1.4 Cart Inverted Pendulum System

Cart Inverted Pendulum Model (No Motor)

Consider the cart inverted pendulum system shown in Figure 4 with parameters given in Table 4.

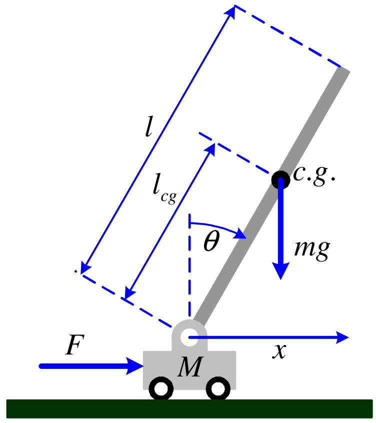

Figure 4: Cart Inverted Pendulum Schematic

	Symbol	Definition	Unit
Variables	x	position of the cart	m
	θ	angle that the pendulum makes with the vertical	rad
	F	force applied on cart (control)	N
Parameters	M	mass of the cart	kg
	m	mass of the pendulum	kg
	l_{cg}	location of the c.g. of the pendulum above the base	m
	I_{cg}	moment of inertia of the pendulum about the c.g.	Nm$/rad^2/s^2$
	g	acceleration due to gravity	m/s^2
	b	friction coefficient between cart and the ground	kg/s
	c	friction coefficient for rotation	kg m^2/s

Table 4: Variables and Parameters for the Cart Pendulum System

The dynamical equations for the above cart inverted pendulum system is given by

$$(M+m)\ddot{x} + ml_{cg}\cos\theta\ddot{\theta} - ml_{cg}\sin\theta\dot{\theta}^2 + b\dot{x} = F, \tag{27}$$

$$(ml_{cg}^2 + I_{cg})\ddot{\theta} + ml_{cg}\cos\theta\ddot{x} - mgl_{cg}\sin\theta + c\dot{\theta} = 0. \tag{28}$$

The above can readily be obtained using a Lagrangian approach.

Defining the state vector as $x_p = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T \stackrel{\text{def}}{=} [x \;\; \theta \;\; \dot{x} \;\; \dot{\theta}]^T$ Equations 27 and 28 can be written as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & M+m & ml_{cg}\cos x_2 \\ 0 & 0 & ml_{cg}\cos x_2 & ml_{cg}^2 + I_{cg} \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ ml_{cg}\sin x_2 x_4^2 - bx_3 \\ mgl_{cg}\sin x_2 - cx_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ F \\ 0 \end{bmatrix}. \tag{29}$$

Linearizing Equation 29 about the equilibrium at $x_{p_e} = [0 \;\; 0 \;\; 0 \;\; 0]^T$ and $F_e = 0$ yields the following descriptor state-space representation for the cart inverted pendulum system:

$$E_p\dot{x}_p = \hat{A}_p x_p + \hat{B}_p F \tag{30}$$

where

$$E_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & M+m & ml_{cg} \\ 0 & 0 & ml_{cg} & ml_{cg}^2 + I_{cg} \end{bmatrix}, \quad \hat{A}_p = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -b & 0 \\ 0 & mgl_{cg} & 0 & -c \end{bmatrix}, \quad \textit{and} \;\; \hat{B}_p = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}. \tag{31}$$

Equation 30 can be written in state-space form as

$$\dot{x}_p = A_p x_p + B_p F = (E_p^{-1}\hat{A}_p)x_p + (E_p^{-1}\hat{B}_p)F. \tag{32}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m^2 l_{cg}^2 g}{\rho} & -\frac{(ml_{cg}^2+I_{cg})b}{\rho} & \frac{ml_{cg}c}{\rho} \\ 0 & \frac{(M+m)mgl_{cg}}{\rho} & \frac{ml_{cg}b}{\rho} & -\frac{(M+m)c}{\rho} \end{bmatrix} x_p + \begin{bmatrix} 0 \\ 0 \\ \frac{(ml_{cg}^2+I_{cg})}{\rho} \\ -\frac{ml_{cg}}{\rho} \end{bmatrix} F. \tag{33}$$

where $\rho = Mml_{cg}^2 + I_{cg}(M+m)$.