The Black-Scholes Model

Turmoli Neogi

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The Black-Scholes[1] differential equation is a fundamental partial differential equation (PDE) in financial mathematics used to model the price dynamics of financial derivatives, particularly European-style options.

The Black-Scholes Differential Equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- $V(S, t)$ denotes the option's price, which varies with the underlying asset price S and the time t.
- S is the current price of the underlying asset.
- r stands for the risk-free interest rate, representing the return on a theoretically risk-free investment.

This is a deterministic equation, even though it is derived from the stochastic differential equation (SDE) that models the underlying asset's price!

[1] developed by economists Fischer Black, Myron Scholes, and later expanded by Robert Merton in the early 1970s

Derivation

The derivation of the Black-Scholes PDE involves several steps, primarily using concepts from stochastic calculus, particularly Ito's Lemma[2], and the notion of constructing a risk-free portfolio.

Assumptions:

1. **Stock Price Behavior:** The stock price follows a GBM.
2. **No Dividends:** It is assumed that the stock does not distribute dividends during the option's life.
3. **Constant Volatility:** The volatility σ remains unchanged throughout the option's duration.
4. **Constant Risk-Free Rate:** The risk-free interest rate r is assumed to be constant over the life of the option.
5. **European Option:** The model applies to European-style options, meaning that the option can only be exercised at its expiration date.
6. **No Arbitrage Opportunities:** There are no possibilities for arbitrage, that is there are no opportunities to make a risk-free profit.
7. **Efficient Markets:** It is assumed that the markets are efficient and frictionless, implying no transaction costs, taxes, or bid-ask spreads.

[2] See my previous lecture notes on Stochastic process and Ito's Lemma

Stochastic Process for the Underlying Asset:

The price S_t of the underlying asset is modelled by a SDE:

$$dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$$

where μ is the drift rate and σ is the volatility of the asset.

This SDE includes a stochastic component dW_t, which is the increment of a Wiener process (Brownian motion), introducing randomness into the asset price evolution.

Derive the SDE for the Option Price Using Ito's Lemma:

Consider a derivative (e.g. an option) whose price $V(S,t)$ depends on the underlying asset price S_t and time t. According to Ito's Lemma, the differential of $V(S,t)$ is given by:

$$dV(S_t,t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS_t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}(dS_t)^2$$

Substituting the SDE for S_t into this expression[3]

$$dV(S_t,t) = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S}\left(\mu S_t\, dt + \sigma S_t\, dW_t\right) + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S_t^2\, dt$$

[3] Refer to my notes on Stochastic Process for calculation of the $(dS_t)^2$ term.

Simplifying, we get

$$dV(S_t, t) = \left(\frac{\partial V}{\partial t} + \mu S_t \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S_t \frac{\partial V}{\partial S} dW_t$$

Construct a Risk-Free Portfolio:

To eliminate the risk (i.e., the stochastic term involving dW_t), consider a portfolio Π consisting of:

- 1 short position in the option $V(S, t)$.
- Δ long positions in the underlying asset S_t.

The value of this portfolio is:

$$\Pi = -V(S_t, t) + \Delta S_t$$

The differential change in the portfolio value is:

$$d\Pi = -dV(S_t, t) + \Delta dS_t$$

Substituting the expressions for $dV(S_t, t)$ and dS_t, we get

$$d\Pi = -\left(\frac{\partial V}{\partial t}dt + \mu S_t \frac{\partial V}{\partial S}dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}dt + \sigma S_t \frac{\partial V}{\partial S}dW_t\right)$$
$$+ \Delta\left(\mu S_t dt + \sigma S_t dW_t\right)$$

Simplifying:

$$d\Pi = \left(-\frac{\partial V}{\partial t} - \mu S_t \frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2} + \Delta \mu S_t\right) dt$$
$$+ \left(-\sigma S_t \frac{\partial V}{\partial S} + \Delta \sigma S_t\right) dW_t$$

Since we want the portfolio to be risk-free, the coefficient of dW_t must be zero:

$$-\sigma S_t \frac{\partial V}{\partial S} + \Delta \sigma S_t = 0$$

This implies that:

$$\Delta = \frac{\partial V}{\partial S}$$

Substituting Δ back into the expression for $d\Pi$:

$$d\Pi = \left(-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 V}{\partial S^2}\right) dt$$

No-Arbitrage Condition:

For the portfolio to be risk-free, it must earn the risk-free rate r. Therefore, the change in the portfolio value must be equal to $r\,\Pi\,dt$.

$$d\Pi = r\,\Pi\,dt = r\left(-V(S_t,t) + \frac{\partial V}{\partial S} S_t\right) dt$$

Substituting for $d\Pi$:

$$-\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2} = r\left(-V(S_t,t) + \frac{\partial V}{\partial S} S_t\right)$$

The Black-Scholes PDE:

Rearranging the terms gives us the Black-Scholes partial differential equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2\, S_t^2\, \frac{\partial^2 V}{\partial S^2} + r\, S_t \frac{\partial V}{\partial S} - r\, V(S_t,t) = 0$$

This is the Black-Scholes PDE.[4]

[4] Note that this equation contains the risk-free rate r instead of the mean μ in the $\frac{\partial V}{\partial S}$ term.

Why Does the Black-Scholes Model Use r Instead of μ ?

The risk-free rate r is not equal to the mean return μ of the underlying asset in general.

- μ: This is the expected or average return of the underlying asset in the real world. It reflects the growth rate of the asset price considering all the risks and rewards in the actual market.
- r: The risk-free rate represents the return on an investment with no risk, such as government bonds. It is the return you would expect if you invested in a perfectly safe asset.

The Black-Scholes model is based on the concept of risk-neutral valuation. In a risk-neutral world, all investors are indifferent to risk, meaning they only care about the expected return without considering the risk associated with it. In such a world, the expected return of any asset, when adjusted for risk, is the risk-free rate r, not the actual expected return μ. This ensures consistent pricing of options and eliminates arbitrage opportunities in the market.

Black-Scholes Equation is Deterministic!

1. **Origin of Black-Scholes Equation:** The Black-Scholes equation starts from a stochastic differential equation (SDE) that models the price dynamics of the underlying asset.
2. **Construction of Hedged Portfolio:** To derive the Black-Scholes equation, a risk-free hedged portfolio is created by combining the option with the underlying asset in a specific ratio.
3. **Elimination of Stochastic Component:** In this hedged portfolio, the stochastic term involving the Wiener process dW_t is eliminated.
4. **Resulting Deterministic PDE:** Removing the randomness through hedging results in a deterministic partial differential equation (PDE) that describes the option's price evolution.
5. **Deterministic Nature of Black-Scholes:** Therefore, while the underlying asset's price follows an SDE, the Black-Scholes equation itself is deterministic, reflecting a risk-neutral valuation approach.

Limitations of the Black-Scholes Model

- **Constant Volatility Assumption:** The model assumes that volatility (σ) is constant over time, which is unrealistic as market volatility often changes.
- **No Dividends:** It does not account for dividend payments on the underlying asset, which can affect option prices.
- **European Options Only:** The model is specifically designed for European options, which can only be exercised at expiration, unlike American options.
- **Constant Risk-Free Rate:** Assumes the risk-free rate (r) is constant, but interest rates can fluctuate over time.
- **No Transaction Costs or Taxes:** Assumes frictionless markets, ignoring transaction costs, taxes, and other market frictions.

There are several generalized models to take into account these limitations.