Homework 5
Due Friday February 11, 2022

1. Consider the statements about symmetric matrices and indicate if the statements are true or false. If a statement is true provide a proof, otherwise give a counterexample.

   (a) The block matrix $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is automatically symmetric.

   Solution: You can write your solution here!

   (b) If A and B are symmetric then AB is symmetric.

   Solution: You can write your solution here!

   (c) If A is not symmetric, then A^{-1} is not symmetric.

   Solution: You can write your solution here!

   (d) If A, B, and C are symmetric, then $(ABC)^T = CBA$

   Solution: You can write your solution here!

2. §3.1 # 10. Which of the following subsets of $\mathbb{R}^3$ are actually subspaces? You should either prove that the set is a subspace or show that the set does not have one of the properties of subspaces.

   (a) The plane of vectors (b_1, b_2, b_3) with $b_1 = b_2$.

   Solution: We can re-write the plane of vectors to more accurately represent our constraints:

$$\begin{bmatrix} b_1 & b_1 & b_2 \end{bmatrix}$$

   This is a subset of $\mathbb{R}^3$ and we can prove it by proving that the sum of any two vectors within the subsetis also in the subset:

$$\begin{bmatrix} b_1 \\ b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} b_3 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} b_1 + b_3 \\ b_1 + b_3 \\ b_2 + b_4 \end{bmatrix}$$

   As we can see, the first two elemetns of the vectors equal eqch other which satisfies this constraint. We also need to prove that any scalar multiple of a vector in the subset is also in the subset.

$$\alpha \begin{bmatrix} b_1 \\ b_1 \\ b_2 \end{bmatrix} + \beta \begin{bmatrix} b_3 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} \alpha \cdot b_1 \\ \alpha \cdot b_1 \\ \alpha \cdot b_2 \end{bmatrix} + \begin{bmatrix} \beta \cdot b_3 \\ \beta \cdot b_3 \\ \beta \cdot b_4 \end{bmatrix}$$

(b) The plane of vectors with $b_1 = 1$.

Solution: You can write your solution here!

(c) The vectors (b_1, b_2, b_3) with $b_1 b_2 b_3 = 0$.

Solution: You can write your solution here!

(d) All linear combinations of $\vec{v} = (1, 4, 0)$ and $\vec{w} = (2, 2, 2)$.

Solution: You can write your solution here!

(e) All vectors (b_1, b_2, b_3) with $b_1 + b_2 + b_3 = 0$.

Solution: You can write your solution here!

(f) All vectors (b_1, b_2, b_3) with $b_1 \leq b_2 \leq b_3$.

Solution: You can write your solution here!

3. An exercise using the outer-product method of matrix multiplication. Every matrix with rank r can be written as the sum of r rank 1 matices. An easy way to write a rank 1 matrix is using an outer-product (recall: $\vec{u}\vec{v}^T$ is an outer-product). Construct a matrix A with rank 2 that has $C(A) = span((1, 2, 4), (2, 2, 1))$ and $C(A^T) = span((1, 0), (1, 1))$, you should use outer-products to find A. Then find a factorization on A into a 3 by 2 matrix times a 2 by 2 matrix, you should think backwards about the outer product method of matrix multiplication to help you.

Solution: You can write your solution here!

4. Suppose that A is a $m \times n$ matrix and $\vec{b}$ is a $m \times 1$ vector. Let B be the $m \times (n + 1)$ matrix formed by adding $\vec{b}$ to A, so $B = [A\,\vec{b}]$. What must be true so that $C(A) = C(B)$? What must be true if $C(A)$ is smaller that $C(B)$? Explain what must be true for $A\vec{x} = \vec{b}$ and $B\vec{x} = \vec{b}$ to have solutions.

Solution: In order for $C(A) = C(B)$, $\vec{b}$ must be a linear combination of some column in matrix A. In order for $C(A) < C(B)$, $\vec{b}$ must be linearly independent of any columns in matrix A. In order for $A\vec{x} = \vec{b}$ *and* $B\vec{x} = \vec{b}$ to both have solutions, $\vec{b}$ simply has to be a linear combination of some column in its corresponding matrix. For example:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} c \\ f \\ i \end{bmatrix}$$

$$B = \begin{bmatrix} a & b & c & j \\ d & e & f & k \\ g & h & i & l \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} j \\ k \\ l \end{bmatrix}$$

5. Suppose that $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -2 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{bmatrix}$

(a) Describe the column space of A by listing a basis for it.

Solution: The column space can be found by finding the pivots after elemination, so let's elimniate!

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -2 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It's clear to see now that our pivots are in x_1 and x_3. The basis for the column space A can be described by the columns that contain the pivots.

$$C(A) = span\left(\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right\}\right)$$

(b) Describe the nullspace of A by listing a basis for it.

Solution: We can start to find the nullspace by referencing our elemination matrix E that we solved above.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -2 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can see by looking at the matrix EA that we have pivots at x_1 and x_3 which means that our free variables are x_2 and x_4. Let's set $x_2 = 1$ and $x_4 = 0$.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{aligned} x_1 + 2x_2 + x_4 &= 0 \\ x_3 + x_4 &= 0 \end{aligned}$$

$$\begin{gathered} x_1 + 2(1) + 0 = 0 \to x_1 = -2 \\ x_3 + 0 = 0 \to x_3 = 0 \end{gathered}$$

We now have one solution. We can switch the values of x_2 and x_4 to get our 2nd solution.

$$\begin{gathered} x_1 + 2(0) + 1 = 0 \to x_1 = -1 \\ x_3 + 1 = 0 \to x_3 = -1 \end{gathered}$$

We can now describe the nullspace with the basis for it:

$$N(A) = span\left(\left\{\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right\}\right)$$

(c) Describe the left nullspace of A by listing a basis for it.

Solution: We can find the left nullspace of A by referencing the elemination we did on A in the problem above.

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -2 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pay attention to row 3 in EA. It is all 0's. Because the left nullspace is just the transpose of the nullspace, we can think of the rows as columns and vice versa. This would mean that row 3 in EA would actually be column 3 in EA^T. This would mean that the third row in our elemination matrix E would be our only solution to the left nullspace.

$$N(A)^T = span\left(\left\{\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}\right)$$

(d) Describe the row space of A by listing a basis for it.

Solution: Much like finding the column space, we can start by finding EA:

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ -1 & -2 & 1 & 0 \\ 2 & 4 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we simply pay attention to where the pivots are located in the rows, and those are our answers. The pivots are located at x_1 and x_3.

$$C(A)^T = span\left(\left\{\begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}\right\}\right)$$