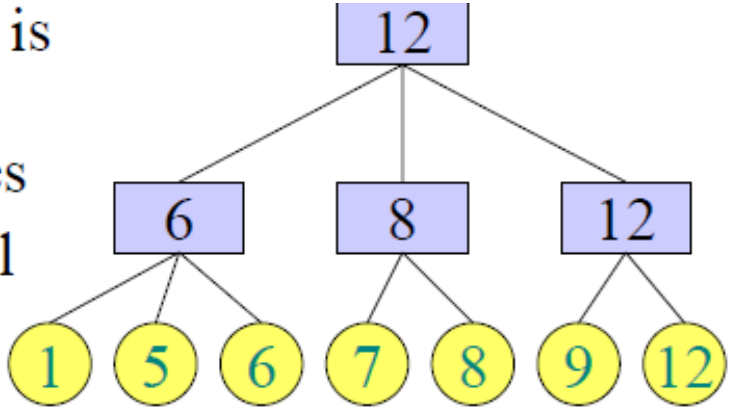
  can restore red-black properties after an insertion
- Instead, we will see 2-3 trees (but will come back to red-black trees at the end)

2-3 and 2-3-4 Trees

2-3 Trees

- The simplest balanced trees on the planet!
- Although a little bit more wasteful

- Degree of each node is either 2 or 3
- Keys are in the leaves
- All leaves have equal depth
- Leaves are sorted
- Each node x contains maximum key in the sub-tree, denoted x.max



Internal nodes

- Internal nodes:
  - Values:
    - x.max: maximum key in the sub-tree
  - Pointers:
    - left[x]
    - mid[x]
    - right[x] : can be null
    - p[x] : can be null for the root
    - …
- Leaves:
  - x.max : the key

Height of 2-3 tree

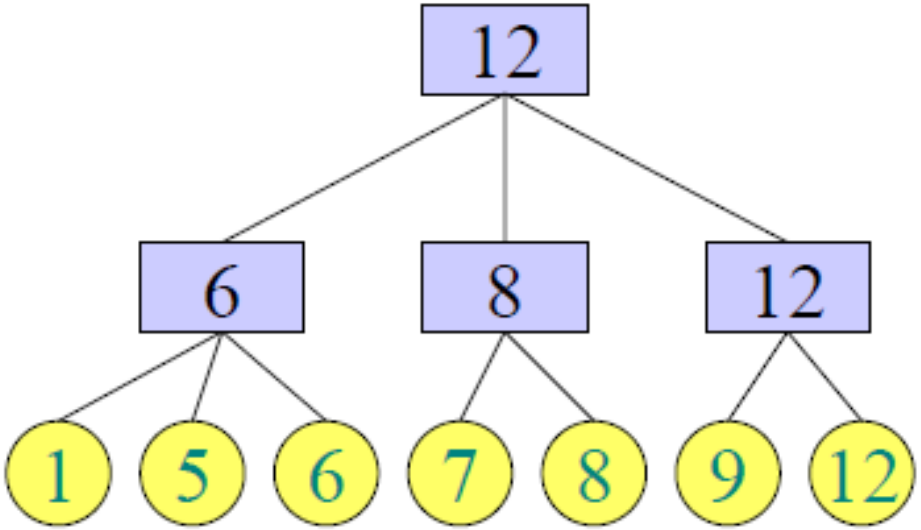
- What is the maximum height h of a 2-3 tree with n nodes ?
- Alternatively, what is the minimum number of nodes in a 2-3 tree of height h ?
- It is $1+2+2^2+2^3+\ldots+2^h = 2^{h+1}-1$
- $n \geq 2^{h+1}-1 \Rightarrow h = O(\log n)$
- Full binary tree is the worst-case example!

Searching

- How can we search for a key k ?

Search(x,k):

- If x=NIL then return NIL
- Else if x is a leaf then
  - If x.max=k then return x
  - Else return NIL
- Else
  - If k ≤ left[x].max then Search(left[x],k)
  - Else if k ≤ mid[x].max then Search(mid[x],k)
  - Else Search(right[x],k)

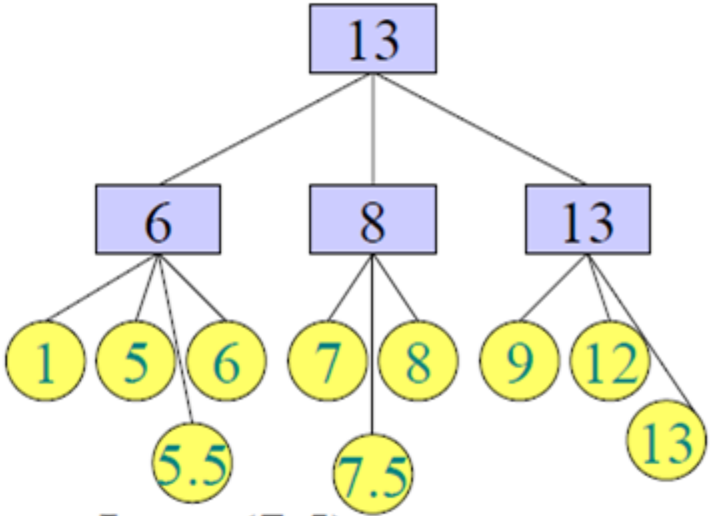


Search(8)

Search(13)

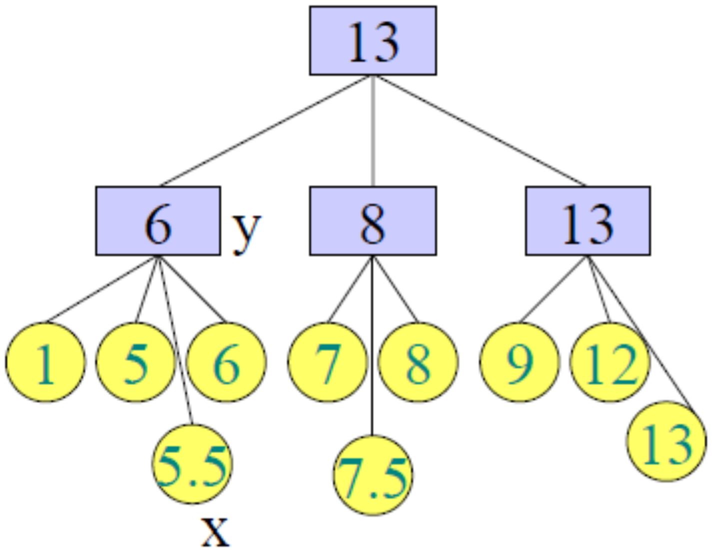
Insertion

- How to insert x ?
- Perform Search for the key of x
- Let y be the last internal node
- Insert x into y in a sorted order
- At the end, update the max values on the path to root



Insert(7.5)
Insert(13)
Insert(5.5)

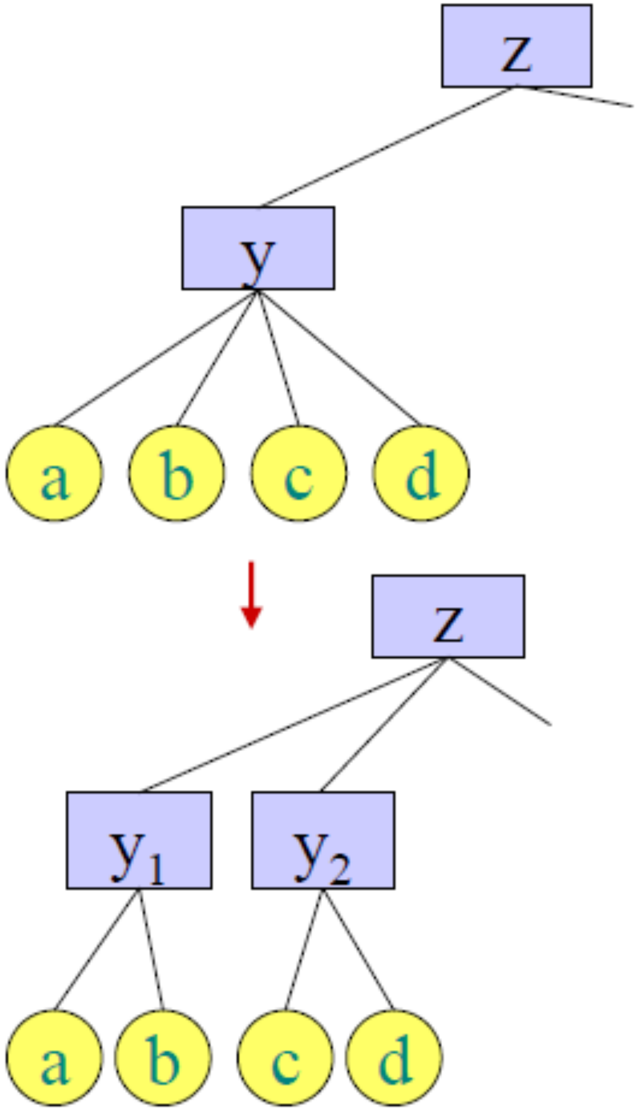
- If y has 4 children, then Split(y)

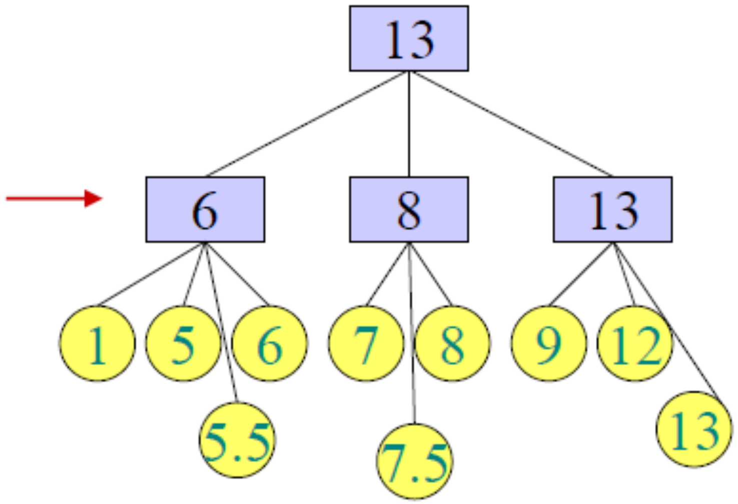


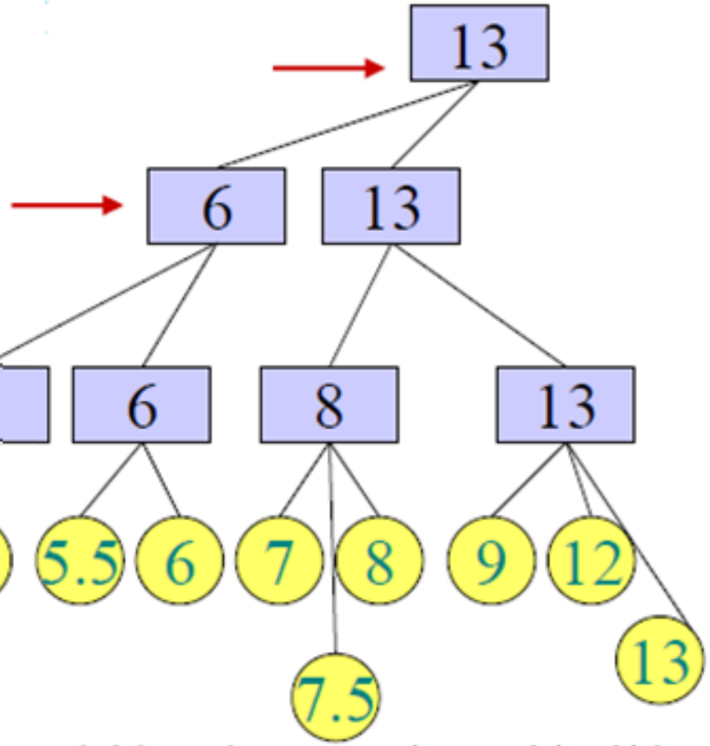
Split

- Split y into two nodes y_1, y_2
- Both are linked to z=parent(y)*
- If z has 4 children, split z

*If y is a root, then create new parent(y)=new root



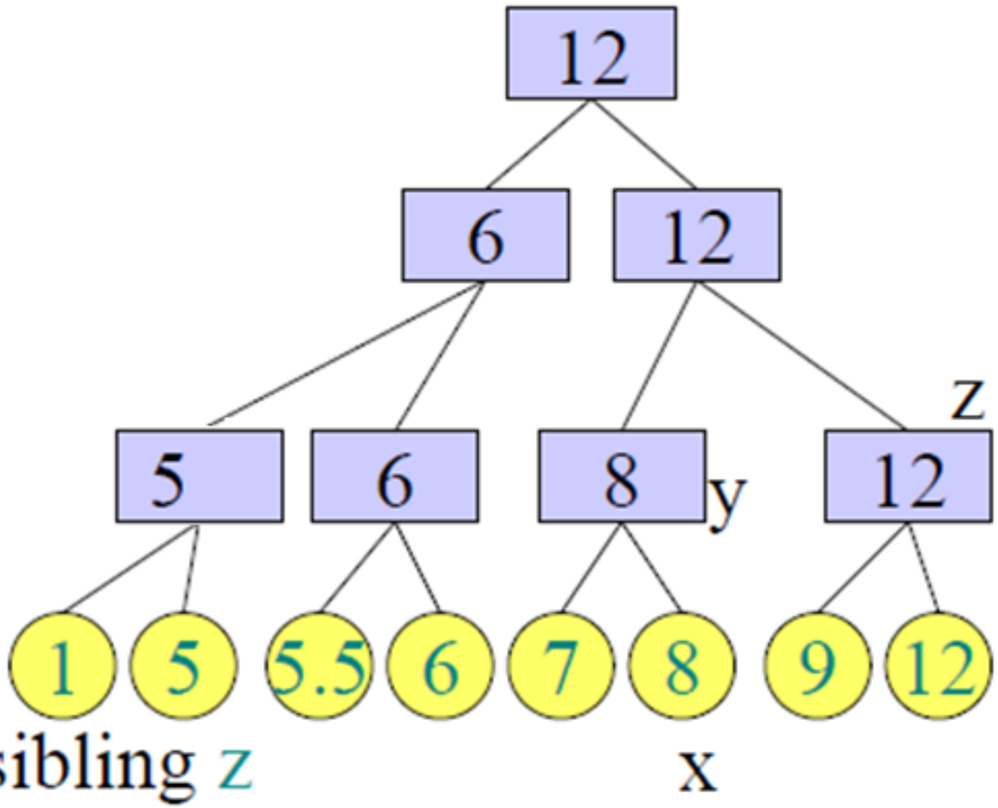




- Insert and Split preserve heights, unless new root is created, in which case all heights are increased by 1
- After Split, all nodes have 2 or 3 children
- Everything takes O(log n) time

Delete

- How to delete x ?
- Let y=p(x)
- Remove x from y
- If y has 1 child:
  - Remove y
  - Attach to y's sibling z



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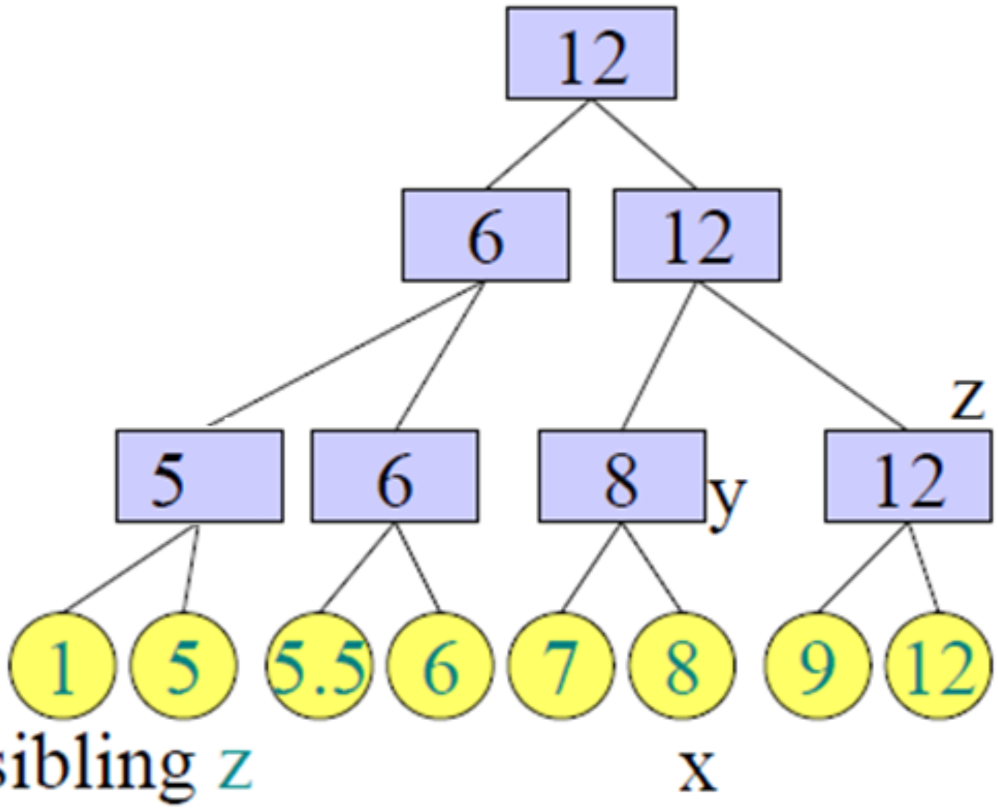


Delete(8)

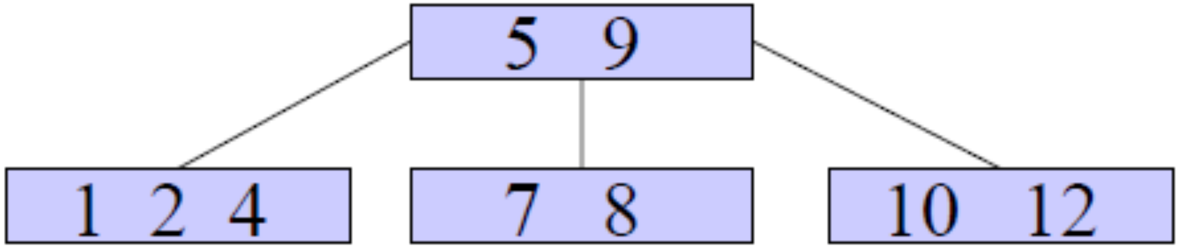
If z has 4 children, then Split(z)

INCOMPLETE – SEE THE END FOR FULL VERSION

Summing up

- 2-3 Trees:
  - O(log n) depth ⇒ Search in O(log n) time
  - Insert, Delete (and Split) in O(log n) time
- We will now see 2-3-4 trees
  - Same idea, but:
    - Each parent has 2,3 or 4 children
    - Keys in the inner nodes
    - More complicated procedures

2-3-4 Trees



Back to Red and Black Trees

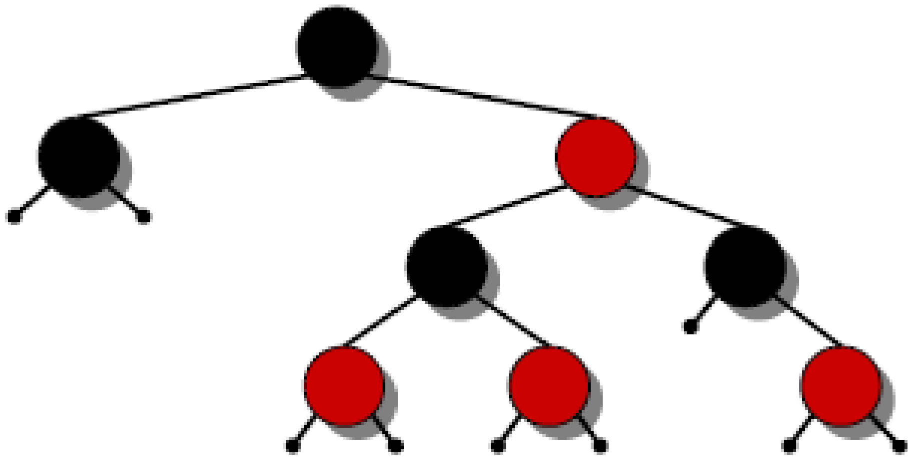
Height of a red-black tree

Theorem. A red-black tree with *n* keys has height

$$h \leq 2 \lg(n + 1).$$

Proof. (The book uses induction. Read carefully.)

INTUITION:
Merge red nodes into their black parents.



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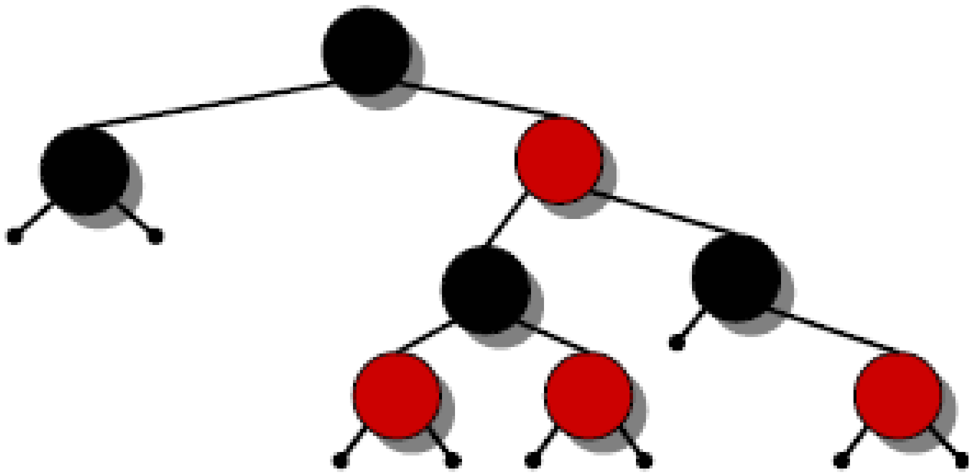

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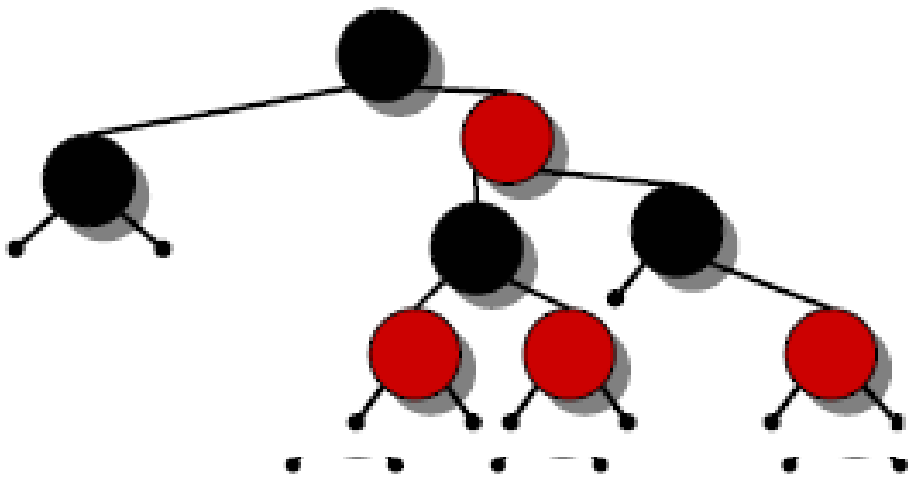

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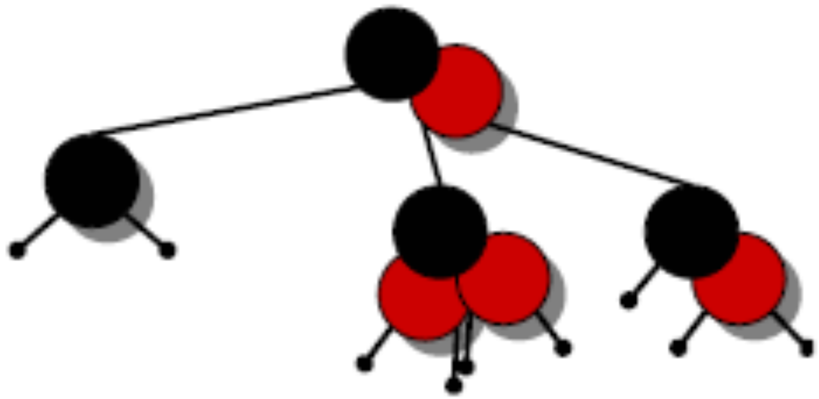

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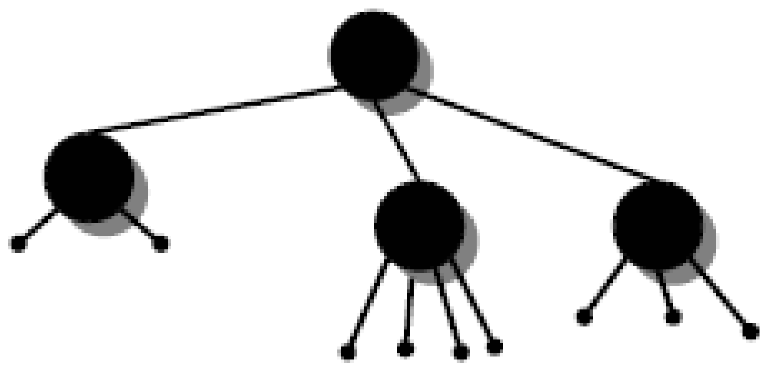

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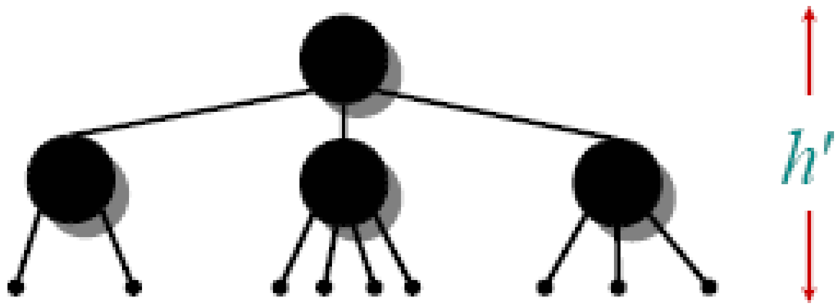


- Merge red nodes into their black parents.
- This process produces a tree in which each node has 2, 3, or 4 children.
- The 2-3-4 tree has uniform depth h' of leaves.

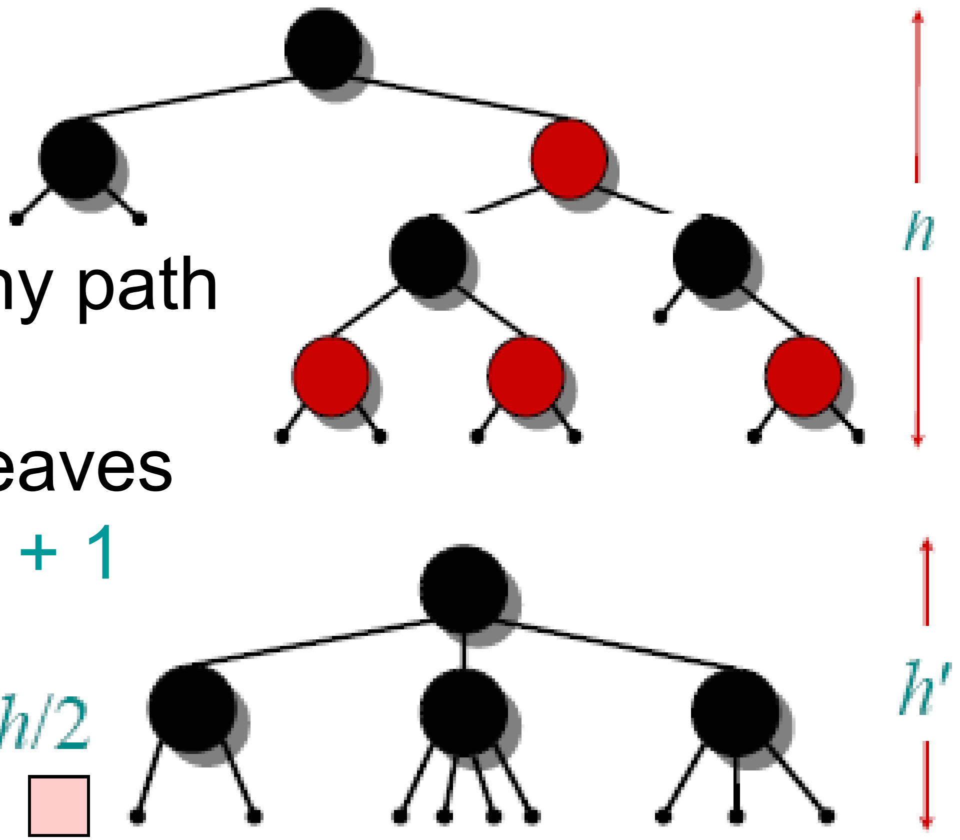
Height of a red-black tree

- We have $h' \geq h/2$, since at most half the leaves on any path are red.
- The number of leaves in each tree is $n + 1$

$\Rightarrow n + 1 \geq 2^{h'}$

$\Rightarrow \lg(n + 1) \geq h' \geq h/2$

$\Rightarrow h \leq 2 \lg(n + 1)$. $\square$



Query operations

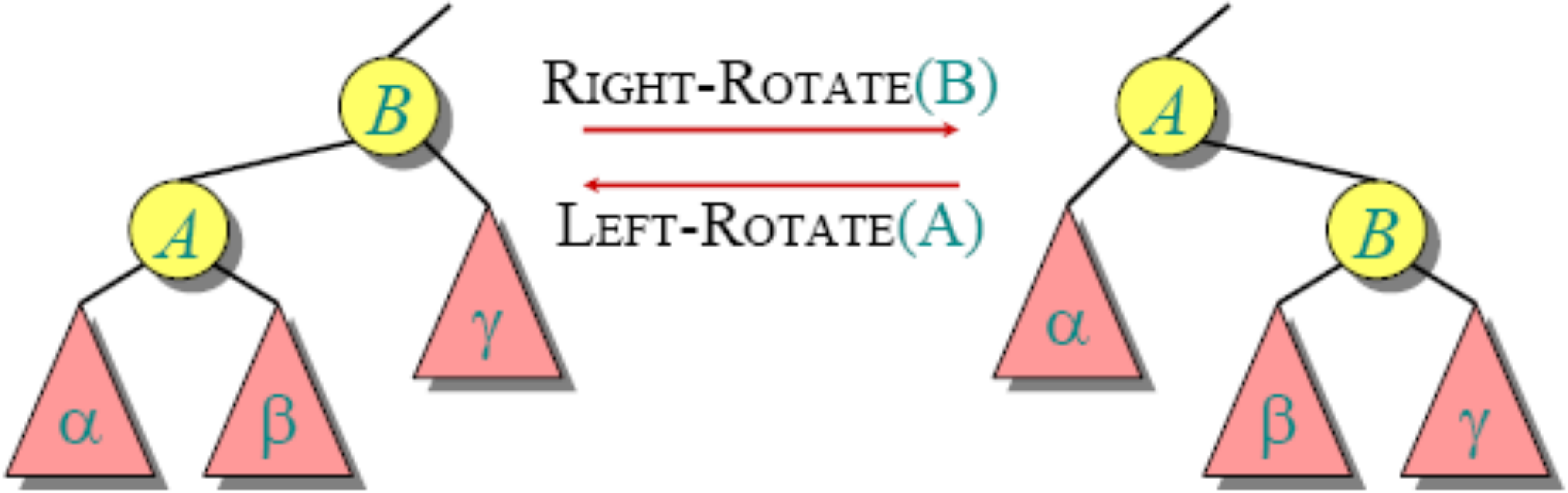
Corollary. The queries SEARCH, MIN, MAX, SUCCESSOR, and PREDECESSOR all run in $O(\lg n)$ time on a red-black tree with n nodes.

Modifying operations

The operations INSERT and DELETE cause modifications to the red-black tree:

- the operation itself,
- color changes,
- restructuring the links of the tree via ***"rotations"***.

Rotations



Rotations maintain the inorder ordering of keys:

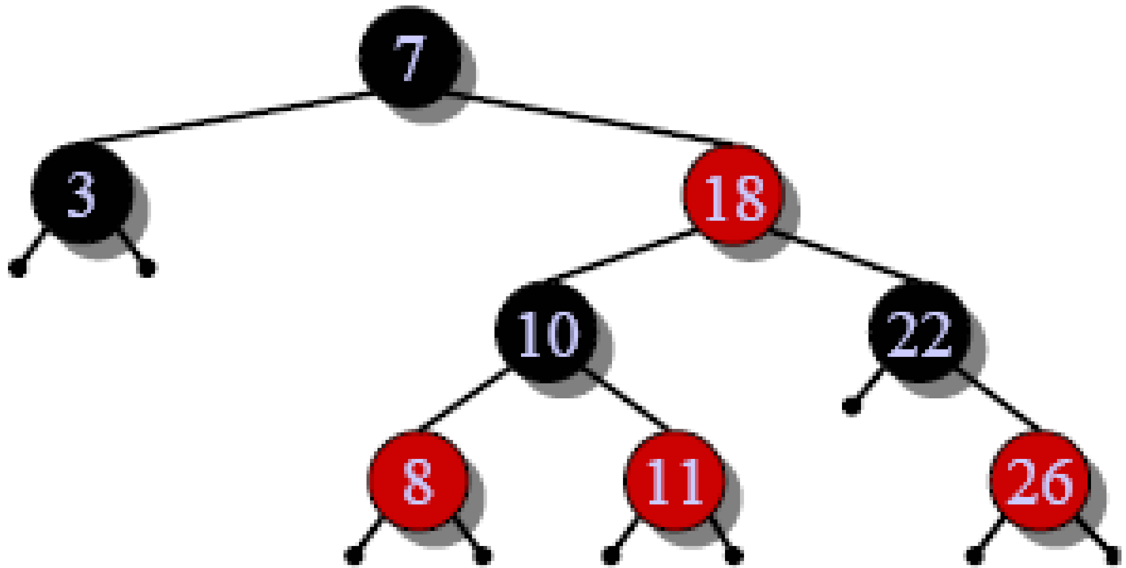
- $a \in \alpha$, $b \in \beta$, $c \in \gamma \Rightarrow a \le A \le b \le B \le c$.

A rotation can be performed in $O(1)$ time.

Insertion into a red-black tree

IDEA: Insert *x* in tree. Color *x* red. Only redblack property 3 might be violated. Move the violation up the tree by recoloring until it canbe fixed with rotations and recoloring.

Example:



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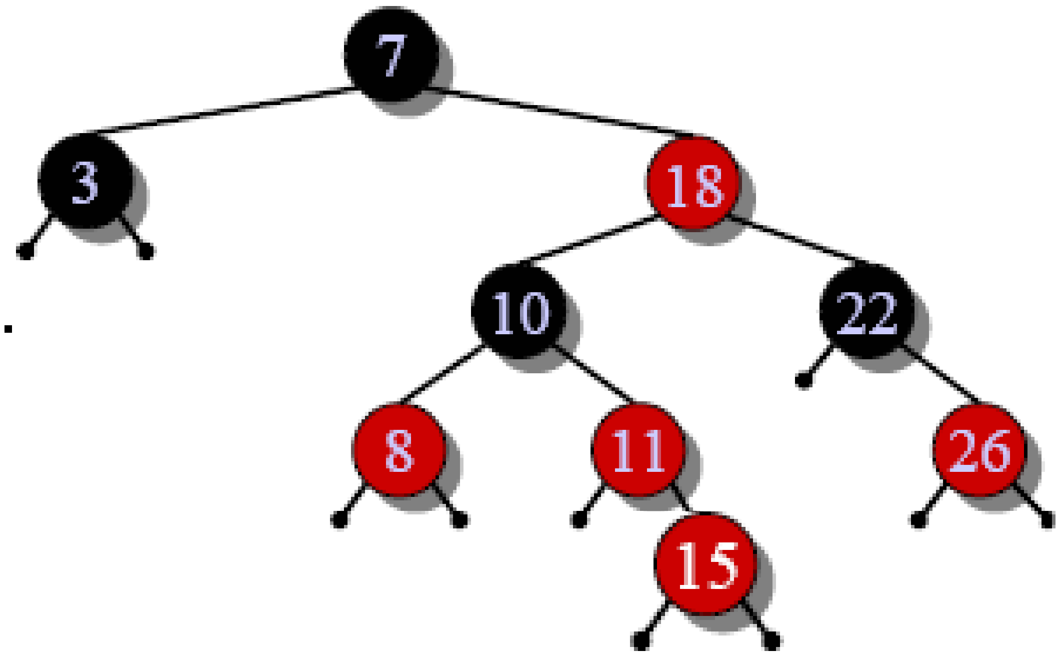
Example:

- Insert *x* =15.
- Recolor, moving the violation up the tree.



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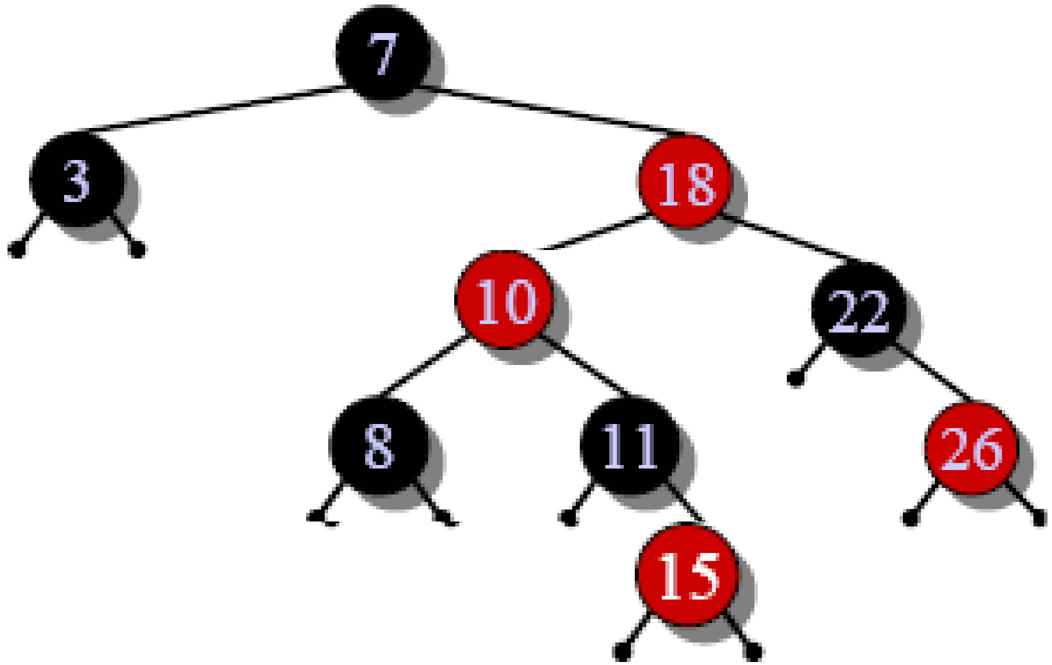
- RIGHT-ROTATE(18)



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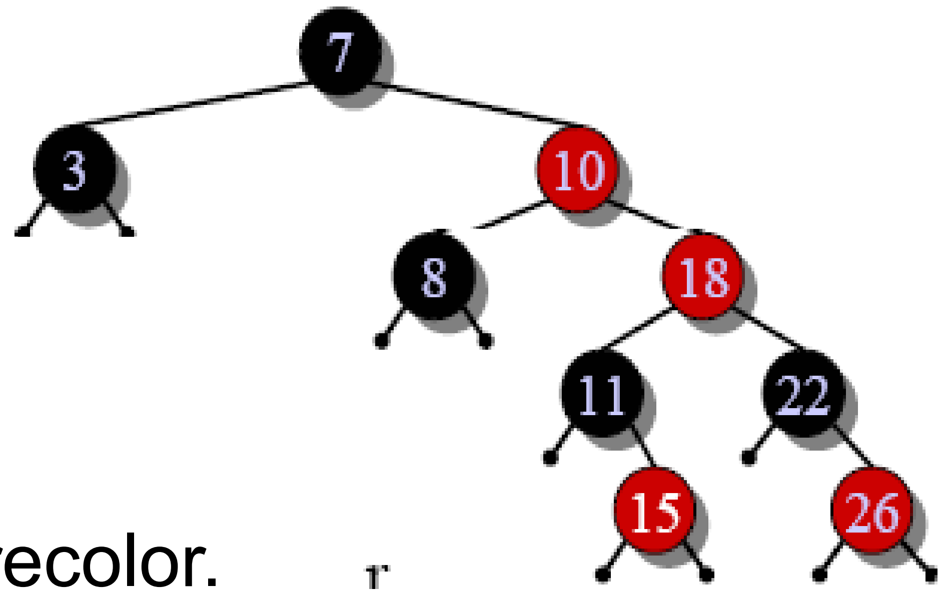

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- LEFT-ROTATE(7) and recolor.

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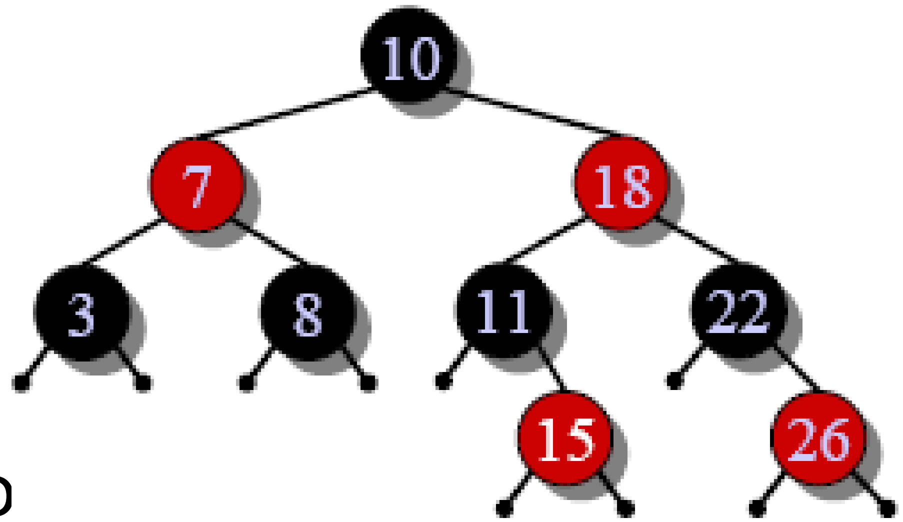
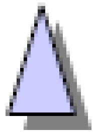
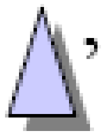


Pseudocode

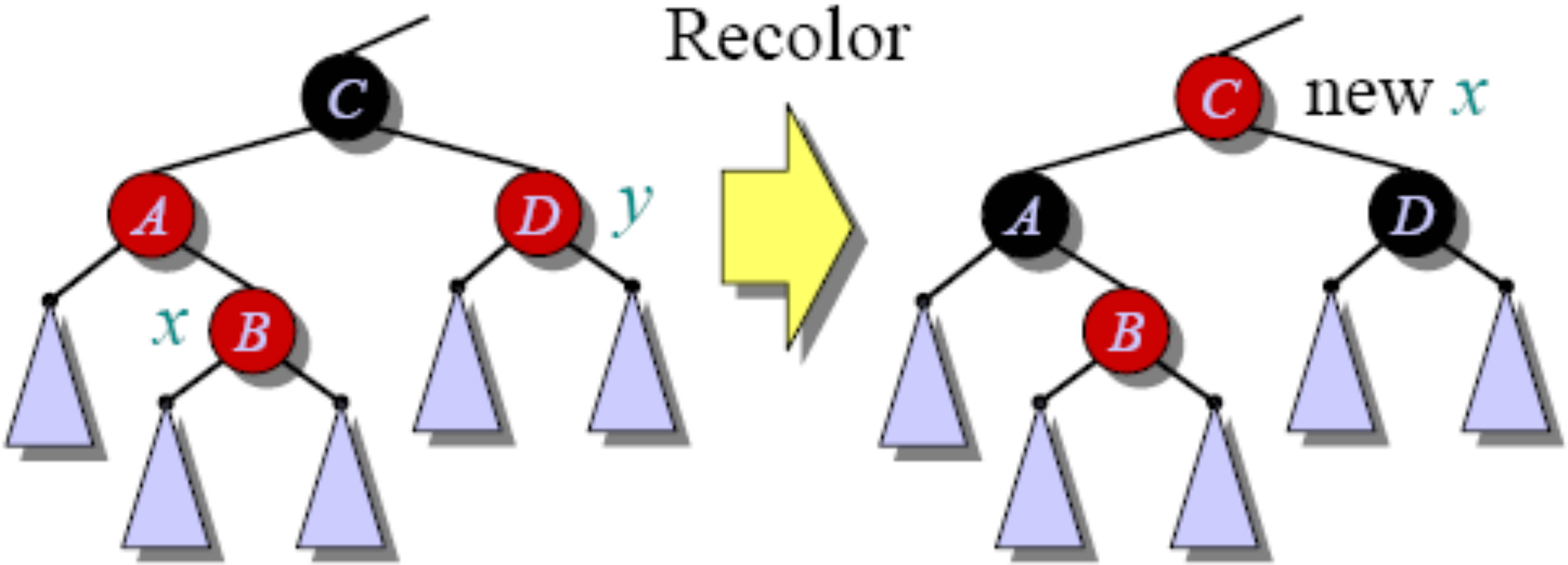
```
RB-INSERT(T, x)
    TREE-INSERT(T, x)
    color[x] ← RED ▻ only RB property 3 can be violated
    while x ≠ root[T] and color[p[x]] = RED
      do if p[x] = left[p[p[x]]
           then y ← right[p[p[x]] ▻ y = aunt/uncle of x
                   if color[y] = RED
                     then 〈Case 1〉
                     else if x = right[p[x]]
                            then 〈Case 2〉 ▻ Case 2 falls into Case 3
                          〈Case 3〉
     else 〈"then" clause with "left" and "right" swapped〉
     color[root[T]] ← BLACK
```

Graphical notation

Let  denote a subtree with a black root.

All 's have the same black-height.

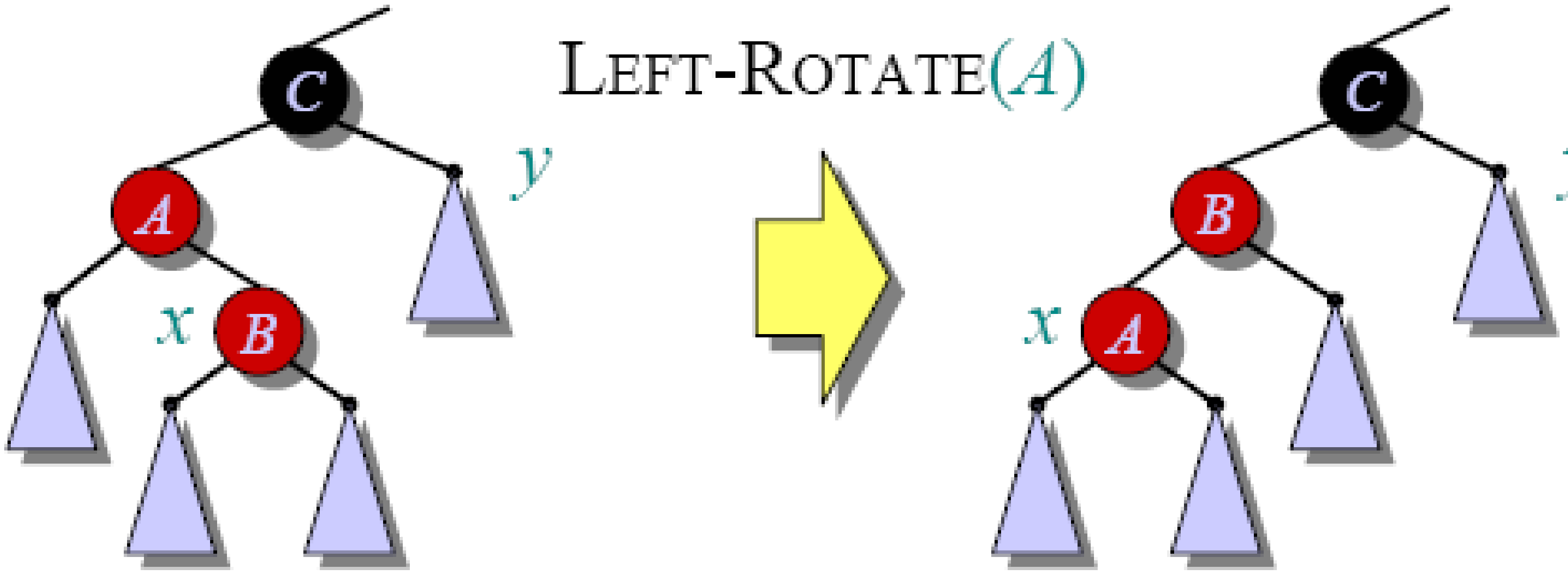
Case 1



(Or, children of *A* are swapped.)

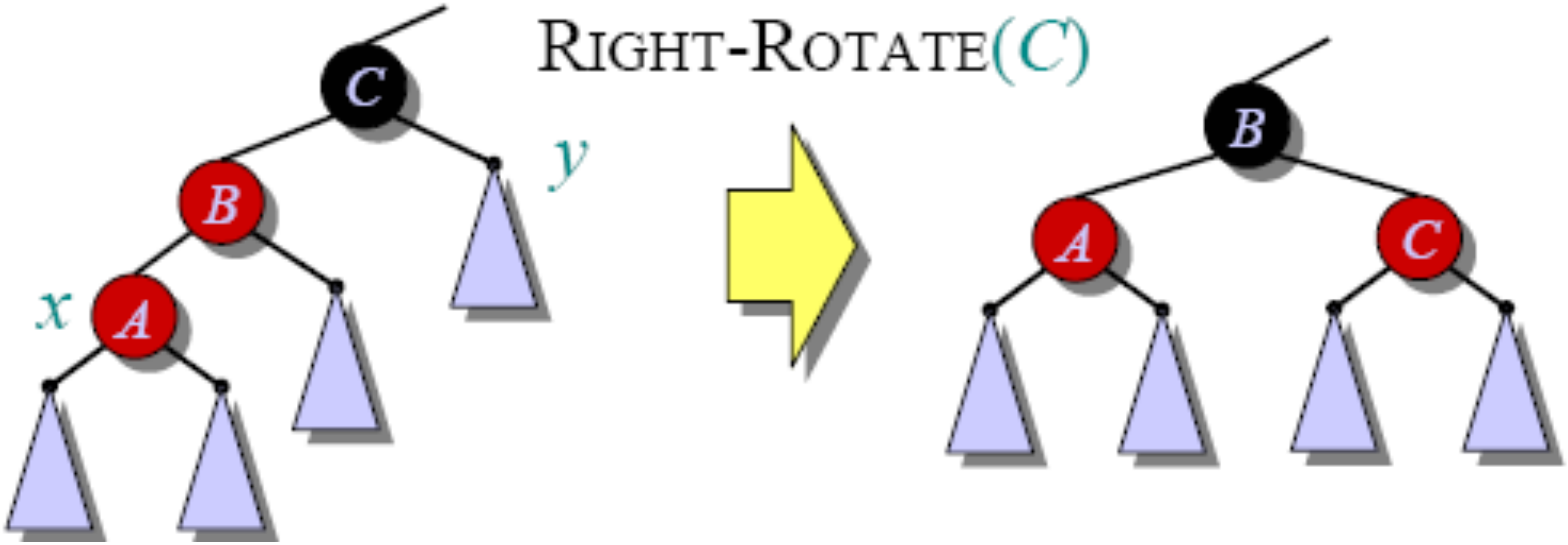
Push *C*'s black onto *A* and *D*, and recurse,
since *C*'s parent may be red.

Case 2



Transform to Case 3.

Case 3



Done! No more violations of RB property 3 are possible.

Analysis

- Go up the tree performing Case 1, which only recolors nodes.
- If Case 2 or Case 3 occurs, perform 1 or 2 rotations, and terminate.

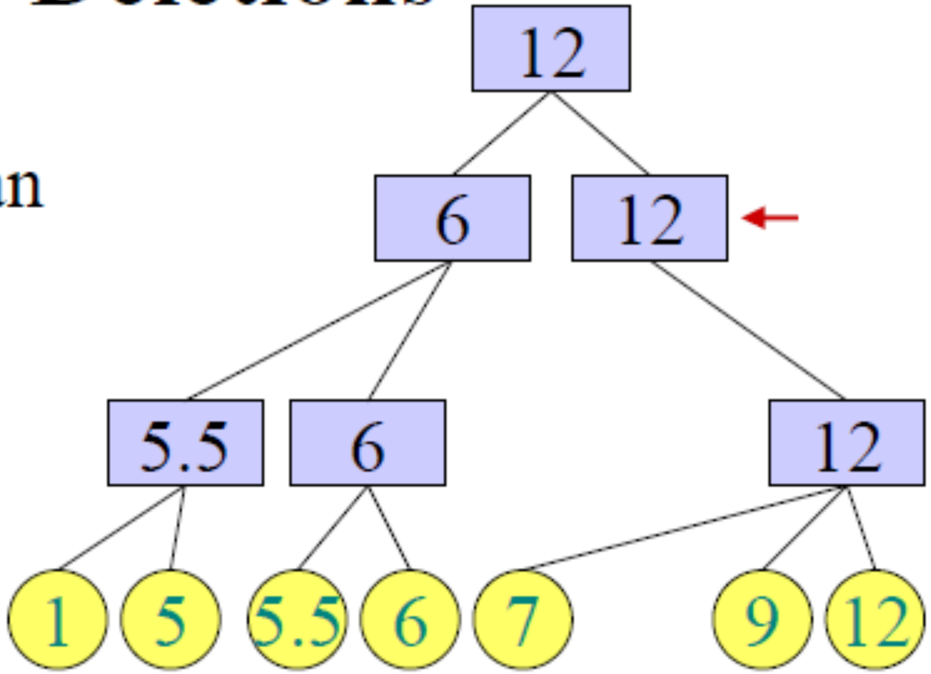
Running time: $O(\lg n)$ with $O(1)$ rotations.

RB-DELETE— same asymptotic running time and number of rotations as RB-INSERT (see textbook).

Back to 2-3 Trees

2-3 Trees: Deletions

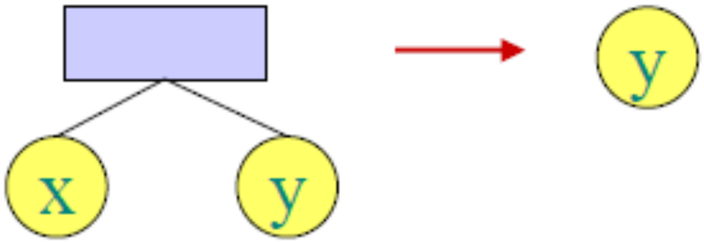
- Problem: there is an internal node that has only 1 child



Full procedure for Delete(x)

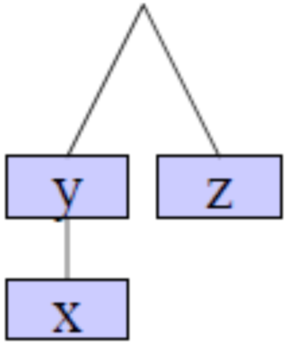
- Special case: x is the only element in the tree: delete everything

- Not-so-special case: x is one of two elements in the tree. In this case, the procedure on the next slide will delete x



- Both NIL and y are special 2-3 trees

Procedure for Delete(x)

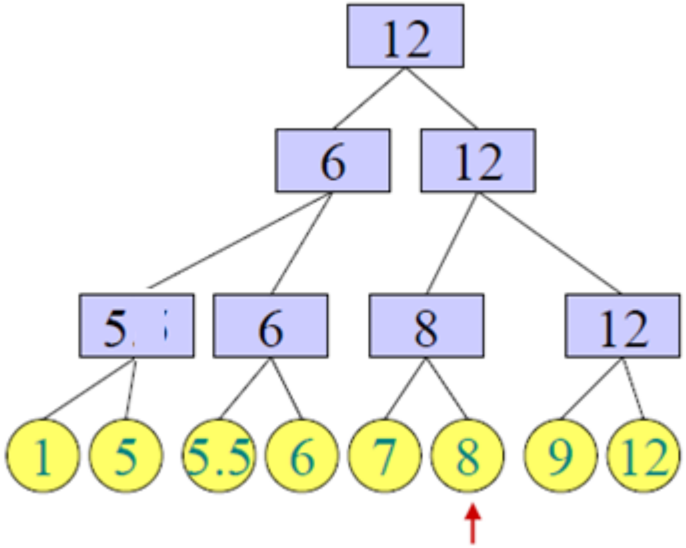


- Let y=p(x)
- Remove x
- If y≠root then
  - Let z be the sibling of y.
  - Assume z is the right sibling of y, otherwise the code is symmetric.
  - If y has only 1 child w left

    Case 1: z has 3 children
    - Attach left[z] as the rightmost child of y
    - Update y.max and z.max

    Case 2: z has 2 children:
    - Attach the child w of y as the leftmost child of z
    - Update z.max
    - Delete(y) (recursively*)
  - Else
    - Update max of y, p(y), p(p(y)) and so on until root
- Else
  - If root has only one child u
    - Remove root
    - Make u the new root

*Note that the input of Delete does not have to be a leaf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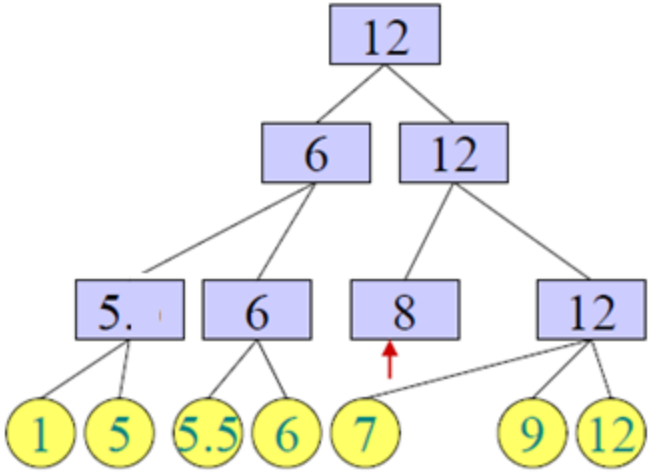
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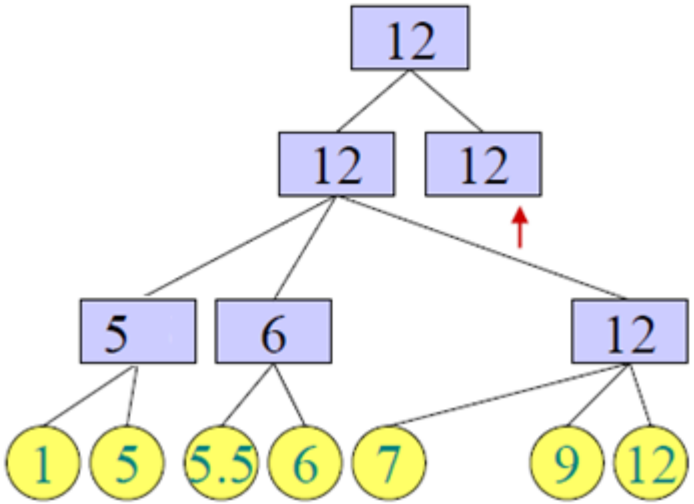
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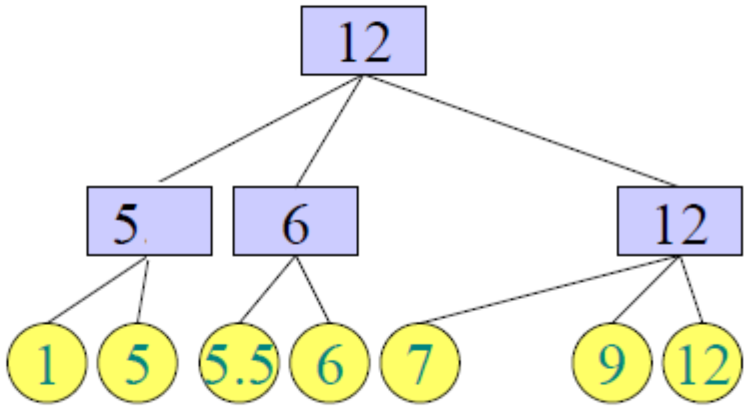
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Summary

Binary Search Tree (BST) review

Red and Black Trees

2-3 and 2-3-4 trees

Operations on Red and Black Trees