Stochastic Process, Ito's Lemma and GBM

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Stochastic Process

A stochastic process is a mathematical framework used to model systems or phenomena that change over time in such a way that involves random elements. Essentially, it is a sequence of random variables, each associated with a specific point in time or space, that represents the development of a system influenced by randomness.

Key Characteristics:

1. **Time or Space Index:**
   - The process is indexed by a variable such as time (t) or space (x).
   - The index can be discrete ($t = 1, 2, 3, ..$) or continuous ($t \geq 0$).

2. **Random Variables:**
   - At each time or space index, there is a random variable that represents the state of the system at that point.
   - These random variables can take values in various domains, such as real numbers (e.g., stock prices) or integers (e.g., the number of arrivals in a queue).

3. **Dependence Structure:**
   - The random variables in a stochastic process may be dependent on each other in various ways, often with some form of temporal or spatial correlation.
   - For example, today's stock price might depend on yesterday's price.

Types of Stochastic Process:

1. **Discrete-Time vs. Continuous-Time:**
   - **Discrete-Time Process:** The index (usually time) takes on discrete values (e.g., daily stock prices).
   - **Continuous-Time Process:** The index is continuous (e.g., stock prices observed continuously over time).

2. **Discrete-State vs. Continuous-State:**
   - **Discrete-State Process:** The random variables can take on a discrete set of values (e.g., number of customers in a queue).
   - **Continuous-State Process:** The random variables can take on continuous values (e.g., the position of a particle in space).

3. **Markov Process:**
   - A special type of stochastic process where the future state depends only on the current state and not on the history of past states (the Markov property).

4. **Martingale:**
   - A process where the expected future value of the process, given all past information, is equal to the current value. This is often used in financial modeling.

5. **Wiener Process (Brownian Motion):**
   - A continuous-time, continuous-state stochastic process that serves as the mathematical model for Brownian motion, used extensively in finance for modeling stock prices.

Examples of Stochastic Process:

1. **Random Walk:**
   - A simple stochastic process where the next state depends on the current state plus some random movement. This can model, for example, the path of a drunkard or stock price movements in a simplified manner.

2. **Poisson Process:**
   - A stochastic process that models the occurrence of random events over time, where the events happen independently and with a constant average rate (e.g., arrivals of customers at a service center).

Importance of Stochastic Process:

1. **Finance:** Modeling stock prices, interest rates, and other financial instruments.
2. **Physics:** Describing systems influenced by random forces, like the movement of particles in fluids.
3. **Engineering:** Modeling noise in communication systems, reliability of systems, and more.
4. **Biology:** Modeling population dynamics, spread of diseases, and more.
5. **Economics:** Understanding random fluctuations in markets and economic indicators.

Stochastic Differential Equation:

A stochastic differential equation (SDE) is a type of differential equation that incorporates randomness, typically through a stochastic process such as Brownian motion. SDEs are used to model systems that are influenced by random effects or noise, making them essential tools in fields like finance, physics, biology, and engineering.

An SDE generally takes the form:

$$dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dW(t)$$

where

- $S(t)$ is the stochastic process we are modelling.
- $\mu(X(t), t)$ is the drift term. It represents the deterministic part of the model, specifically the expected rate of change of $X(t)$ over time.
- $\sigma(X(t), t)$ is the diffusion term, representing the random fluctuations or volatility of the process.
- $dW(t)$ is the differential of a Wiener process (or Brownian motion), which represents the source of randomness.

Brownian Motion:

We encountered Brownian motion in the last section. So let us now discuss it in more details.

Brownian motion, also known as the Wiener process, is a continuous-time stochastic process characterized by the following key properties:

- **Randomness:** The process evolves unpredictably, with each step being independent of the previous one. This lack of memory in the process is referred to as the Markov property.
- **Continuity:** The paths of Brownian motion are continuous, with no abrupt jumps. However, despite this smoothness, the paths are highly irregular and almost nowhere differentiable.
- **Normally Distributed Increments:** For any time interval Δt, the change in the process over that interval is normally distributed. If $W(t)$ represents the process at time t, then the difference $W(t + dt) - W(t)$ is normally distributed with a mean of 0 and a variance of Δt. This is expressed mathematically as

$$W(t + \Delta t) - W(t) \sim \mathcal{N}(0, \Delta t)$$

- **Independence of Increments:** The increments of Brownian motion over non-overlapping time intervals are independent.
- **Initial Condition:** The process generally starts at zero, meaning $W(0) = 0$.

Variance of Brownian Motion

For a Wiener process, the increment $\Delta W(t)$ over a small time interval Δt is normally distributed with variance Δt. Mathematically:

$$\mathrm{Var}(\Delta W(t)) = \mathbb{E}[(\Delta W(t))^2] = \Delta t$$

This tells us that the expected value of the square of $\Delta W(t)$ is proportional to Δt.

Order of Magnitude:
$\Delta W(t)$ is of the order $\sqrt{\Delta t}$, so when squared, it gives a term of order Δt.

$$(\Delta W(t))^2 \sim (\sqrt{\Delta t})^2 = \Delta t$$

As we will see soon, this property is very important in the context of stochastic calculus!

Itô's Lemma:

Statement:

Let $X(t)$ be a stochastic process that satisfies the following stochastic differential equation (SDE):

$$dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dW(t)$$

where $W(t)$ is a Wiener process (or Brownian motion), $\mu(X(t), t)$ is the drift, and $\sigma(X(t), t)$ is the diffusion term.
Let $f(X(t), t)$ be a twice-differentiable function of $X(t)$ and t. Itô's Lemma states that the differential of $f(X(t), t)$, denoted $df(X(t), t)$, is given by:

$$df(X(t), t) = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial X}\, dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\, (dX(t))^2$$

This formula includes a second-order term $(dX(t))^2$, which arises from the quadratic variation of the Brownian motion.

Derivation:

Let's start by expanding the function $f(X(t), t)$ using a Taylor expansion, similar to how we would in standard calculus. However, we must account for the stochastic nature of $X(t)$.

1. Taylor Expansion of $f(X(t), t)$:

For a function $f(X(t), t)$ that depends on both the stochastic process $X(t)$ and time t, the Taylor expansion up to second order in dt and $dX(t)$ is

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial X}\, dX(t) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\, (dX(t))^2 + \text{higher-order terms}$$

2. Substitute $dX(t)$:

Using the SDE for $X(t)$, we know that:

$$dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dW(t)$$

Substitute this into the Taylor expansion of f:

$$df = \frac{\partial f}{\partial t}\, dt + \frac{\partial f}{\partial X}\, (\mu(X(t), t)\, dt + \sigma(X(t), t)\, dW(t)) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\, (dX(t))^2$$

3. Compute $(dX(t))^2$:

Since $dX(t)$ contains both dt and $dW(t)$, we need to calculate $(dX(t))^2$. Using the fact that $(dW(t))^2 = dt$ (a property of Brownian motion) and neglecting higher-order terms like dt^2, we have:

$$(dX(t))^2 = (\mu(X(t), t)\,dt + \sigma(X(t), t)\,dW(t))^2$$
$$= \mu^2(X(t), t)\,dt^2 + 2\mu(X(t), t)\sigma(X(t), t)\,dt\,dW(t) + \sigma^2(X(t), t)\,(dW(t))^2$$

Now, use the following approximations:

- dt^2 is negligible (order dt^2).
- $dt\,dW(t)$ is also negligible (order $dt^{3/2}$).
- $(dW(t))^2 = dt$ (property of Brownian motion).

Thus, the only surviving term is:

$$(dX(t))^2 = \sigma^2(X(t), t)\,dt$$

4. Substitute $(dX(t))^2$ Back into the Expansion

Now that we have $(dX(t))^2 = \sigma^2(X(t), t)\,dt$, substitute this back into the Taylor expansion.

5. Final Simplified Formula

Collecting terms, we get:

$$df = \frac{\partial f}{\partial t}\,dt + \frac{\partial f}{\partial X}\mu(X(t),t)\,dt + \frac{\partial f}{\partial X}\sigma(X(t),t)\,dW(t) + \frac{1}{2}\frac{\partial^2 f}{\partial X^2}\sigma^2(X(t),t)\,dt$$

This is Itô's Lemma.

Interpretation of Itô's Lemma:

- The term $\frac{\partial f}{\partial t}\,dt$ comes from the explicit dependence of f on time.
- The term $\frac{\partial f}{\partial X}\mu(X(t),t)\,dt$ represents the deterministic part of the change in $X(t)$ (the drift).
- The term $\frac{\partial f}{\partial X}\sigma(X(t),t)\,dW(t)$ accounts for the stochastic or random part of the change in $X(t)$ (the diffusion).
- The final term $\frac{1}{2}\frac{\partial^2 f}{\partial X^2}\sigma^2(X(t),t)\,dt$ represents the impact of the variance of the stochastic process.

Geometric Brownian Motion

We discussed the generic form of SDE:

$$dX(t) = \mu(X(t), t)\,dt + \sigma(X(t), t)\,dW(t)$$

A common example of an SDE is the Geometric Brownian Motion:[1]

$$dS(t) = \widetilde{\mu} S(t)\,dt + \widetilde{\sigma} S(t)\,dW(t)$$

In GBM, the drift and diffusion terms are both proportional to the current value of the process $S(t)$.

The drift part is $\mu(X(t), t) = \widetilde{\mu} S(t)$, where $\widetilde{\mu}$ is a constant representing the expected rate of return (or growth rate).

The diffusion part is $\sigma(X(t), t) = \widetilde{\sigma} S(t)$, where $\widetilde{\sigma}$ is a constant representing the volatility or the standard deviation of the returns.

This proportionality in both the drift and diffusion terms is what makes GBM suitable for modeling stock prices, as it reflects the idea that larger values of $S(t)$ will naturally experience larger changes (both in terms of expected return and volatility).

[1] Here I have expressed the constants $\widetilde{\mu}$ and $\widetilde{\sigma}$ with 'tilde', to distinguish them from the generic $\mu(X(t), t)$ and $\sigma(X(t), t)$.

Applying Itô's Lemma to GBM

The logarithm of stock price is important in finance for several reasons. Let us find it using Itô's Lemma.

So we define the function $f(S(t), t) = \ln(S(t))$.

We now compute the partial derivatives, $\frac{\partial f}{\partial S} = \frac{1}{S(t)}$ and $\frac{\partial^2 f}{\partial S^2} = -\frac{1}{S(t)^2}$.

We now apply these results to Itô's Lemma formula (discussed before), noting that in this case $\frac{\partial f}{\partial t} = 0$ as $f(S(t), t) = \ln(S(t))$ does not explicitly depend on time, and substituting the expressions for μ and σ.

We finally get the expression

$$d \ln(S(t)) = \left(\widetilde{\mu} - \frac{1}{2}\widetilde{\sigma}^2\right) dt + \widetilde{\sigma}\, dW(t)$$

This result tells us that the log of the stock price $\ln(S(t))$ follows a stochastic process with:

- A drift term $\widetilde{\mu} - \frac{1}{2}\widetilde{\sigma}^2$, which adjusts the mean rate of return by subtracting half the variance. This adjustment is due to the continuous compounding effect in the log-normal model.
- A volatility term $\widetilde{\sigma}$ multiplied by the Brownian motion $dW(t)$, which introduces randomness into the process.

Exponential Form and Stock Price Dynamics:

To find the process for $S(t)$ directly, we can integrate the SDE we derived for $\ln(S(t))$:

$$\ln(S(t)) = \ln(S(0)) + \left(\widetilde{\mu} - \frac{1}{2}\widetilde{\sigma}^2\right) t + \widetilde{\sigma} W(t)$$

Exponentiating both sides gives:

$$S(t) = S(0) \exp\left[\left(\widetilde{\mu} - \frac{1}{2}\widetilde{\sigma}^2\right) t + \widetilde{\sigma} W(t)\right]$$

This exponential form shows that $S(t)$ is log-normally distributed, with a mean adjusted by $\frac{1}{2}\widetilde{\sigma}^2$ and a random component driven by the Brownian motion. [2]

Application in Finance

In finance, this result is essential for option pricing, particularly in the Black-Scholes model. Itô's Lemma allows us to move from a model for stock price dynamics to a model for the price of derivatives (like options) by transforming variables.

[2] See my previous lecture notes to understand how Monte Carlo simulations use GBM.