ISSN 2070-0482, Mathematical Models and Computer Simulations, 2017, Vol. 9, No. 3, pp. 339–348. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © E.A. Ayryan, A.D. Egorov, D.S. Kulyabov, V.B. Malyutin, L.A. Sevastyanov, 2016, published in Matematicheskoe Modelirovanie, 2016, Vol. 28, No. 11, pp. 113–125.

Application of Functional Integrals to Stochastic Equations

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Received May 12, 2015

Abstract—Representing a probability density function (PDF) and other quantities describing a solution of stochastic differential equations by a functional integral is considered in this paper. Methods for the approximate evaluation of the arising functional integrals are presented. Onsager–Machlup functionals are used to represent PDF by a functional integral. Using these functionals the expression for PDF on a small time interval Δt can be written. This expression is true up to terms having an order higher than one relative to Δt. A method for the approximate evaluation of the arising functional integrals is considered. This method is based on expanding the action along the classical path. As an example the application of the proposed method to evaluate some quantities to solve the equation for the Cox–Ingersol–Ross type model is considered.

Keywords: stochastic differential equations, Onsager-Machlup functionals, functional integrals

DOI: 10.1134/S2070048217030024

1. INTRODUCTION

Various physical, chemical, and biological systems with the presence of fluctuations or noise are described by stochastic differential equations in the sense of Ito, of the form

$$dx(t) = f(x,t)dt + g(x,t)dw(t) \tag{1}$$

with the initial condition $x(t_0) = x_0$, where $w(t)$ is the Wiener process.

Models of interacting populations such as "predator–victim," symbiosis, and competition, and their modifications, are described through stochastic differential equations in [1, 2].

In many applications we often need to find, in analytic or approximate form, the transition probability density function (TPDF) for the stochastic variable $x(t)$, moments for TPDF, mathematical expectation of a function of solution of Eq. (1), and other quantities. We can use methods for solving stochastic differential equations [3, 4], the Fokker–Planck equation, and numerical methods of solving these equations [5] to compute these quantities. For these purposes we can also use the method of functional integration.

For a long time, functional integrals have been successfully used in quantum field theory, statistical mechanics [6–10], and also in the theory of stochastic differential equations [7, 11, 12]. They provide a convenient tool for the analytic and approximate computation of various characteristics of stochastic models.

In this paper we consider representing the TPDF through functional integral and approximate computation methods of the arising functional integrals.

In the second Section, we study the Onsager–Machlup functional to represent a TPDF through a functional integral. In the third Section, we consider a method for the approximate computation of functional integrals based on the decomposition of the action near the classical trajectory. In the fourth Section, we propose, as an example, an analog of the Cox–Ingersol–Ross model.

2. REPRESENTATION OF QUANTITIES BY A FUNCTIONAL INTEGRAL

In order to write a TPDF through a functional integral we use the Onsager–Machlup functional [5, 11–13]. In the general case, we cannot find a TPDF on a small time interval Δt corresponding to an arbitrary stochastic differential equation. However, we can find an expression for a TPDF on a small time interval Δt which is true up to summands of orders higher than one with respect to Δt. This expression for Eq. (1), in the case of the Ito scheme and functions not depending on time, has the form [12]

$$p(x,t+\Delta t,y,t) \approx \frac{1}{\sqrt{2\pi g^2(y)\Delta t}} \exp\left\{ -\frac{\left(x-y-\left(f(y)-\frac{1}{2}g'(y)g(y)\right)\Delta t\right)^2}{2g^2(y)\Delta t} \right\}. \tag{2}$$

Using expression (2) we can write a TPDF in the form

$$\begin{aligned} p(x,t,x_0,t_0) &= \lim_{N\to\infty}\int\ldots\int\prod_{i=1}^{N-1}dx_i\prod_{i=1}^{N}\frac{1}{\sqrt{2\pi g^2(x_{i-1})\Delta t}} \\ &\times \exp\left\{-\sum_{i=0}^{N-1}\frac{\left(x_{i+1}-x_i-\left(f(x_i)-\frac{1}{2}g'(x_i)g(x_i)\right)\Delta t\right)^2}{2g^2(x_i)\Delta t}\right\}. \end{aligned} \tag{3}$$

In the limit case formula (3) reads

$$p(x,t,x_0,t_0) = Z[0] = \int D[x]\exp\left\{-\int_{t_0}^{t}\frac{\left(\dot{x}(\tau)-\left(f(x(\tau))-\frac{1}{2}g'(x(\tau))g(x(\tau))\right)\right)^2}{2g^2(x(\tau))}d\tau\right\}, \tag{4}$$

where $D[x] = \lim\limits_{N\to\infty}\prod_{i=1}^{N-1}\frac{dx_i}{\sqrt{2\pi g^2(x_i)\Delta t}}\frac{1}{\sqrt{2\pi g^2(x)\Delta t}}$.

The expression $D[x]$ formally diverges and makes sense only together with the exponent under the sign of the integral in (4), and in a rigorous mathematical form the functional integral in the right hand side of equality (4) is defined as the limit of finite multiplicity integrals.

The functional under the exponent in formula (4) is called the Onsager–Machlup functional. The moments for TPDF are written out as follows:

$$\left\langle\prod_{i=1}^{n}x(\tau_i)\right\rangle = Z[0]^{-1}\int D[x]\prod_{i=1}^{n}x(\tau_i)\exp\left\{-\int_{t_0}^{t}\frac{\left(\dot{x}(\tau)-\left(f(x(\tau))-\frac{1}{2}g'(x(\tau))g(x(\tau))\right)\right)^2}{2g^2(x(\tau))}d\tau\right\}. \tag{5}$$

From formulas (4) and (5), we can pass to formulas with a constant coefficient before $\dot{x}(\tau)^2$. This transformation in the case of stochastic differential equations is analogous to the Lamperti transformation [14] or to the change of function by the Ito formula, after which the initial equation is reduced to an equation with a constant diffusion coefficient.

Let functions G and φ be such that $dG(y)/dy = 1/g(y)$ and $G(\varphi(y)) = y$. Then, if we make the replacement $x(\tau) = \varphi(y(\tau))$, we obtain $\dot{\varphi}(y(\tau))/g(\varphi(y(\tau))) = \dot{y}(\tau)$. After this replacement formula (4) takes the form

$$p(y,t,y_0,t_0) = Z[0] = \int D[y]\exp\left\{-\frac{1}{2}\int_{t_0}^{t}\left(\dot{y}(\tau) - \frac{f\left(\varphi(y(\tau))) - \frac{1}{2}g'(\varphi(y(\tau)))g(\varphi(y(\tau))\right)}{g(\varphi(y(\tau)))}\right)^2 d\tau\right\},$$

where $y = \varphi^{-1}(x)$, $y_0 = \varphi^{-1}(x_0)$, $D[y] = \lim\limits_{N\to\infty}\prod_{i=1}^{N-1}\frac{dy_i}{\sqrt{2\pi\Delta t}}\frac{1}{\sqrt{2\pi g^2(\varphi(y))\Delta t}}$.

$Z[0]$ can be written in the form $Z[0] = \frac{1}{g(\varphi(y))}\overline{Z}[0]$, where

$$\overline{Z}[0] = \int D[y]\exp\left\{-\frac{1}{2}\int_{t_0}^{t}\left(\dot{y}(\tau) - \frac{f\left(\varphi(y(\tau))) - \frac{1}{2}g'(\varphi(y(\tau)))g(\varphi(y(\tau))\right)}{g(\varphi(y(\tau)))}\right)^2 d\tau\right\},$$
$$D[y] = \lim_{N\to\infty}\prod_{i=1}^{N-1}\frac{dy_i}{\sqrt{2\pi\Delta t}}\frac{1}{\sqrt{2\pi\Delta t}}. \tag{6}$$

3. COMPUTATION OF FUNCTIONAL INTEGRALS

Let us proceed to the method of computing functional integrals of form (6). The functional integral in formula (6) is an integral over functions or trajectories satisfying the conditions $y(t_0) = \varphi^{-1}(x_0)$ and $y(t) = \varphi^{-1}(x)$. The expression

$$\frac{1}{2}\left(\dot{y}(\tau) - \frac{f(\varphi(y(\tau))) - \frac{1}{2}g'(\phi(y(\tau)))g(\varphi(y(\tau)))}{g(\varphi(y(\tau)))}\right)^2$$

in formula (6) can be considered as the Lagrangian $L(\dot{y}, y, \tau)$ of the system, and the quantity $S = \int_{t_0}^{t} L(\dot{y}, y, \tau)d\tau$ can be considered as its action. Using the minimal action principle [6], we can distinguish among all trajectories the classical trajectory $y_{\rm cl}$ for which the action S takes the extreme value. The classical trajectory is found as the solution of the Euler–Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0.$$

Further, to compute the integral, we can use the decomposition of action S with respect to the classical trajectory $y_{\rm cl}$:

$$S[y(\tau)] \approx S[y_{\rm cl}(\tau)] + \frac{1}{2}\delta^2 S[y_{\rm cl}(\tau)].$$

The second order variation $\delta^2 S[y_{\rm cl}(\tau)]$ can be written in the form

$$\delta^2 S[y_{\rm cl}(\tau)] = \int_{t_0}^{t}\delta y \Lambda \delta y d\tau,$$

where $y = y_{\rm cl} + \delta y$,

$$\Lambda = \left(\frac{\partial^2 L}{\partial y^2}\right)_{y_{\rm cl}} + \left(\frac{\partial^2 L}{\partial y \partial \dot{y}}\right)_{y_{\rm cl}}\frac{d}{dt} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{y}\partial y}\right)_{y_{\rm cl}} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot{y}^2}\right)_{y_{\rm cl}}\frac{d}{dt}.$$

After these transformations integral (6) is written in the form

$$\overline{Z}[0] = \int D[x]\exp\left\{-S[y_{\mathrm{cl}}(\tau)] - \frac{1}{2}\int\limits_{t_0}^{t} x\Lambda x d\tau\right\}, \tag{7}$$

where integration is made over the trajectories $x = \delta y$ satisfying the conditions $x(t_0) = 0$ and $x(t) = 0$, and

$$D[x] = \lim_{N\to\infty}\prod_{i=1}^{N-1}\frac{dx_i}{\sqrt{2\pi\Delta t}}\frac{1}{\sqrt{2\pi\Delta t}}.$$

For computing integral (7) let us use the decomposition

$$x = \sum_{j=1}^{\infty} a_j u_j,$$

where the functions u_j are solutions of the Sturm–Liouville problem associated with the operator Λ, i.e.,

$$\Lambda u_j = \lambda_j u_j, \tag{8}$$

$\int_{t_0}^{t} u_j(\tau)u_i(\tau)d\tau = \delta_{ji}$, $u(t_0) = 0$, $u(t) = 0$, λ_j are the eigenvalues.

Then integral (7) will be written in the form

$$\overline{Z}[0] = \exp\{-S[y_{\mathrm{cl}}(\tau)]\}J\int D[a]\exp\left\{-\frac{1}{2}\sum_{j=1}^{\infty}\lambda_j a_j^2\right\}, \tag{9}$$

where J is the Jacobian of the transition from variable x to variable a:

$$D[a] = \lim_{N\to\infty}\prod_{i=1}^{N-1} da_i.$$

Denote

$$K(t_0, t) = J\int D[a]\exp\left\{-\frac{1}{2}\sum_{j=1}^{\infty}\lambda_j a_j^2\right\}. \tag{10}$$

Since the Jacobian J is invariant with respect to the operators Λ [6, 12], we have

$$K(t_0, t)\prod_{j=1}^{\infty}\lambda_j^{1/2} = K_{\mathrm{free}}(t_0, t)\prod_{j=1}^{\infty}\lambda_{\mathrm{free},j}^{1/2},$$

where

$$K_{\mathrm{free}}(t_0, t) = J\int D[a]\exp\left\{-\frac{1}{2}\sum_{j=1}^{\infty}\lambda_{\mathrm{free},j} a_j^2\right\},$$

and $\lambda_{\mathrm{free},j}$ are the eigenvalues of the Sturm–Liouville problem associated with the operator $\Lambda_{\mathrm{free}} = -d^2/dt^2$.

$$K_{\mathrm{free}}(t_0, t) = \int D[x]\exp\left\{-\frac{1}{2}\int\limits_{t_0}^{t} x\Lambda_{\mathrm{free}} x d\tau\right\}$$

$$= \lim_{N\to\infty}\int\ldots\int\prod_{i=1}^{N-1}\frac{dx_i}{\sqrt{2\pi\Delta t}}\frac{1}{\sqrt{2\pi\Delta t}}\exp\left\{-\sum_{i=0}^{N-1}\frac{(x_{i+1}-x_i)^2}{2\Delta t}\right\} = \frac{1}{\sqrt{2\pi(t-t_0)}}.$$

Therefore,

$$K(t_0,t) = K_{\text{free}}(t_0,t)\prod_{j=1}^{\infty}\frac{\lambda_{\text{free},j}^{1/2}}{\lambda_j^{1/2}} = \frac{1}{\sqrt{2\pi(t-t_0)}}\prod_{j=1}^{\infty}\frac{\lambda_{\text{free},j}^{1/2}}{\lambda_j^{1/2}}. \tag{11}$$

Thus, formulas (9)–(11) imply that

$$\overline{Z}[0] = \exp\{-S[y_{\text{cl}}(\tau)]\}\frac{1}{\sqrt{2\pi(t-t_0)}}\prod_{j=1}^{\infty}\frac{\lambda_{\text{free},j}^{1/2}}{\lambda_j^{1/2}}. \tag{12}$$

In our case

$$L(\dot{y},y,\tau) = \frac{1}{2}\left(\dot{y}(\tau) - \frac{f(\varphi(y(\tau))) - (1/2)g'(\varphi(y(\tau)))g(\varphi(y(\tau)))}{g(\varphi(y(\tau)))}\right)^2 = \frac{1}{2}(\dot{y}(\tau) - F(y(\tau)))^2.$$

The trajectory y_{cl} satisfies the equation

$$\ddot{y}_{\text{cl}}(\tau) - F(y_{\text{cl}}(\tau))F'(y_{\text{cl}}(\tau)) = 0. \tag{13}$$

Therefore,

$$\int_{t_0}^{t}\dot{y}_{\text{cl}}^2(\tau)d\tau = \dot{y}_{\text{cl}}(t)y_{\text{cl}}(t) - \dot{y}_{\text{cl}}(t_0)y_{\text{cl}}(t_0) - \int_{t_0}^{t}F(y_{\text{cl}}(\tau))F'(y_{\text{cl}}(\tau))y_{\text{cl}}(\tau)d\tau,$$

$$S[y_{\text{cl}}(\tau)] = \frac{1}{2}\int_{t_0}^{t}F(y_{\text{cl}}(\tau))(F(y_{\text{cl}}(\tau)) - F'(y_{\text{cl}}(\tau))y_{\text{cl}}(\tau))d\tau \tag{14}$$

$$+\frac{1}{2}(\dot{y}_{\text{cl}}(t)y_{\text{cl}}(t) - \dot{y}_{\text{cl}}(t_0)y_{\text{cl}}(t_0)) - \int_{t_0}^{t}F(y_{\text{cl}}(\tau))dy_{\text{cl}}(\tau).$$

Operator Λ in Eq. (8) is of the form

$$\Lambda = F'(y_{\text{cl}}(\tau))^2 + F(y_{\text{cl}}(\tau))F''(y_{\text{cl}}(\tau)) - d^2/dt^2. \tag{15}$$

For the approximate computation of eigenvalues λ_j of the operator Λ let us replace the functions $u_j(\tau)$ and $y_{\text{cl}}(\tau)$, $t_0 \le \tau \le t$, by the vectors $\mathbf{u}_j = (u_j(\Delta t), u_j(2\Delta t), \ldots, u_j((N-1)\Delta t))$, $\mathbf{y}_{\text{cl}} = (y_{\text{cl}}(\Delta t), y_{\text{cl}}(2\Delta t), \ldots, y_{\text{cl}}((N-1)\Delta t))$, $\Delta tN = t - t_0$, and let us replace the operator $\Lambda = F'(y_{\text{cl}}(\tau))^2 + F(y_{\text{cl}}(\tau))F''(y_{\text{cl}}(\tau)) - d^2/dt^2$ by the following matrix of dimension $(N-1)\times(N-1)$

$$\overline{\Lambda} = \frac{1}{\Delta t^2}\begin{pmatrix} 2+\Delta t^2 f_1 & -1 & 0 & \cdots & 0 \\ -1 & 2+\Delta t^2 f_2 & -1 & \cdots & 0 \\ 0 & -1 & 2+\Delta t^2 f_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2+\Delta t^2 f_{N-1} \end{pmatrix},$$

where $f_j = F'(y_{\text{cl}}(j\Delta t))^2 + F(y_{\text{cl}}(j\Delta t))F''(y_{\text{cl}}(j\Delta t))$, $1 \le j \le N-1$.

Let us replace the operator $\Lambda_{\text{free}} = -d^2/dt^2$ by the matrix of dimension $(N-1)\times(N-1)$

$$\overline{\Lambda}_{\text{free}} = \frac{1}{\Delta t^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix}. \tag{16}$$

For computing eigenvalues λ_j and $\lambda_{\text{free},j}$ of the matrices $\overline{\Lambda}$ and $\overline{\Lambda}_{\text{free}}$, we can use the Sturm sequences method [15]. We can compute $\prod_{j=1}^{N-1}\lambda_j = \det\overline{\Lambda}$ and $\prod_{j=1}^{N-1}\lambda_{\text{free},j} = \det\overline{\Lambda}_{\text{free}}$.

4. THE ANALOG OF THE COX–INGERSOLL–ROSS MODEL

As an example, let us consider the analog of the Cox–Ingersoll–Ross model described by the stochastic differential equation

$$dx(t) = (ax(t) + b)dt + \sigma\sqrt{x(t)}dw(t). \tag{17}$$

The TPDF corresponding to the stochastic differential equation (17), for a small time interval Δt, is

$$p(x, t+\Delta t, y, t) \approx \frac{1}{\sqrt{2\pi\sigma^2 y\Delta t}}\exp\left\{-\frac{\left(x - y - \left(ay + b - \frac{1}{2}\frac{\sigma^2}{2}\right)\Delta t\right)^2}{2\sigma^2 y\Delta t}\right\}. \tag{18}$$

The TPDF can be written in the form

$$\begin{aligned} p(x,t,x_0,t_0) = \lim_{N\to\infty}\int\ldots\int\prod_{i=1}^{N-1}dx_i\prod_{i=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2 x_i\Delta t}} \\ \times\exp\left\{-\sum_{i=0}^{N-1}\frac{(x_{i+1} - x_i - (ax_i + b - \sigma^2/4)\Delta t)^2}{2\sigma^2 x_i\Delta t}\right\}. \end{aligned} \tag{19}$$

Let us make the change of variables $x = \varphi(y) = \frac{\sigma^2}{4}y^2$ and $dx = \frac{\sigma^2}{2}ydy$. We obtain

$$\frac{dx}{\sqrt{2\pi\sigma^2 x\Delta t}} = \frac{dy}{\sqrt{2\pi\Delta t}}, \quad \frac{\left(\dot{x} - \left(ax + b - \sigma^2/4\right)\right)^2}{2\sigma^2 x} = \frac{1}{2}\left(\dot{y} - \left(\frac{ay}{2} + \frac{4b - \sigma^2}{2\sigma^2 y}\right)\right)^2.$$

Thus,

$$\begin{aligned} p(y,t,y_0,t_0) = \lim_{N\to\infty}\int\ldots\int\prod_{i=1}^{N-1}\frac{dy_i}{\sqrt{2\pi\Delta t}}\frac{1}{\sqrt{2\pi\sigma^2\varphi(y)\Delta t}} \\ \times\exp\left\{-\frac{1}{2}\sum_{i=0}^{N-1}\Delta t\left(\frac{y_{i+1} - y_i}{\Delta t} - \left(\frac{ay_i}{2} + \frac{4b - \sigma^2}{2\sigma^2 y_i}\right)\right)^2\right\}, \end{aligned} \tag{20}$$

where $y = \varphi^{-1}(x) = 2\sqrt{x}/\sigma$ and $y_0 = \varphi^{-1}(x_0) = 2\sqrt{x_0}/\sigma$.

Consider the case $\sigma = 2\sqrt{b}$. Expression (20) can be written out in the form of a functional integral with respect to the Wiener measure $dw(y)$:

$$p(x,t,x_0,t_0) = \frac{1}{\sigma\sqrt{\varphi(y)}}\exp\left\{\frac{a}{4}(y^2(t) - y^2(t_0))\right\}\int\exp\left\{-\frac{a^2}{8}\int_{t_0}^{t}y^2(\tau)d\tau\right\}dw(y).$$

For computing integrals with respect to the Wiener measure, we can use the methods worked out in [10, 16, 17].

Following the scheme proposed above, let us write the integral in (20) in the form

$$p(y,t,y_0,t_0) = Z[0] = \frac{1}{\sigma\sqrt{\varphi(y)}}\overline{Z}[0] = \frac{1}{\sigma\sqrt{\varphi(y)}}\int D[y]\exp\left\{-\frac{1}{2}\int_{t_0}^{t}\left(\dot{y}(\tau) - \frac{a}{2}y(\tau)\right)^2 d\tau\right\}, \tag{21}$$

where $D[y] = \lim_{N\to\infty}\prod_{i=1}^{N-1}\frac{dy_i}{\sqrt{2\pi\Delta t}}\frac{1}{\sqrt{2\pi\Delta t}}$.

In this case the Lagrangian is

$$L(\dot{y}, y, \tau) = \frac{1}{2}\left(\dot{y}(\tau) - \frac{a}{2}y(\tau)\right)^2.$$

Equation (13) implies that the Euler–Lagrange equation for the classical trajectory is

$$\ddot{y}_{\mathrm{cl}}(\tau) - (a/2)^2 y_{\mathrm{cl}}(\tau) = 0. \tag{22}$$

Equation (14) implies that

$$S[y_{\mathrm{cl}}(\tau)] = \frac{1}{2}(\dot{y}_{\mathrm{cl}}(t)y_{\mathrm{cl}}(t) - \dot{y}_{\mathrm{cl}}(t_0)y_{\mathrm{cl}}(t_0)) - \int_{t_0}^{t}\frac{a}{2}y_{\mathrm{cl}}(\tau)dy_{\mathrm{cl}}(\tau)$$

$$= \frac{1}{2}(\dot{y}_{\mathrm{cl}}(t)y_{\mathrm{cl}}(t) - \dot{y}_{\mathrm{cl}}(t_0)y_{\mathrm{cl}}(t_0)) - \frac{a}{4}(y_{\mathrm{cl}}^2(t) - y_{\mathrm{cl}}^2(t_0))$$

$$= \frac{1}{2}(\dot{y}_{\mathrm{cl}}(t)y - \dot{y}_{\mathrm{cl}}(t_0)y_0) - \frac{a}{4}(y^2 - y_0^2).$$

From this equality and from (12), we obtain the following expression for $\overline{Z}[0]$,

$$\overline{Z}[0] = \exp\left\{-\frac{1}{2}(\dot{y}_{\mathrm{cl}}(t)y - \dot{y}_{\mathrm{cl}}(t_0)y_0) + \frac{a}{4}(y^2 - y_0^2)\right\}\frac{1}{\sqrt{2\pi(t-t_0)}}\prod_{j=1}^{N-1}\frac{\lambda_{\mathrm{free},j}^{1/2}}{\lambda_j^{1/2}},$$

where λ_j and $\lambda_{\mathrm{free},j}$ are the eigenvalues of the matrices $\overline{\Lambda}$ and $\overline{\Lambda}_{\mathrm{free}}$ of dimension $(N-1)\times(N-1)$.

Equation (15) implies that

$$\Lambda = a^2/4 - d^2/dt^2,$$

and matrix $\overline{\Lambda}$ is

$$\overline{\Lambda} = \frac{1}{\Delta t^2}\begin{pmatrix} 2+\Delta t^2\omega^2 & -1 & 0 & \cdots & 0 \\ -1 & 2+\Delta t^2\omega^2 & -1 & \cdots & 0 \\ 0 & -1 & 2+\Delta t^2\omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2+\Delta t^2\omega^2 \end{pmatrix},$$

where $\omega = a/2$ and $\Delta t N = t - t_0$. Matrix $\overline{\Lambda}_{\mathrm{free}}$ is determined by equality (16).

Computing determinants of the matrices $\overline{\Lambda}$ and $\overline{\Lambda}_{\mathrm{free}}$, we obtain that for $t_0 = 0$, $t = 1$, $\omega = 1$,

$$\text{and } N = 20, \quad \frac{1}{\sqrt{2\pi(t-t_0)}}\prod_{j=1}^{N-1}\frac{\lambda_{\mathrm{free},j}^{1/2}}{\lambda_j^{1/2}} = 0.36808.$$

In this case the function $y_{\mathrm{cl}}(\tau)$, $t_0 \le \tau \le t$, can be found precisely, since we can find analytically the solution of Eq. (22). This solution is

$$y_{\mathrm{cl}}(\tau) = \frac{1}{\sinh(\omega(t-t_0))}(x_0\sinh(\omega(t-\tau)) + x\sinh(\omega(\tau - t_0))).$$

However, since in the general case, we cannot analytically find the solution of the Euler–Lagrange equation, let us compute function $y_{\mathrm{cl}}(\tau)$ approximately. We find the approximate values of function $y_{\mathrm{cl}}(\tau)$, $t_0 \le \tau \le t$, solving Eq. (22) by the net method for the solution of nonlinear boundary problems [18].

Table 1. Approximate and exact values of $\overline{Z}[0]$: $a = 2, \sigma = 1, b = 1/4, t_0 = 0, t = 1, y_0 = 0$

y	−3	−2	−1	−0.6	−0.3	0	0.3	0.6	1	2	3
$\overline{Z}[0]_\Pi$	0.093	0.199	0.316	0.348	0.363	0.368	0.363	0.348	0.316	0.199	0.093
$\overline{Z}[0]_T$	0.090	0.196	0.315	0.348	0.363	0.368	0.363	0.348	0.315	0.196	0.090

Table 2. Approximate and exact values of $\overline{Z}[0]$: $a = 2, \sigma = 1, b = 1/4, t_0 = 0, t = 1, y_0 = 1$

y	$e-3$	$e-2$	$e-1$	$e-0.6$	$e-0.3$	e	$e+0.3$	$e+0.6$	$e+1$	$e+2$	$e+3$
$\overline{Z}[0]_\Pi$	0.090	0.198	0.320	0.356	0.374	0.381	0.379	0.359	0.327	0.209	0.098
$\overline{Z}[0]_T$	0.090	0.197	0.315	0.348	0.363	0.368	0.363	0.348	0.315	0.197	0.090

Using the obtained approximate values for $\frac{1}{\sqrt{2\pi(t-t_0)}}\prod_{j=1}^{N-1}\frac{\lambda_{\text{free},j}^{1/2}}{\lambda_j^{1/2}}$ and $y_{\text{cl}}(\tau)$, we find the approximate values for $\overline{Z}[0] = Z[0]\sigma^2 y/2$. Tables 1 and 2 give the approximate and exact values of $\overline{Z}[0]$ for variable y related to variable x by the equality $x = \varphi(y) = \sigma^2 y^2/4$. The exact values of $\overline{Z}[0]$ are obtained from the explicit expression

$$\overline{Z}[0] = \frac{\sqrt{\omega}}{\sqrt{2\pi\sinh(\omega(t-t_0))}}\exp\left\{\omega\frac{x^2-x_0^2}{2} + \omega\frac{4x_0x - 2\cosh(\omega(t-t_0))(x_0^2+x^2)}{4\sinh(\omega(t-t_0))}\right\}. \tag{23}$$

Expression (23) is obtained from the explicit expression for the kernel of the operator $\exp\left\{-t\frac{1}{2}\left(\left(\frac{d}{dt}\right)^2 + x^2 - 1\right)\right\}$, given by the Mehler formula [9]

$$p_t(x_0, x) = \frac{e^{t/2}}{\sqrt{2\pi\sinh(t)}}\exp\left\{\frac{4x_0x - 2\cosh(t)(x_0^2+x^2)}{4\sinh(t)}\right\}.$$

Let us consider the approximate computation of the mathematical expectation of the solution of Eq. (17) by the obtained approximate values for $\overline{Z}[0]$, taking into account the normalizing factor $\exp(-\omega(t-t_0)/2)$.

From (19)–(21) we obtain that approximate values of the mathematical expectation are computed by the formula

$$Mx = \lim_{N\to\infty}\int\ldots\int x_N\prod_{i=1}^{N}dx_i\prod_{i=1}^{N}\frac{1}{\sqrt{2\pi\sigma^2x_i\Delta t}}$$
$$\times\exp\left\{-\sum_{i=0}^{N-1}\frac{(x_{i+1}-x_i-ax_i\Delta t)^2}{2\sigma^2x_i\Delta t}\right\}\exp\left\{-\frac{\omega(t-t_0)}{2}\right\}$$
$$= \lim_{N\to\infty}\int\ldots\int\frac{\sigma^2}{4}y_N^2\frac{1}{\sigma\sqrt{\frac{\sigma^2}{4}y_N^2}}\frac{\sigma^2}{2}y_N\prod_{i=1}^{N}\frac{dy_i}{\sqrt{2\pi\Delta t}}$$
$$\times\exp\left\{-\frac{1}{2}\sum_{i=0}^{N-1}\Delta t\left(\frac{y_{i+1}-y_i}{\Delta t}-\frac{ay_i}{2}\right)^2\right\}\exp\left\{-\frac{\omega(t-t_0)}{2}\right\}$$
$$= \exp\left\{-\frac{\omega(t-t_0)}{2}\right\}\int\frac{\sigma^2}{4}y^2\overline{Z}[0]dy.$$

The exact value of the mathematical expectation is computed from the equation

$$dMx(t) = \left(aMx(t) + \sigma^2/4\right)dt, \quad Mx(0) = x_0,$$

and equals

$$Mx(t) = x_0 \exp\{a(t - t_0)\} + \frac{\sigma^2}{4a}(\exp\{a(t - t_0)\} - 1).$$

For $a = 2$, $\sigma = 1$, $b = 1/4$, $t_0 = 0$, $t = 1$, and $x_0 = 0$, the approximate value of the mathematical expectation equals 0.824, and the exact value equals 0.799. For $a = 2$, $\sigma = 1$, $b = 1/4$, $t_0 = 0$, $t = 1$, and $x_0 = 1/4$, the approximate value of the mathematical expectation equals 2.81, and the exact value equals 2.65.

5. CONCLUSIONS

There are many mathematical models in natural science giving rise to stochastic differential equations (SDEs). One of the most frequently occurring problems, in practice, in constructing models of this kind is introducing a dynamical model, described by differential equations, of the random fluctuations caused by the presence of noise in a system (as, instance.g., in radio engineering, thermodynamics, and kinetics of chemical reactions). Another source of models based on SDEs is taking a limit in the description of a system (instance.g., models of gene diffusion and a diffusion approximation of mass service systems).

Stochastic differential equations are often used to describe the stochastic behavior of a system; however, as a rule, the stochastic term of the equation is represented as an exterior action on the system [21]. From the point of view of mathematical modeling, one of the most fundamental problems is constructing a stochastic differential equation for the modeled system such that the stochastic part is related to the structure of the system [1, 2]. A possible solution of this problem is obtaining the stochastic and deterministic parts from the same initial equation. For these purposes it is convenient to use the main kinetic equation.

The systems in which time evolves because of the interaction of their elements can be conveniently described using the main kinetic equation (another name is the managing equation [19], and in the English language literature, it is called the Master equation [5]). Thus, it raises the question on how to describe the studied system described by one-step processes by a stochastic differential equation in the form of the Langevene equation from the main kinetic equation. Hence, to solve this problem, we assume to approximate the main kinetic equation by the Fokker–Planck equation, for which we can write out the equivalent to it stochastic differential equation in the form of the Langevene equation.

In the papers of some of the authors [1, 2, 20], the problem of constructing SDEs whose stochastic term is related with the structure of the studied system is partially solved. Now we have a manuscript concluding this research. Representing a TPDF to solve the obtained SDE through a functional integral, as well as methods for the approximate computation of this functional integral, will be considered in a subsequent publication.

ACKNOWLEDGMENTS

This research was partially supported by Russian Foundation for Basic Research grants nos. 15-07-08795 and 16-07-00556, also by the Ministry of Education and Science of the Russian Federation (the Agreement No 02.A03.21.0008), and also by a grant of the Belarussian Republic's Fund for Basic Research (project F14D-002).

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Translated by A. Stoyanovsky