Monte Carlo Simulation in Finance

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What is it?

Monte Carlo simulation is a statistical technique used to model the probability of different outcomes in a process that cannot easily be predicted due to the presence of random variables. In the context of stock prices, it involves generating a large number of possible future price paths based on certain assumptions to estimate the likelihood of various outcomes.

Why is it Necessary?

- **Risk Assessment:** Helps in understanding the range of possible future stock prices and associated risks.
- **Option Pricing:** Used in financial models to estimate the value of options and other derivatives.
- **Portfolio Management:** Assists in evaluating the potential performance of investment portfolios under different scenarios.
- **Scenario Analysis:** Provides insights into how different market conditions might impact stock prices.

Key Assumptions

1. **Geometric Brownian Motion (GBM):** Assumes stock prices follow a geometric Brownian motion, where the logarithm of the stock prices is normally distributed.

$$S(t) = S(0) \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right)$$

   - $S(t) =$ Stock price at time t
   - $S(0) =$ Initial stock price
   - $\mu =$ Expected return of the stock
   - $\sigma =$ Volatility of the stock returns
   - $W(t)$ represents the Wiener process or standard Brownian motion, which introduces randomness into the model.

2. **Normal Distribution:** Assumes that the stock returns are normally distributed.
3. **No Arbitrage:** Assumes that markets are efficient and there are no opportunities for arbitrage.
4. **Constant Volatility:** Assumes constant volatility over the time period considered.

Steps in Monte Carlo Simulation

1. **Define the Model:** Set up the parameters for the stock price model (e.g., initial price, expected return, volatility).
2. **Generate Random Paths:** Simulate multiple random price paths using the GBM model.
3. **Calculate Outcomes:** For each simulated path, calculate the relevant metrics (e.g., final price, option payoff).
4. **Analyze Results:** Aggregate the outcomes to estimate probabilities, value distributions, or other metrics of interest.

Benefits

- **Flexibility:** Can model complex financial instruments and scenarios.
- **Comprehensive Analysis:** Provides a range of possible outcomes rather than a single estimate.
- **Risk Management:** Enhances understanding of potential risks and uncertainties.

Limitations

- **Computationally Intensive:** Requires significant computational power for large numbers of simulations.
- **Assumptions:** Reliant on assumptions that may not hold in real markets (e.g., constant volatility).

Deriving the Discrete Time GBM

The Geometric Brownian Motion (GBM) formula for modeling stock prices is derived from the assumption that the stock prices follow a stochastic process with a constant drift and volatility.

As we stated earlier, the integral form of the GBM model from time 0 to time t is

$$S(t) = S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right)$$

For a discrete time step Δt, the price $S(t)$ at time $t = n\Delta t$ can be written as

$$S_{t+\Delta t} = S_t \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}Z\right]$$

where Z is a standard normal random variable $Z \sim \mathcal{N}(0, 1)$. [1]
Here we are analyzing daily stock data, so $Deltat = 1$ day, and the formula simplifies to

$$S_{t+1} = S_t \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) + \sigma Z\right]$$

The relevant part in the code is

$prices.append(prices[-1] * np.exp((mu - 0.5 * sigma * *2) + sigma * np.random.normal()))$

The $np.random.normal()$ function generates random variables with a normal distribution, which aligns with the assumption that returns are normally distributed.

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[1] The square root $\sqrt{\Delta t}$ appears because the increment of Brownian motion over a small time interval Δt is normally distributed with a variance of Δt. To represent this increment in discrete time, we multiply a standard normal variable Z by $\sqrt{\Delta t}$.

Constant Volatility assumption: This model assumes constant volatility as it uses a single value of σ for all simulations. (In real-world scenarios, volatility may change over time.)

No Arbitrage is implicitly assumed by using the GBM model.

Can Monte Carlo be done with other distributions?

Monte Carlo simulations can employ a variety of distributions depending on the nature of the problem being modeled. While Gaussian distributions are common, especially in finance, alternative distributions are used when dealing with non-normal data, modeling rare events, or capturing behaviors that Gaussian distributions cannot adequately represent. The choice of distribution should align with the characteristics of the data or phenomena being simulated.

Some other commonly used distributions are Heavy-Tailed Distributions, Jump Diffusion Models, Lognormal Distribution, Beta Distribution, Poisson Distribution, etc.