8 Linearization of Nonlinear Systems

Nonlinear Dynamical System

$$\begin{aligned} \dot{x}(t) =& f(x(t), u(t)) \qquad x(0) = x_o \\ y(t) =& g(x(t), u(t)) \end{aligned}$$

$$x(t) \in \mathcal{R}^n; \quad x_o \in \mathcal{R}^n; \quad u(t) \in \mathcal{R}^m; \quad y(t) \in \mathcal{R}^p; \quad f(x(t), u(t)) \in \mathcal{R}^n; \quad g(x(t), u(t)) \in \mathcal{R}^p$$

Equilibrium
A point (x_e, u_e) is sa id to be an *equilibrium* of the above system at $(t = 0)$ if

$$f(x(t), u(t)) = 0 \qquad \forall t \geq 0$$

Linearization

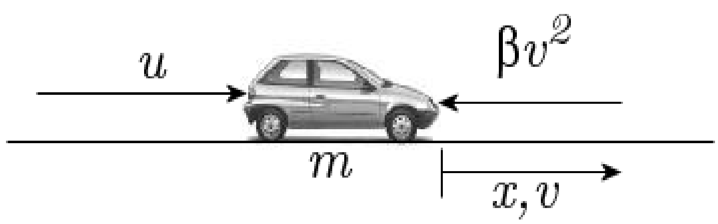
$$\begin{aligned} \delta\dot{x}(t) =& A\delta x(t) + B\delta u(t)) \qquad \delta x(0) = \delta x_o \\ \delta y(t) =& C\delta x(t) + D\delta u(t)) \end{aligned}$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(x_e, u_e)} \qquad B = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{(x_e, u_e)}$$

$$C = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \cdots & \frac{\partial g_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial x_1} & \frac{\partial g_n}{\partial x_p} & \cdots & \frac{\partial g_p}{\partial x_n} \end{bmatrix}_{(x_e, u_e)} \qquad D = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \cdots & \frac{\partial g_2}{\partial u_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial u_1} & \frac{\partial g_p}{\partial u_2} & \cdots & \frac{\partial g_p}{\partial u_m} \end{bmatrix}_{(x_e, u_e)}$$

$$\delta x(t) \triangleq x(t) - x_e; \quad \delta u(t) \triangleq u(t) - u_e; \quad \delta y(t) \triangleq y(t) - y_e; \quad \delta x_o \triangleq x_o - x_e; \quad y_e \triangleq g(x_e, u_e)$$

Nonlinear Car Dynamics
Consider a car of mass m with horizontal force u due to the engine and aerodynamic drag force βv^2 $(\beta > 0)$.



The nonlinear differential equation which relates the velocity v to the input force u:

$$\dot{v} = -\frac{\beta}{m}v^2 + \frac{1}{m}u$$

Selected equilibrium velocity v_e results in

$$0 = -\frac{\beta}{m}v^2 + \frac{1}{m}u \Rightarrow u_e = \beta v_e^2$$

Small signal:

$$v \triangleq v_e + \delta v \qquad and \qquad u \triangleq u_e + \delta u \qquad (Note: \delta\dot{v} = \dot{v} \quad and \quad \delta\dot{u} = \dot{u})$$

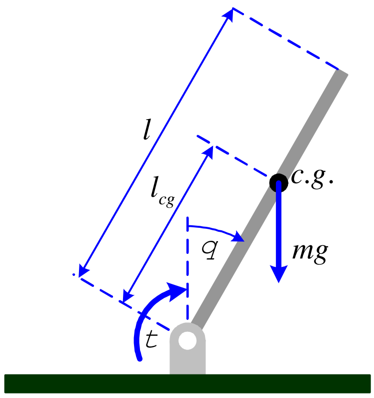
Linearize nonlinear term (Taylor series approximation)

$$v^2 = v_e^2 + 2v_e\delta v$$

$$\begin{aligned}\delta\dot{v} =& \dot{v} = -\frac{\beta}{m}v_e^2 - \frac{2\beta v_e}{m}\delta v + \frac{1}{m}u_e + \frac{1}{m}\delta u \\ =& -\frac{2\beta v_e}{m}\delta v + \frac{1}{m}\delta u + \underbrace{\frac{1}{m}\left(-\beta v_e^2 + u_e\right)}_{=0}\end{aligned}$$

Nonlinear Fixed-Base Inverted Pendulum Dynamics

Consider the fixed-base inverted pendulum system shown in Figure below with parameters given in Table below.



	Symbol	Definition	Unit
Variables	θ	angle that the pendulum makes with the vertical	rad
	τ	torque applied at the hinge (control)	Nm
Parameters	m	mass of the pendulum	kg
	l_{cg}	location of the c.g. of the pendulum above the base	m
	I_{cg}	moment of inertia of the pendulum about the c.g.	$\mathrm{Nm}/rad^2/s^2$
	g	acceleration due to gravity	$\mathrm{m/s^2}$
	c	friction coefficient for rotation	$\mathrm{kg\ m^2/s}$

The dynamical equation for the fixed-base inverted pendulum system is given by

$$(ml_{cg}^2 + I_{cg})\ddot{\theta} + c\dot{\theta} - mgl_{cg}\sin\theta = \tau$$

Define

$$x_1 = \theta \quad and \quad x_2 = \dot{\theta}$$

Rewrite the dynamics

$$\begin{aligned}\dot{x}_1 =& f_1(x,\tau) = x_2 \\ \dot{x}_2 =& f_2(x,\tau) = \frac{1}{ml_{cg}^2 + I_{cg}}\left(mgl_{cg}\sin x_1 - cx_2 + \tau\right)\end{aligned}$$

Equilibrium

$$x_e = [0 \;\; 0]^T \quad and \quad \tau_e = 0$$