A Brief Introduction to Computational Analytic Combinatorics

Andrew Luo · Steve Melczer

University of Waterloo

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Recall (Generating Function)

A formal power series of the form

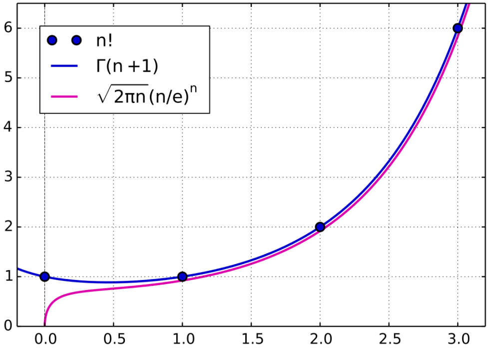
$$F(x) = f_0 + f_1 x + f_2 x^2 + ... + f_k x^k + ...$$

Example. A generating function for the sequence $1, 2, 4, 8, ...$ is given by $F(x) = 1 + 2x + 4x^2 + 8x^3 + ... = \frac{1}{1-2x}$

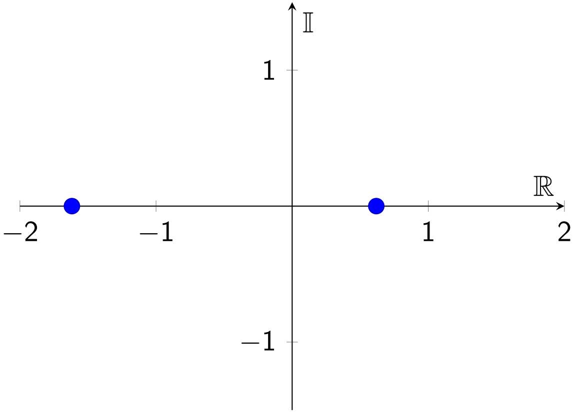
We write $[x^k]F(x) = f_k = 2^k$.

In general...

- Extracting coefficients is *difficult*.
- Instead, we use complex analysis to compute *asymptotic approximations*.



Example. Consider the Fibonacci GF $F(x) = \frac{x}{1-x-x^2}$.



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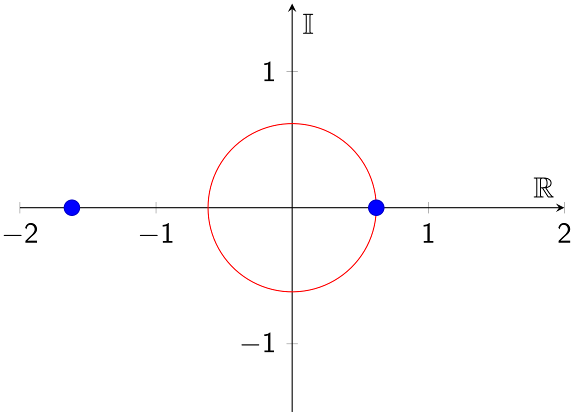

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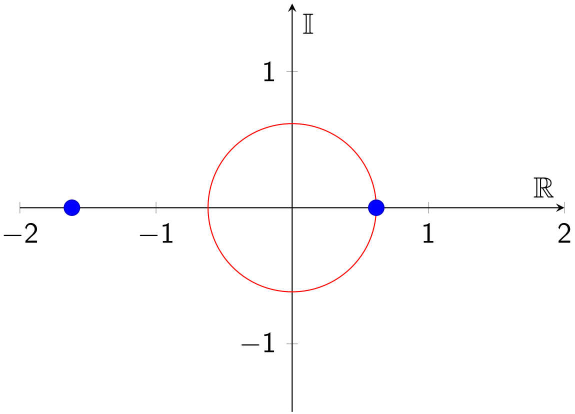


The radius of convergence is ϕ^{-1}. Consequently, $\mathrm{Fib}(n) \sim \frac{1}{\sqrt{5}}\phi^n$.

Steps for determining asymptotics for $F = G/H$...

- Find all the poles of F.
- Determine the pole(s) of minimum modulus z_0.
- For some (computable) polynomial $P(k)$, we have

$$[x^k]F(x) \sim P(k)|z_0|^{-k}.$$

- If the pole is simple (multiplicity one), then

$$P(k) = \frac{-G(z_0)}{z_0 H'(z_0)}.$$

Multivariate Generating Functions

Consider $F(\boldsymbol{z}) = F(x, y) = \frac{1}{1-x-y} = 1 + x + y + x^2 + 2xy + y^2 + \ldots$.

Specify a direction:

- $\boldsymbol{r} = [1, 1]$

  $1x^0y^0 + x + y + x^2 + 2x^1y^1 + y^2 + x^3 + \ldots + y^3 + \ldots + 6x^2y^2 + \ldots$

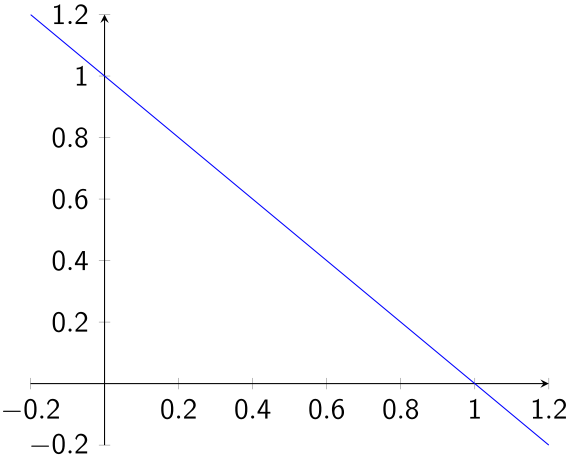
  $$[z^{n\mathbf{r}}]F(x, y) = [x^n y^n]F(x, y) = \binom{2n}{n}$$

- $\boldsymbol{r} = [2, 1]$

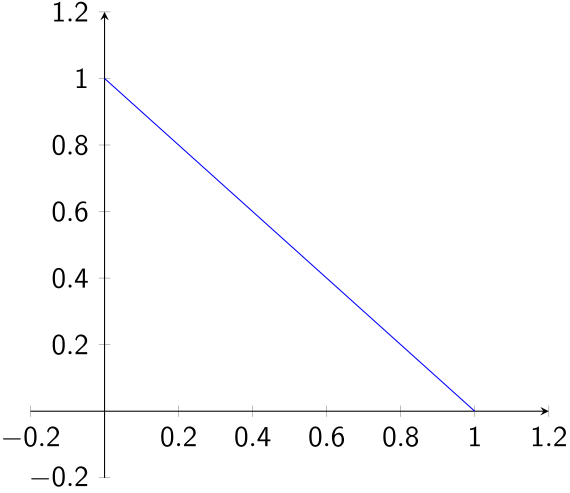
  $1x^0y^0 + x + y + x^2 + 2xy + y^2 + x^3 + 3x^2y^1 + \ldots + 15x^4y^2 + \ldots$

  $$[z^{n\mathbf{r}}]F(x, y) = [x^{2n} y^n]F(x, y) = \binom{3n}{n}$$

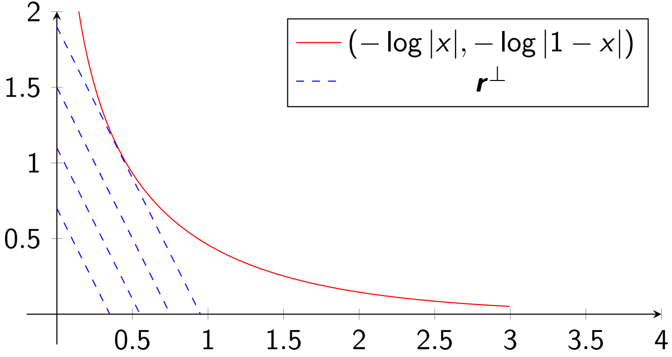
Singularities of F



We define a singularity $\boldsymbol{w} = (w_1, \ldots, w_d) \in \mathbb{C}^d$ of F to be *minimal* if there does not exist a singularity $\boldsymbol{w'} \in \mathbb{C}^d$ such that $|w_i'| < |w_i|$ for $i \in \{1, \ldots, d\}$.



The *contributing* points of $F = G/H$ are the minimal points where the function $h_{\boldsymbol{r}}(\boldsymbol{w}) := -\sum_{j=1}^{d} r_j \log |w_j|$ is minimized.



Under certain smoothness conditions, they are the minimal solutions to the system

$$\begin{aligned} H^{\mathfrak{s}}(\boldsymbol{w}) &= 0 \\ r_j w_1 H^{\mathfrak{s}}_{z_1}(\boldsymbol{w}) - r_1 w_j H^{\mathfrak{s}}_{z_j}(\boldsymbol{w}) &= 0 \qquad (2 \le j \le d) \end{aligned}$$

where $H^{\mathfrak{s}}$ is the square-free part of H.

Theory of ACSV

Theorem

Let $F(\boldsymbol{z}) = G(\boldsymbol{z})/H(\boldsymbol{z}) \in \mathbb{Q}[z]$ *be coprime polynomials such that* $F(\boldsymbol{z})$ *is analytic at the origin. If the system*

$$\begin{aligned} H^{\mathfrak{s}}(\boldsymbol{w}) &= 0 \\ r_j w_1 H^{\mathfrak{s}}_{z_1}(\boldsymbol{w}) - r_1 w_j H^{\mathfrak{s}}_{z_j}(\boldsymbol{w}) &= 0 \qquad (2 \leq j \leq d) \end{aligned}$$

admits a finite number of solutions with $\boldsymbol{w}$ *minimal and* $H_{z_d} \neq 0$*, then there exists a computable matrix* $\mathcal{H}$ *such that as* $n \to \infty$,

$$f_{n\boldsymbol{r}} = \boldsymbol{w}^{n\boldsymbol{r}} n^{(1-d)/2} \frac{(2\pi r_d)^{(1-d)/2}}{\sqrt{\det(\mathcal{H})}} \frac{-G(\boldsymbol{w})}{w_d H_{z_d}(\boldsymbol{w})} \left(1 + O\left(\frac{1}{n}\right)\right)$$

Recall. We want the minimal solutions to

$$\begin{aligned} H^{\mathfrak{s}}(\boldsymbol{w}) &= 0 \\ r_j w_1 H^{\mathfrak{s}}_{z_1}(\boldsymbol{w}) - r_1 w_j H^{\mathfrak{s}}_{z_j}(\boldsymbol{w}) &= 0 \qquad (2 \le j \le d). \end{aligned}$$

Assume H is a square-free polynomial, and that G/H is combinatorial. Introduce a new variable λ

$$\begin{aligned} H(\boldsymbol{w}) &= 0 \\ z_j H_{z_j} - r_j \lambda &= 0 \qquad (1 \leq j \leq d). \end{aligned}$$

We can think of λ as a substitution for $\frac{w_1 H(\boldsymbol{w})}{r_1}$.

Introduce a variable t

$$
\begin{aligned}
H(\boldsymbol{w}) &= 0 \\
H(t\boldsymbol{w}) &= 0 \\
z_j H_{z_j} - r_j \lambda &= 0 \qquad (1 \leq j \leq d).
\end{aligned}
$$

When F is combinatorial, a solution $(\boldsymbol{w}, \lambda, t)$ to the system is minimal **iff** $t = 1$ and there does not exist $t' \in (0,1)$ such that $(\boldsymbol{w}, \lambda, t')$ is a solution to the system.

How do we solve this system?

Definition

Let $\mathcal{V} = \{\boldsymbol{z} : f_1(\boldsymbol{z}) = ... = f_d(\boldsymbol{z}) = 0\}$ *be the solutions to a polynomial system. If* $\mathcal{V}$ *is finite, the Kronecker Representation of* $\mathcal{V}$ *consists of*

- *A variable* $u = \boldsymbol{\kappa} \cdot \boldsymbol{z}$ *such that* u *takes on distinct values for* $z \in \mathcal{V}$*.*
- *A square-free polynomial* $P \in \mathbb{Z}[u]$
- *Polynomials* $Q_1, ..., Q_d \in \mathbb{Z}[u]$ *such that*

$$z_i = Q_i(u)/P'(u)$$

Theorem

The Kronecker Representation of a polynomial system can be computed in polynomial time.

Example.
Consider the system

$$\begin{aligned} x^3 + y^3 - 10 &= 0 \\ y^2 - 2 &= 0 \end{aligned}$$

A Kronecker Representation for this system is:

$$\begin{aligned} P(u) &= u^6 - 6u^4 - 20u^3 + 36u^2 - 120u + 100 \\ Q_x(u) &= 60u^3 - 72u^2 + 360u - 600 \\ Q_y(u) &= 12u^4 - 72u^2 + 240u \end{aligned}$$

Input: $F(\boldsymbol{z}) = G(\boldsymbol{z})/H(\boldsymbol{z})$, H square-free, direction $\boldsymbol{r}$

- Create the system

$$\{H, z_1 H_{z_1} - r_1\lambda, ..., z_d H_{z_d} - r_d\lambda, H(tz_1, ..., tz_d)\}.$$

- Compute P and Q, the Kronecker Representation of the system.
- Find the factors of P, and determine solutions to the critical point system.
- Return the points $(\boldsymbol{w}, \lambda, t)$ where $t = 1$ and there does not exist a point $(\boldsymbol{w}, \lambda, t')$ with $t' \in (0, 1)$.

Examples

Live code demo.

If code doesn't start up, see output file **here**.

Future Work

- Homotopy techniques to deal with geenrating functions in the non-combinatorial case
- Extending to Laurent series

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- Homotopy techniques to deal with geenrating functions in the non-combinatorial case
- **Extending to Laurent series**

Definition

Given a singular set

$$\mathcal{V} = \{w \in \mathbb{C} : H(w) = 0\},$$

the amoeba of

$$\mathcal{V}$$

is the set

$$\{\log(|w_1|), ..., \log(|w_d|) : \boldsymbol{w} \in \mathcal{V}\}.$$

Thank you!