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Constrained Hamiltonian approach to the Maxwell theory

Dmitry S. Kulyabov[a,b], Anna V. Korolkova[a], Migran N. Gevorkyan[a], and Leonid A. Sevastianov[a,c]

[a]Department of Applied Probability and Informatics,
Peoples' Friendship University of Russia (RUDN University),
6 Miklukho-Maklaya st., Moscow, 117198, Russian Federation
[b]Laboratory of Information Technologies
Joint Institute for Nuclear Research
6 Joliot-Curie, Dubna, Moscow region, 141980, Russian Federation
[c]Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research
6 Joliot-Curie, Dubna, Moscow region, 141980, Russian Federation

ABSTRACT

The most common physical formalisms are the Lagrangian formalism and the Hamiltonian formalism. From the superficial point of view, they are one and the same, but rewritten in other terms. However, it seems that the Hamiltonian formalism has a richer structure and is more convenient for studying the electromagnetic field, especially in the formalization of its geometrization. Unfortunately for field problems, there is a whole set of Hamiltonian formalisms. The authors study the applicability of different variants of the Hamiltonian formalism to the problems of electrodynamics. In this paper we consider the Hamiltonian formalism with constraints.

Keywords: Lagrangian formalism, Hamiltonian formalism, Hamiltonian formalism with constraints, Maxwell equations

1. INTRODUCTION

In the study of electromagnetic and optical phenomena Hamiltonian formalism is often used. The main drawback of Hamiltonian formalism it seems to be poorly developed for field systems. For descriptions of field systems, we propose several variants of the Hamiltonian formalism. Previously, we considered the possibility of building symplectic[1] and multipulse[2] Hamiltonian formalism. In this paper we consider the construction of a Hamiltonian formalism with constraints.

2. NOTATIONS AND CONVENTIONS

1. We will adhere to the following agreements. Greek indices (α, β) will refer to the four-dimensional space. Latin indices from the middle of the alphabet (i, j, k) will refer to the three-dimensional space.

2. In the theoretical description, Latin indices will refer to the space of arbitrary dimension.

3. The comma in the index denotes a partial derivative with respect to corresponding coordinate ($f_{,i} := \partial_i f$); the semicolon denotes a covariant derivative ($f_{;i} := \nabla_i f$).

4. The CGS symmetrical system[3] is used for notation of the equations of electrodynamics.

Further author information: (Send correspondence to Dmitry S. Kulyabov)
Dmitry S. Kulyabov: E-mail: kulyabov-ds@rudn.ru
Anna V. Korolkova: E-mail: korolkova-av@rudn.ru
Migran N. Gevorkyan: E-mail: gevorkyan-mn@rudn.ru
Leonid A. Sevastianov: E-mail: sevastianov-la@rudn.ru

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3. HAMILTONIAN FORMALISM

There are several variants of the Hamiltonian formalism.

- The symplectic Hamiltonian formalism.[1,4]
- The Dirac–Bergman Hamiltonian formalism for systems with constraints.[5,6]
- The Hamilton–De Donder Hamiltonian formalism.[7]
- The multimomentum Hamiltonian formalism.[7–9]

Consider major points of the Hamiltonian formalism.

Let the system be described by some quantity called the action:

$$S[q^i] = \int \mathrm{d}^4 \mathcal{L}(x^i, q^i, \dot{q}^i).$$

Lagrangian (Lagrangian density) $(x^I, q^i, \dot{q}^i)$ depends on the generalized coordinates q and their first derivatives of $\dot{q}$.

In the transition to the Hamiltonian formalism the system is described by generalized coordinates q^I and generalized momentum:

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}. \tag{1}$$

One can construct the Legendre transform in velocities:

$$\mathcal{H}(q^i, p_i) := p_i \dot{q}^i - \mathcal{L}(q^i, \dot{q}^i).$$

The function $\mathcal{H}$ is called a Hamiltonian (Hamiltonian density). The Hamiltonian of the system depends only on generalized coordinates and momenta.

Using the connection between Hamiltonian and Lagrangian, we consider the action:

$$S[q^i, p_i] = \int \mathrm{d}^4 \big(p_i \dot{q}^i - \mathcal{H}(q^i, p_i) \big).$$

The corresponding system of Euler–Lagrange equations has the form:

$$\begin{aligned} \frac{\delta S}{\delta q^i} &= -\dot{p}_i - \frac{\delta \mathcal{H}}{\delta q^i} = 0, \\ \frac{\delta S}{\delta p_i} &= \dot{q}^i - \frac{\delta \mathcal{H}}{\delta p_i} = 0. \end{aligned} \tag{2}$$

Let us define the Poisson brackets:

$$[f, g] := \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}.$$

Then we can rewrite equation (2) as:

$$\begin{aligned} \dot{q}^i &= \big[q^i, \mathcal{H} \big] = \frac{\delta \mathcal{H}}{\delta p_i}, \\ \dot{p}_i &= [p_i, \mathcal{H}] = -\frac{\delta \mathcal{H}}{\delta q^i}. \end{aligned}$$

4. HAMILTONIAN DYNAMICS WITH CONSTRAINTS

If the Lagrangian is singular by velocities:

$$\det\left[\frac{\partial^2\mathcal{L}}{\partial\dot{q}^i\partial\dot{q}^j}\right]=0,$$

it is not possible to express all momentum by formula (1). In this case one gets only possible momentum and for the rest the concept of constraints[5] is used.

We will consider the system with Lagrangian $\mathcal{L}(x^k,q^k,p_k)$, $I=\overline{1,n}$. Also consider set of constraints:

$$\varphi^a(x^k,q^k,\dot{q}^k),\quad a=\overline{1,m},\quad m\leqslant n.$$

Action minimum, in case the trajectory satisfy the equations of connection, is interpret also on trajectories without constraints, but with Lagrangian with constraints:

$$L(x^k,q^k,\dot{q}^k)=\mathcal{L}(x^k,q^k,\dot{q}^k)-\lambda_a(x^k,q^k,\dot{q}^k)\varphi^a(x^k,q^k,\dot{q}^k).$$

Hamilton's equations take the following form:

$$\dot{q}^i=\frac{\delta\mathcal{H}}{\delta p_i}+\lambda_a\frac{\partial\varphi^a}{\partial p_i},$$
$$\dot{p}_i=-\frac{\delta\mathcal{H}}{\delta q^i}-\lambda_a\frac{\partial\varphi^a}{\partial q^i}.$$

Lagrange multipliers are found from the condition of preserving constraints:

$$\dot{\varphi^a}=[\varphi^a,\mathcal{H}]+\lambda_a\varphi^a=0.$$

5. THE HAMILTONIAN OF THE ELECTROMAGNETIC FIELD WITH CONSTRAINTS

We consider the construction of a Hamiltonian formalism with constraints for the case electromagnetic field.

Write the Lagrangian of the electromagnetic field:[10]

$$\mathcal{L}(x^\alpha,A_\beta,A_{\alpha,\beta})=-\frac{1}{16\pi c}F_{\alpha\beta}F^{\alpha\beta}\sqrt{-g}-\frac{1}{c^2}A_\alpha j^\alpha\sqrt{-g}.$$

Since $F_{00}=0$,

$$\frac{\partial^2\mathcal{L}}{\partial\dot{A}_0^2}=0.$$

That is, the Lagrangian is irregular. Therefore, we need to find constraints for the case p^0. Because

$$p^0=\frac{\partial\mathcal{L}}{\partial\dot{A}_0}=0,$$

then enter the constraints

$$\varphi:=p^0\approx 0. \tag{3}$$

The $\approx$ symbol indicates that the equality must be performed on surfaces of all constraints.

Let's write down expressions for momentum:

$$p_i=\frac{\partial\mathcal{L}}{\partial\dot{A}^i}=\frac{1}{c}\frac{\partial\mathcal{L}}{\partial A^i_{,0}}=-\frac{\sqrt{-g}}{4\pi c^2}[A_{i,0}-A_{0,i}].$$

Let us express time derivatives of A_i by momentum:

$$\dot{A}_i = cA_{i,0} = -\frac{4\pi c^3}{\sqrt{-g}} p_i + cA_{0,i}.$$

Let us construct the Hamiltonian

$$H = p^i \dot{A}_i - \mathcal{L} + \lambda p^0 = p^i \left(-\frac{4\pi c^3}{\sqrt{-g}} p_i + cA_{0,i} \right) + \frac{\sqrt{-g}}{16\pi c} F_{\alpha\beta} F^{\alpha\beta} + \frac{\sqrt{-g}}{c^2} A_\alpha j^\alpha + \lambda p^0$$

Hamilton's equations take the following form:

$$\dot{A}_0 = \frac{\partial H}{\partial p^0} = \lambda,$$

$$\dot{A}_i = \frac{\partial H}{\partial p^i} = -\frac{4\pi c^3}{\sqrt{-g}} p_i + cA_{0,i}, \tag{4}$$

$$\dot{p}^0 = -\frac{\delta H}{\delta A_0} = -\frac{\partial H}{\partial A_0} + \partial_\alpha \frac{\partial H}{\partial A_{0,\alpha}} = -cp^i_{,i} + \frac{\sqrt{-g}}{c^2} j^0, \tag{5}$$

$$\dot{p}^i = -\frac{\delta H}{\delta A_i} = -\frac{\partial H}{\partial A_i} + \partial_\alpha \frac{\partial H}{\partial A_{i,\alpha}} = -\frac{\sqrt{-g}}{c^2} j^i + \frac{\sqrt{-g}}{16\pi c} F^{\alpha i}_{,\alpha}. \tag{6}$$

We show that the resulting system of equations is equivalent to Maxwell equations (which seems obvious).

From the equation (4) we obtain

$$p_i = \frac{\sqrt{-g}}{4\pi c^3} \left(cA_{0,i} - \dot{A}_i \right) = \frac{\sqrt{-g}}{4\pi c^2} (A_{0,i} - A_{i,0}) = \frac{\sqrt{-g}}{4\pi c^2} F_{i0}. \tag{7}$$

We raise the indices in the equation (7) and substitute the result in the equation (6). Writing the tensor components $F^{\alpha\beta}$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H_3 & H_2 \\ D^2 & H_3 & 0 & -H_1 \\ D^3 & -H_2 & H_1 & 0 \end{pmatrix},$$

we get the equation

$$\frac{1}{\sqrt{^3 g}} [\partial_j H_k - \partial_k H_j] = -\frac{1}{c} \partial_t D^i + \frac{4\pi}{c} j^i.$$

Similarly from (5) taking into account equation (7) and connection (3) we obtain another Maxwell equation.

$$\frac{1}{\sqrt{^3 g}} \partial_i \left(\sqrt{^3 g} D^i \right) = 4\pi\rho.$$

Thus, we showed that the resulting Hamiltonian gives inhomogeneous Maxwell equations. It is obvious that the homogeneous Maxwell equations is not obtained from the Hamiltonian and from the Bianchi identities for tensor $F_{\alpha\beta}$.

6. CONCLUSION

Полученные уравнения The authors applied the Dirac–Bergman method to the Maxwell equations. Obtained equations is equivalent to inhomogeneous Maxwell equations. The main feature of our approach is that Maxwell's equations is considered in covariant form and in an arbitrary Riemannian coordinates.

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REFERENCES

1. D. S. Kulyabov, A. V. Korolkova, L. A. Sevastianov, E. G. Eferina, T. R. Velieva, and I. S. Zaryadov, "Maxwell's Equations Instantaneous Hamiltonian," in *Proceedings of SPIE. Saratov Fall Meeting 2016: Laser Physics and Photonics XVII and Computational Biophysics and Analysis of Biomedical Data III*, V. L. Derbov and D. E. Postnov, eds., *Proceedings of SPIE - The International Society for Optical Engineering* **10337**, pp. 103370L.1–9, SPIE, 2017.
2. D. S. Kulyabov, A. V. Korolkova, L. A. Sevastianov, E. G. Eferina, and T. R. Velieva, "A Geometric Approach to the Lagrangian and Hamiltonian Formalism of Electrodynamics," in *Proceedings of SPIE. Saratov Fall Meeting 2016: Laser Physics and Photonics XVII and Computational Biophysics and Analysis of Biomedical Data III*, V. L. Derbov and D. E. Postnov, eds., *Proceedings of SPIE - The International Society for Optical Engineering* **10337**, pp. 103370M.1–6, SPIE, 2017.
3. D. V. Sivukhin, "The international system of physical units," *Soviet Physics Uspekhi* **22**, pp. 834–836, oct 1979.
4. G. Vilasi, *Hamiltonian Dynamics*, World Scientific Pub Co Inc, 2001.
5. P. A. M. Dirac, *Lectures on Quantum Mechanics*, Courier Corporation, 2001.
6. E. Newman and P. G. Bergmann, "Lagrangians Linear in the "Velocities"," *Physical Review* **99**, pp. 587–592, jul 1955.
7. G. A. Sardanashvily, *Generalized Hamiltonian Formalism for Field Theory*, World Scientific Publishing Company, Singapore, 1995.
8. G. Giachetta, L. Mangiarotti, and G. A. Sardanashvily, "Covariant Hamilton Equations for Field Theory," *J Phys A* **32**, pp. 6629–6642, sep 1999.
9. G. Giachetta, L. Mangiarotti, and G. A. Sardanashvily, *Advanced Classical Field Theory*, World Scientific Publishing Company, Singapore, 2009.
10. J. A. Stratton, *Electromagnetic Theory*, MGH, 1941.