Automatically Computing Asymptotics of Sequences with Multivariate Rational Generating Functions

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Generating Functions

Recall (Generating Function)

The **generating function** *for* $(f_i)_{i\geq 0}$ *is*

$$F(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_k x^k + \cdots$$

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Example. The Fibonacci sequence $0, 1, 1, 2, \ldots$ has generating function

$$F(x) = x + x^2 + 2x^3 + ... = \frac{x}{1 - x - x^2}.$$

Extraction of Coefficients

Theorem (Cauchy Integral Formula for Coefficients)

Let $\mathcal{C}$ be a suitable closed curve about the origin. Then

$$f_n = [z^n]F(z) = \frac{1}{2\pi i}\int_{\mathcal{C}} F(z)\frac{dz}{z^{n+1}}.$$

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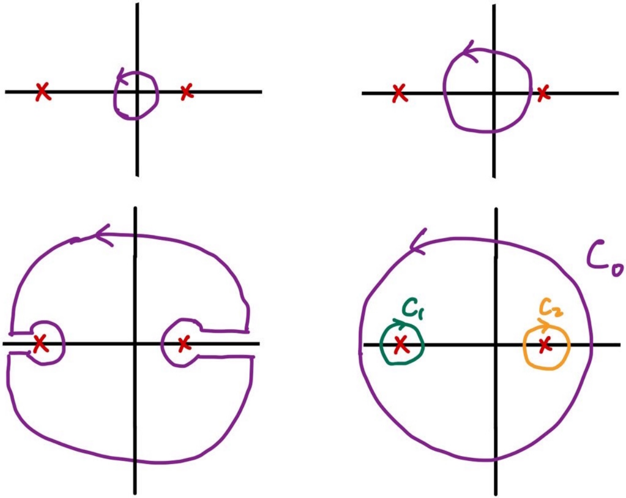
Example. The Fibonacci generating function $F(x) = \frac{x}{1-x-x^2}$ has poles at ϕ^{-1} and $-\phi$. We have

$$f_n = \frac{1}{2\pi i}\int_{|x|=\epsilon} \frac{F(x)}{x^{n+1}}dx$$

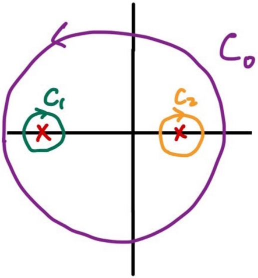
for $\epsilon < \phi^{-1}$.

Extraction of Coefficients

We may deform the curve of integration...



Extraction of Coefficients



Taking $\mathcal{C}_0$ arbitarily far from the origin,

$$\int_{\mathcal{C}_0} \frac{F(x)}{x^{n+1}} dx \to 0.$$

Hence,

$$f_n = -\mathrm{Res}_{z=-\phi} \frac{F(z)}{z^{n+1}} - \mathrm{Res}_{z=\phi^{-1}} \frac{F(z)}{z^{n+1}} = \frac{1}{\sqrt{5}} \left(-\phi^{-n} + \phi^{n}\right).$$

The Multivariate Case

Assume

$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{H(\mathbf{z})} = \sum_{\mathbf{i} \in \mathbb{N}^d} f_{\mathbf{i}} \mathbf{z}^{\mathbf{i}} = \sum_{i_1, \dots, i_d \geq 0} f_{i_1, \dots, i_d} z_1^{i_1} \cdots z_d^{i_d}$$

is a rational function in d variables.

We consider the $\mathbf{r}$-diagonal

$$[\mathbf{z}^{n\mathbf{r}}] F(\mathbf{z}) = f_{nr_1, \dots, nr_d}.$$

Example. We can consider the $(1,1)$-diagonal of

$$F(x,y) = f_{0,0} + f_{1,0}x + f_{0,1}y + f_{1,1}xy + \cdots$$

$$\begin{array}{cccccc}
f_{0,5} & f_{1,5} & f_{2,5} & f_{3,5} & f_{4,5} & f_{5,5} \\
f_{0,4} & f_{1,4} & f_{2,4} & f_{3,4} & f_{4,4} & f_{5,4} \\
f_{0,3} & f_{1,3} & f_{2,3} & f_{3,3} & f_{4,3} & f_{5,3} \\
f_{0,2} & f_{1,2} & f_{2,2} & f_{3,2} & f_{4,2} & f_{5,2} \\
f_{0,1} & f_{1,1} & f_{2,1} & f_{3,1} & f_{4,1} & f_{5,1} \\
f_{0,0} & f_{1,0} & f_{2,0} & f_{3,0} & f_{4,0} & f_{5,0}
\end{array}$$

Multivariate Generating Function

Why use more variables?

- Tracking parameters.
- Capturing a wider range of possible sequences.

The number of *horizontally convex polyominoes* with n cells and k rows is $[x^n y^k]F(x,y)$ where

$$F(x,y) = \frac{xy(1-x)^3}{(1-x)^4 - xy(1-x-x^2+x^3+x^2y)}.$$

Multivariate Generating Function

Why use more variables?

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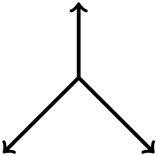
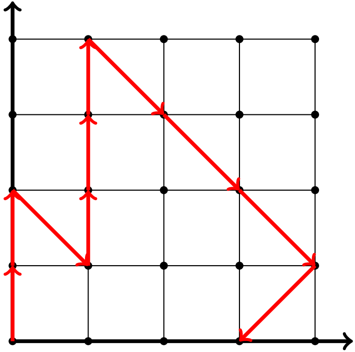
The number of lattice paths of length n with directions $\{N, SW, SE\}$ that stay in the non-negative quadrant is $[x^n y^n z^n]F(x,y)$ where

$$F(x,y,z) = \frac{(1+x)(1-2zy^2(1+x^2))}{(1-y)(1-z(x^2y^2+y^2+x))(1-zy^2(1+x^2))}.$$

Lattice Paths

Let s_n be the number of lattice paths starting from the origin and taking n steps in $\{(-1,-1),(1,-1),(0,1)\}$ without ever leaving the non-negative quadrant.





Lattice Paths

In (Bostan and Kauers, 2009), it was shown that the generating function

$$S(t) = \sum_{n \geq 0} s_n t^n$$

satisfies a linear ODE of order 43. We can approximate that

$$s_n = C \cdot 3^n n^\alpha \log^\beta(n) \sum_{k \geq 0} C_k n^{-k} + O((2\sqrt{2})^n)$$

where $C = 0.000\ldots$. Whether $C = 0$ remained open for the next 7 years.

Lattice Paths

Techniques from the *kernel method* for lattice path enumeration implies s_n is the $(1,1,1)$-diagonal of the coefficients of

$$F(x,y,z) = \frac{(1+x)(1-2zy^2(1+x^2))}{(1-y)(1-z(x^2y^2+y^2+x))(1-zy^2(1+x^2))}.$$

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```
sage: F = (1 + x)*(2*z*x^2*y^2 + 2*z*y^2 - 1)/((-1 + y)*(z*x
    ^2*y^2 + z*y^2 + z*x - 1)*(z*x^2*y^2 + z*y^2 - 1))
sage: diagonal(F)
O(2.828427124746190?^n*n^(-2))
```

Showing $s_n = O((2\sqrt{2})^n)$...?

Lattice Paths

But we can increase the precision!

```
sage: diagonal(F, expansion_precision=2)
0.9705627484771406?/pi*(-2.828427124746190?)^n*n^(-2)
  + 32.97056274847714?/pi*2.828427124746190?^n*n^(-2)
  + O(2.828427124746190?^n*n^(-3))
```

Now we've verified that

$$s_n \sim (12\sqrt{2} - 16)\frac{(-2\sqrt{2})^n}{\pi n^2} + (12\sqrt{2} + 16)\frac{(2\sqrt{2})^n}{\pi n^2}.$$

Our analysis begins with a familiar result...

Theorem (Multivariate Cauchy Integral Formula for Coefficients)

Given a suitable product of circles $\mathcal{C} \subset \mathbb{C}^d$ about the origin,

$$f_{\mathbf{i}} = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{\mathbf{i}+\mathbf{1}}}.$$

Finding Contributing Points

The integrand of

$$f_{n\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_{\mathcal{C}} F(\mathbf{z}) \frac{d\mathbf{z}}{\mathbf{z}^{n\mathbf{r}+\mathbf{1}}}$$

has the maximum modulus when $|\mathbf{z}^{\mathbf{r}}|$ is minimized.

Definition

The **height function** *in the direction* $\mathbf{r}$ *is*

$$h_{\mathbf{r}}(\mathbf{z}) = -\sum_{j=1}^{d} r_j \log |z_j|.$$

The Multivariate Case

Candidates for minimizing the height come from **critical points** of h.

When $V(H)$ is *smooth*, these can be obtained by solving

$$\begin{aligned} z_i H_{z_i}(\mathbf{z}) r_d - \lambda r_i =& 0 \quad (1 \leq i \leq d), \\ H(\mathbf{z}) =& 0. \end{aligned}$$

Finding Contributing Points

Not all critical points are minimizers of the height function.

Definition

Given $V \subset \mathbb{C}^d$, we say $\mathbf{w} \in V$ is **minimal** *if there does not exist $\mathbf{z} \in V$ such that $|z_i| < |v_i|$ for all i.*

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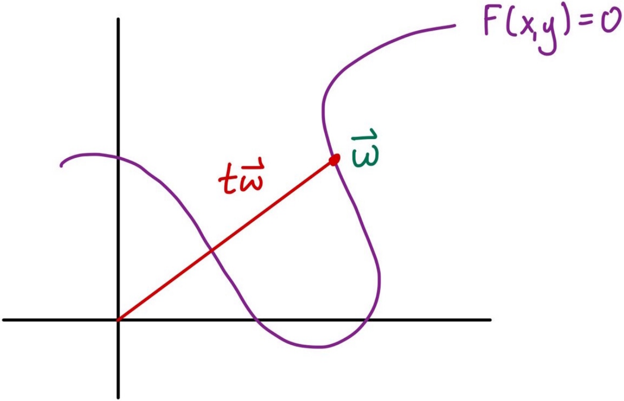
Testing minimality is *hard*.

Finding Contributing Points

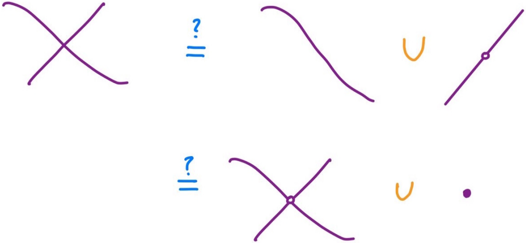
We say $F(\mathbf{z})$ is **combinatorial** if only finitely many coefficients are negative.

Combinatorial functions have an "easy" test for minimality.



Non-Smooth Points

When $V(H)$ is non-smooth, we want to decompose it into smooth manifolds.



Whitney Stratifications

Definition

A **Whitney Stratification** *of the variety* X *is a decomposition*

$$X = X_d \supset X_{d-1} \supset \cdots \supset X_1 \supset X_0$$

such that each $X_k \setminus X_{k-1}$ *is a manifold of dimension* k *preserving local geometric properties.*

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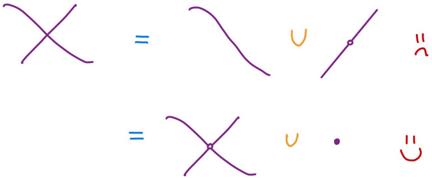

Whitney Stratification - Examples

Example. The Whitney cusp is defined by the equation $y^2 + z^3 - x^2z^2 = 0$.

Non-smooth along the x-axis, but behaves differently at the origin.

Critical Points

Critical points on $V(p_1, \dots, p_s)$ come from the maximal minors of

$$N = N_{\mathbf{w}}(p_1, ..., p_s) = \begin{pmatrix} \nabla_{\log} p_1(\mathbf{w}) \\ \cdots \\ \nabla_{\log} p_s(\mathbf{w}) \\ \mathbf{r} \end{pmatrix},$$

where $\nabla_{\log} f = (z_1 f_{z_1}, \dots, z_d f_{z_d})$.

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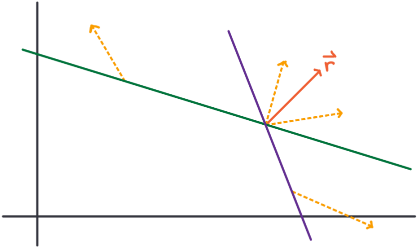
where $\nabla_{\log} f = (z_1 f_{z_1}, \ldots, z_d f_{z_d})$.

If $V(p_1, ..., p_s)$ is non-smooth, additional algebraic techniques are used to clear extraneous solutions. One option is to take an ideal saturation.

Contributing Points

When H admits a *transverse factorization* $H(\mathbf{z}) = H_1(\mathbf{z}) \cdots H_m(\mathbf{z})$, we can characterize contributing points by looking at certain cones.



Asymptotic Contributions

Given a contributing point $\mathbf{w}$, we want to

- Deform our domain of integration and use residues to reduce our Cauchy integral to a local integral around $\mathbf{w}$.
- Use an appropriate change of variables to reduce the local integral to that of a Fourier Laplace integral.

Asymptotic Contributions

Deform our domain of integration and use residues to reduce our Cauchy integral to a local integral around $\mathbf{w}$.

This step introduces a matrix

$$\Gamma_{\mathbf{w}} = \begin{pmatrix} (\nabla_{\log} H_1)(\mathbf{w}) \\ \vdots \\ (\nabla_{\log} H_s)(\mathbf{w}) \\ w_1 \mathbf{e}^{(1)} \\ \vdots \\ w_{d-s} \mathbf{e}^{(d-s)} \end{pmatrix},$$

where $\mathbf{e}^{(i)}$ is the i-th elementary basis vector.

Asymptotic Contributions

Use an appropriate change of variables to reduce the local integral to that of a Fourier Laplace integral.

We obtain an integral of the form

$$\int_{\mathcal{N}} A(\boldsymbol{\theta}) \exp(-n\phi(\mathbf{r}, \boldsymbol{\theta})) d\boldsymbol{\theta}$$

where we can compute the Hessian $\mathcal{H}$ of ϕ.

Main Asymptotic Result

Theorem

Let $F(\mathbf{z})$ admit a square-free transverse factorization. If $\mathbf{w}$ is the unique contributing point of $F(\mathbf{z})$. Then, under verifiable assumptions,

$$f_{n\mathbf{r}} = \mathbf{w}^{-n\mathbf{r}}(2\pi n)^{(s-d)/2} \sum_{k \geq 0} C_k n^k$$

for computable constants C_k.

Main Asymptotic Result

In particular,

$$C_0 = \frac{G(\mathbf{w}) \prod_{1 \leq j \leq s} |\mathbf{w}_j|}{u(\mathbf{w}) \sqrt{\mathsf{det}(r_d \mathcal{H})} |\mathsf{det} \Gamma_{\mathbf{w}}|}.$$

When $s = d$,

$$C_0 = \frac{(2\pi)^{(s-d)/2} G(\mathbf{w})}{u(\mathbf{w}) |\mathsf{det} \Gamma_{\mathbf{w}}|}.$$

A Software Package

The `sage_acsv` package provides the first rigorous implementation of asymptotic computations of rational generating functions.

To install the package, run the command

```
sage -pip install sage_acsv
```

The source code can be found on

https://github.com/ACSVMath/sage_acsv

along with official documentation

https://acsvmath.github.io/sage_acsv/.

Central Binomial Coefficients

The $(1,1)$-direction of the generating function

$$F(x,y) = \frac{1}{1-x-y} = \sum_{i,j\geq 0} \binom{i+j}{j} x^i y^j$$

forms the central binomial coefficients

$$[x^n y^n]F(x,y) = \binom{2n}{n}.$$

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```
sage: from sage_acsv import (get_expansion_terms,
....:     diagonal_asymptotics_combinatorial as diagonal)
sage: var('w x y z')
(w, x, y, z)
sage: diagonal(1/(1 - x - y))
1/sqrt(pi)*4^n*n^(-1/2) + O(4^n*n^(-3/2))
```

Central Binomial Coefficients

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$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

We can also compute higher order terms.

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```

verifies that

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}.$$

We can also compute higher order terms. Running

```
sage: diagonal(1/(1 - x - y), expansion_precision=3)
1/sqrt(pi)*4^n*n^(-1/2) - 1/8/sqrt(pi)*4^n*n^(-3/2) + 1/128/
    sqrt(pi)*4^n*n^(-5/2) + O(4^n*n^(-7/2))
```

verifies that

$$\binom{2n}{n} = \frac{4^n}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \frac{1}{128n^2}\right) + O(4^n n^{-7/2}).$$

A Sequence Alignment Problem

The sequence alignment problem from molecular biology is concerned with the ways a string may evolve via substitutions, insertions, and deletions. One particular case of interest yields the generating function

$$F(x,y) = \frac{x^2y^2 - xy + 1}{1 - x - y - xy + xy^2 + x^2y - x^2y^3 - x^3y^2}.$$

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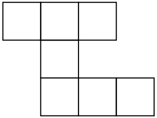
```
sage: G = x^2*y^2 - x*y + 1
sage: H = 1 - (x + y + x*y - x*y^2 - x^2*y + x^2*y^3 + x^3*y
    ^2)
sage: asm_vals = diagonal(G/H)
0.9430514023983397?*4.518911369262258?^n/(sqrt(pi)*sqrt(n))
    + O(4.518911369262258?*n^(-3/2))
```

Horizontally Convex Polyominoes

A *horizontally convex polyomino* is a union of cells $[a, a+1] \times [b, b+1]$ in $\mathbb{Z}^2$ such that the interior of the figure is connected and every row is connected.



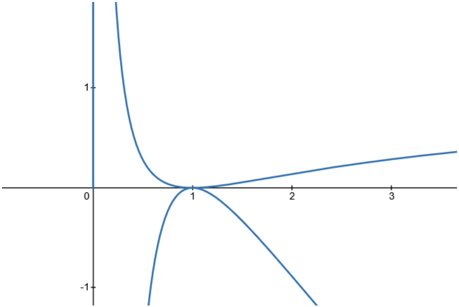
The number of HCPs with n cells and k rows can be represented by $[x^n y^k]F(x, y)$ where

$$F(x, y) = \frac{xy(1-x)^3}{(1-x)^4 - xy(1-x-x^2+x^3+x^2y)}.$$

Horizontally Convex Polyominoes

```
sage: F = x*y*(1 - x)^3/((1 - x)^4 - x*y*(1 - x - x^2 + x^3
    + x^2*y), r=[2,1])
0.2974174421957931?*10.11457259449030 4?^n/(sqrt(pi)*sqrt(n))
     + O(10.11457259449030 4?^n*n^(-3/2))
```

We can only use a direction (n, k) where $n > k$. As $\frac{n}{k} \to 1$, the critical point approaches $(0, \infty)$.



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$$\frac{1}{(1-x)(1-y)+(1-x)(1-z)+(1-y)(1-z)}$$

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- Non-transverse intersections (e.g. cone points).

$$\frac{1}{(1-x)(1-y)+(1-x)(1-z)+(1-y)(1-z)}$$

- Non-rational generating functions (e.g. algebraic functions).

$$T(z_1,z_2)=\frac{1-\sqrt{1-4z_1z_2}}{2z_1}.$$

Other Functionality

- Compute critical points of a general rational function.

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- Compute critical points of a general rational function.
- Compute asymptotics in a non-rational algebraic direction.
- Option to use `msolve` and `macaulay2` backends for Groebner basis computations.
- Implementations of useful utility functions such as Whitney stratifications and univariate representations.

Future Work

- Compute higher order term expansions in more cases.

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- Compute higher order term expansions in more cases.
- Using algebraic and cohomological techniques to decompose non-transverse intersections.
- Factoring over the power series ring to compute asymptotics of points that are locally transverse.

References

[1] Benjamin Hackl et al. *A SageMath Package for Analytic Combinatorics in Several Variables: Beyond the Smooth Case*. 2025. arXiv: 2504.09790 [math.CO]. URL: https://arxiv.org/abs/2504.09790.

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