# Derangements, The Exponential Function, and The Principle of Inclusion Exclusion

Andrew Luo

University of Waterloo

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# Introduction

**Situation.** You are exchanging gifts for Secret Santa in a group of $n$ people. You each bring a gift, then you shuffle them and hand them out randomly.

Let's assume a shuffle is a random permutation.

Recall that a random shuffle is just a permutation.

**Example.** Suppose $n = 10$. We permutate the numbers 1 to 10 as follows.

| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| 9 | 3 | 1 | 2 | 10 | 8 | 7 | 9 | 4 | 5 |

Person 1 receives person 9's gift, person 2 receives person 3's gift, and so on.

Is there a problem?

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|---|---|---|---|---|---|---|---|---|---|
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Person 1 receives person 9's gift, person 2 receives person 3's gift, and so on.

Person 7 gets their own gift - this can't be right!

What we really want there is a *derangement*, not a permutation!

**Definition**

Let $P$ be a permutation of the numbers from 1 to $n$. We say that $k$ is a *fixed point* of $P$ if $k$ appears in the $k$-th position of the permutation.

**Example**

In the permutation $12345 \to 54321$, 3 is a fixed point.

**Definition**

A *derangement* is a permutation without any fixed points.

**Problem.** If someone gets their own gift, they can exchange it. But that gives away their anonymity (their gift is no longer secret).

**Janky Solution.** Anonymously vote if someone got their own gift. If so, redistribute them.

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| 9 | 3 | 1 | 2 | 10 | 8 | 7 | 9 | 4 | 5 |
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| 6 | 5 | 9 | 1 | 10 | 7 | 8 | 3 | 2 | 4 |

How many times do we have to do this on average?

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# An Interesting Limit

Consider the following expression

$$f(n) = \left(1 + \frac{1}{n}\right)^n$$

What happens as $n$ gets very large?

When $n = 1$, $f(n) = 2$.

When $n = 2$, $f(n) = 2.25$.

When $n = 3$, $f(n) \approx 2.37$.

When $n = 10$, $f(n) \approx 2.59$.

When $n = 100$, $f(n) \approx 2.70$.

When $n = 1000$, $f(n) \approx 2.717$.

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When $n = 1000$, $f(n) \approx 2.717$.

Consider the following expression

$$f(n) = \left(1 + \frac{1}{n}\right)^n$$

As $n \to \infty$, $f(n) \to 2.71828182846...$

We call this number $e$, for Euler's constant.

Euler's constant takes on many interesting forms. One of them is its Taylor Series representation:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$$

This is a very useful representation.

Note that $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} \approx 2.708$. It took us until $n = 100$ for the $\left(1 + \frac{1}{n}\right)^n$ expression to reach an approximation nearly as good.

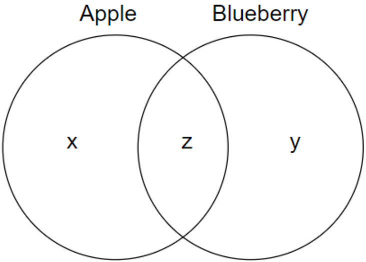
In fact, a major result from calculus gives us that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

Using this formula, we can approximate any power of $e$. For example,

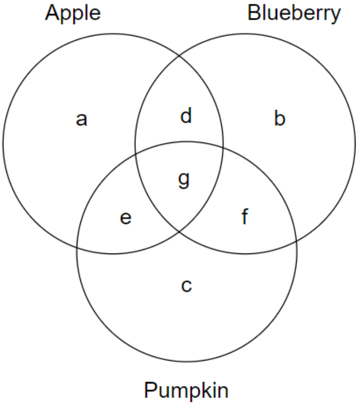
$$\frac{1}{e} = e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots$$

**Problem.** Suppose in a class of $n$ people, $x$ of them like blueberry pie and $y$ of them like apple pie. $z$ of them like both. How many people like either blueberry pie or apple pie?




We can represent the set of people as a Venn Diagram. If we add up $x$ and $y$, then we added the middle region twice. So we subtract by $z$ to account for that. This gives $x + y - z$

What if we add a third flavour? The idea is the same.




Here, $a$ is the people who like apple, $d$ is the people who like apple and blueberry, and $g$ is the people who like all three. The other letters are similar. The number of people who like any of the three is $a + b + c - d - e - f + g$.

In general, we have:

**Theorem (Principle of Inclusion Exclusion)**

*Suppose we had some sets $A_1$, $A_2$, ..., $A_n$. The number of things that belong in any of these sets can be obtained by adding the sizes of each set, subtracting the intersection of any two sets, adding the intersection of any three sets, and so on.*

**Proof.**

Left as an exercise. □

Recall the problem we want to solve - on average, how many permutations of 1 to $n$ are derangements?

We'll instead consider the complement - how many permutations of 1 to $n$ are **not** derangements? A permutation is not a derangement if it has a fixed point.

Let $P_k$ be the number of permutations that fixes $k$. For example, $1234 \to 1243$ belongs in $P_1$ and $P_2$ since it fixes 1 and 2 (but swaps 3 and 4). The size of $P_n$ is $(n-1)!$ since one of the numbers does not get permuted.

Then a permutation is not a derangement if and only if it belongs to one of $P_1$, $P_2$, ..., $P_n$.

How many permutations belong to both $P_1$ and $P_2$?

How many permutations belong to $P_1$ and $P_2$? If 1 and 2 are fixed, there are $n-2$ numbers left to permute. So there are $(n-2)!$ permutations that belong to $P_1$ and $P_2$.

Similarly, $(n-2)!$ permutations belong to $P_1$ and $P_3$, $P_1$ and $P_4$, and so on...

There are ${}_nC_2$ many pairs of sets.

Similarly, there are $(n-3)!$ permutations that belong to all of $P_1$, $P_2$, and $P_3$. This is the same for any choice of three sets.

There are ${}_nC_3$ triples of sets.

In general, there are ${}_nC_k$ ways to choose $k$ of the sets $P_1$, ..., $P_n$, and there are $(n-k)!$ permutations belonging to all $k$ them.

By P.I.E., the number of permutations with a fixed point is

$$
\begin{aligned}
& n(n-1)! - {}_nC_2(n-2)! + {}_nC_3(n-3)! - \ldots \\
&= n! - \frac{n!}{2!(n-2)!}(n-2)! + \frac{n!}{3!(n-3)!}(n-3)! - \ldots \\
&= \frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \frac{n!}{4!} + \frac{n!}{5!} - \ldots + (-1)^{n+1}\frac{n!}{n!}
\end{aligned}
$$

By P.I.E., the number of permutations with a fixed point is

$$\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \frac{n!}{4!} + \frac{n!}{5!} - \ldots + (-1)^{n+1}\frac{n!}{n!}$$

Hence the number of derangements is

$$\begin{aligned}
& n! - \left(\frac{n!}{1!} - \frac{n!}{2!} + \frac{n!}{3!} - \frac{n!}{4!} + \frac{n!}{5!} - \ldots + (-1)^{n+1}\frac{n!}{n!}\right) \\
&= n! - \frac{n!}{1!} + \frac{n!}{2!} - \frac{n!}{3!} + \frac{n!}{4!} - \frac{n!}{5!} + \ldots + (-1)^{n}\frac{n!}{n!} \\
&= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots + (-1)^{n}\frac{1}{n!}\right) \\
&\approx \frac{n!}{e}
\end{aligned}$$

So it will take, on average, about $e$ shuffles before we get a derangement!

Python Simulation

Average Number of Trials

Big Picture: The probability that any particular number $k$ is a fixed point is $1/n$. So the probability that a permutation is a derangement is about $\left(1-\frac{1}{n}\right)^n$. This is similar to the limit for $e$, and in fact, it goes to $\frac{1}{e}$.

This is not exact since the probabilities are not independent, but it becomes a better and better approximation as $n$ gets larger. This also suggests that the probability that any $k$ is not a fixed point becomes more independent from the other numbers as $n$ gets larger.