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The Riemannian geometry is not sufficient for the geometrization of the Maxwell's equations

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ABSTRACT

The transformation optics uses geometrized Maxwell's constitutive equations to solve the inverse problem of optics, namely to solve the problem of finding the parameters of the medium along the paths of the electromagnetic field propagation. The quadratic Riemannian geometry is usually used for the geometrization of Maxwell's constitutive equations, because of the usage of the general relativity approaches. However, the problem of the insufficiency of the Riemannian structure for describing the constitutive tensor of the Maxwell's equations arises. The authors analyze the structure of the constitutive tensor and correlate it with the structure of the metric tensor of Riemannian geometry. It was concluded that the use of the quadratic metric for the geometrization of Maxwell's equations is insufficient, since the number of components of the metric tensor is less than the number of components of the constitutive tensor. The possible solution to this problem may be a transition to Finslerian geometry, in particular, the use of the Berwald-Moor metric to establish the structural correspondence between the field tensors of the electromagnetic field.

Keywords: Maxwell's equations, permeability tensor, Riemannian geometry, Finsler geometry, transformation optics

1. INTRODUCTION

The problem of geometrization of Maxwell's equations arose from the interest in Einstein's general relativity as an element of the unified field theory. This scientific direction was divided into two parts: the geometrization of Maxwell's field equations and the geometrization of Maxwell's constitutive equations.[1–6] Attempts to geometrize the field equations merged later into the gauge field approach. The geometrization of the constitutive equations for a long time did not find application and was almost forgotten. Interest in geometrization was woken up by the study of metamaterials[7,8] and led to the appearance of transformation optics.

The main motivation for using the geometric approach to Maxwell's equations is that the inverse problem of optics becomes the direct problem of geometrized optics. The inverse problem of optics is the problem of calculating the parameters of the medium from the known paths of the electromagnetic waves propagation.

Transformation optics uses the geometrization of Maxwell's constitutive equations on the basis of Riemannian geometry from the general relativity. However, this formalism is applicable only for a small range of problems, when $\varepsilon_{ij} = \mu_{ij}$, that is, only impedance matched medium are investigated and the parameters of the medium can be obtained only in the form of the refractive index n_{ij}.

In the general case, when solving the inverse problem of optics, we need to obtain independent permittivity and magnetic permeability.

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The structure of the paper is as follows. In section 2 we provide the basic notation and conventions used in the article. In the section 3 the structure of Maxwell's and Minkowski's tensors is described in the framework of the fiber bundles. Also the structure of the constitutive tensor is shown. In the section 4 the geometrization of Maxwell's constitutive equations is described on the basis of the Yang–Mills Lagrangian. We also did a comparison of this geometrization with the Plebanski's geometrization. The conclusion is drawn that the Riemannian geometry is not sufficient for the geometrization of Maxwell's equations.

2. NOTATIONS AND CONVENTIONS

1. We will use the notation of abstract indices.[9] In this notation tensor as a complete object is denoted merely by an index (e.g., x^i). Its components are designated by underlined indices (e.g., $x^{\underline{i}}$).
2. We will adhere to the following agreements. Greek indices (α, β) will refer to the four-dimensional space, in the component form it looks like: $\underline{\alpha} = \overline{0,3}$. Latin indices from the middle of the alphabet (i, j, k) will refer to the three-dimensional space, in the component form it looks like: $\underline{i} = \overline{1,3}$.
3. The CGS symmetrical system[10] is used for notation of the equations of electrodynamics.

3. THE CONSTITUTIVE TENSOR

In what follows, we will rely on the fiber bundles theory.[11] The Maxwell's tensor $F_{\alpha\beta}$ is an element of the cotangent bundle T^*M, i.e. a 2-form, and the Minkowski's tensor $G^{\alpha\beta}$ is an element of the tangent bundle TM, that is a bivector. We will consider the case of Riemannian geometry, so the connection between tangents and cotangent layers is set by effective metric $g_{\alpha\beta}$ on the basis of the bundle:

$$\begin{gathered}
F = \frac{1}{2} F_{\underline{\alpha}\,\underline{\beta}} \, \mathrm{d}x^{\underline{\alpha}} \wedge \mathrm{d}x^{\underline{\beta}} \,, \quad F \in \Lambda^2, \\
G = \frac{1}{2} G^{\underline{\alpha}\,\underline{\beta}} \, \partial_{\underline{\alpha}} \wedge \partial_{\underline{\beta}} \,, \quad G \in \Lambda_2, \\
j = j^{\underline{\alpha}} \, \partial_{\underline{\alpha}} \,, \quad j \in \Lambda_1.
\end{gathered}$$

Here Λ^2 is the space of 2-forms, Λ_2 is the space of bivectors, Λ_1 is the space of vectors, j^α is the current.

The tensors $F_{\alpha\beta}$ and $G^{\alpha\beta}$ have the sense of curvature in the cotangent T^*M and the tangent TM bundles. The connection between these quantities can be defined by means of some functional λ:

$$G^{\alpha\beta} = \lambda(F_{\gamma\delta}).$$

To clarify the relationship between the tensors $F_{\alpha\beta}$ and $G^{\alpha\beta}$, we will write the Maxwell equations by using the exterior calculus formal description:

$$\begin{gathered}
\mathrm{d}F = 0, \\
\delta G = \frac{4\pi}{c} j,
\end{gathered} \tag{1}$$

where c is the speed of light.

The Riemannian metric is usually explicitly used in the definition of the Hodge duality operator, so we will write down the divergence δ not through the Hodge duality operator:

$$\begin{gathered}
* : \Lambda^k \to \Lambda^{n-k}, \\
\delta = (-1)^k *^{-1} \mathrm{d} *,
\end{gathered}$$

but through the Poincar duality operator:

$$\begin{gathered}
\sharp : \Lambda^k \to \Lambda_{n-k}, \\
\delta = (-1)^k \sharp^{-1} \mathrm{d} \, \sharp.
\end{gathered}$$

Let's write the constitutive equations as follow:

$$G = \lambda(F).$$

Then the equation (1) takes the form:

$$\mathrm{d}\sharp\lambda(F) = \frac{4\pi}{c}\sharp j. \tag{2}$$

In addition, let us obtain the Hodge duality operator without an explicit metric specification. For this we define the isomorphism:

$$\begin{aligned} * &: \Lambda^2 \to \Lambda^2, \\ * &: F \mapsto \sharp\lambda(F). \end{aligned} \tag{3}$$

Then the equation (2) takes the form:

$$\mathrm{d}^* F = \frac{4\pi}{c}\sharp j,$$

and the operator (3) is the Hodge duality operator, defined not via the Riemannian metric, but through the functional λ.

By the virtue of the fact that the most practical problems consider linear media, for simplicity we will make the following assumptions. We will assume that the mapping $\lambda : \Lambda^2 \to \Lambda_2$ is a linear (the connection can be defined by means of tensors) and a local map (all tensors are considered at the same point). Then it can be represented in the following:[?]

$$G^{\alpha\beta} = \lambda^{\alpha\beta\gamma\delta} F_{\gamma\delta}, \tag{4}$$

here $\lambda^{\alpha\beta\gamma\delta}$ is constitutive tensor that contains information about the permittivity and the permeability, as well as about electromagnetic constitutive relations in Maxwell's equations.[2,4,12]

A nonlinear nonlocal case in the presence of translational symmetry is reduced to a linear local case by means of the Fourier transform. We may write the nonlocal linear relation between F and G as follows:

$$G(x) = \int \lambda(x,s) \wedge F(s)\,\mathrm{d}s\,, \quad x, s \in M. \tag{5}$$

Then, assuming the existence of translational invariance $\lambda(x,s) = \lambda(x-s)$, we may write the connection between F and G in equation (5):

$$G^{\alpha\beta}(\omega, k_i) = \lambda^{\alpha\beta\gamma\delta}(\omega, k_i) F_{\gamma\delta}(\omega, k_i).$$

From (4) may be seen that $\lambda^{\alpha\beta\gamma\delta}$ has the following symmetry:

$$\lambda^{\alpha\beta\gamma\delta} = \lambda^{[\alpha\beta][\gamma\delta]}.$$

To refine the symmetry, the tensor $\lambda^{\alpha\beta\gamma\delta}$ can be represented in the following form:[13–16]

$$\begin{gathered} \lambda^{\alpha\beta\gamma\delta} = {}^{(1)}\lambda^{\alpha\beta\gamma\delta} + {}^{(2)}\lambda^{\alpha\beta\gamma\delta} + {}^{(3)}\lambda^{\alpha\beta\gamma\delta}, \\ {}^{(1)}\lambda^{\alpha\beta\gamma\delta} = {}^{(1)}\lambda^{([\alpha\beta][\gamma\delta])}, \\ {}^{(2)}\lambda^{\alpha\beta\gamma\delta} = {}^{(2)}\lambda^{[[\alpha\beta][\gamma\delta]]}, \\ {}^{(3)}\lambda^{\alpha\beta\gamma\delta} = {}^{(3)}\lambda^{[\alpha\beta\gamma\delta]}. \end{gathered}$$

Let's write out the number of independent components:

- $\lambda^{\alpha\beta\gamma\delta}$ has 36 independent components,
- ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$ has 20 independent components,

- ${}^{(2)}\lambda^{\alpha\beta\gamma\delta}$ has 15 independent components,
- ${}^{(3)}\lambda^{\alpha\beta\gamma\delta}$ has 1 independent component.

Usually only part ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$ is considered, since ${}^{(2)}\lambda^{\alpha\beta\gamma\delta}$ and ${}^{(3)}\lambda^{\alpha\beta\gamma\delta}$ make it impossible to record the electromagnetic field Lagrangian:

$$L = -\frac{1}{16\pi c} F_{\alpha\beta} G^{\alpha\beta} \sqrt{-g} - \frac{1}{c^2} A_\alpha j^\alpha \sqrt{-g}. \tag{6}$$

That is, when we use parts ${}^{(2)}\lambda^{\alpha\beta\gamma\delta}$ and ${}^{(3)}\lambda^{\alpha\beta\gamma\delta}$, the tensor $F_{\alpha\beta}$ must be self-anticommute.

4. GEOMETRIZATION OF MAXWELL'S EQUATIONS

Typically, for geometrized optics the geometrization is based on the approach proposed by Plebanski.[5–8] Briefly the program of geometrization by Plebanski can be described as follows:

1. One should write the Maxwell equations in the Minkowski space.
2. One should write vacuum Maxwell equations in the effective Riemannian space.
3. One should equate the corresponding members of the equations.

As a result, we get the expression of the permittivity and the permeability using geometric objects, namely through the metric tensor of the effective Riemannian space. This approach is somewhat similar to a mathematical trick. In addition, it still doesn't shed light on the actual geometrization mechanism.

The Lagrangian of the electromagnetic field (6) we will write in the form of a Lagrangian of Yang-Mills:

$$L = -\frac{1}{16\pi c} g^{\alpha\gamma} g^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} \sqrt{-g} - \frac{1}{c^2} A_\alpha j^\alpha \sqrt{-g}.$$

The geometrization based on Maxwell Lagrangian in the form of Yang–Mills Lagrangian, we will call Tamm geometrization approach.[2–4]

We will construct tensor $\lambda^{\alpha\beta\gamma\delta}$ as follows:[17]

$$\lambda^{\alpha\beta\gamma\delta} = 2\sqrt{-g} g^{\alpha\beta} g^{\gamma\delta} = \sqrt{-g}\left(g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma}\right) + \sqrt{-g}\left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}\right). \tag{7}$$

Then by taking into account the symmetry of tensors $F_{\alpha\beta}$ and $G^{\alpha\beta}$, equation (4), with respect of (7), will be as follows:

$$G^{\alpha\beta} = \frac{1}{2}\sqrt{-g}\left(g^{\alpha\gamma} g^{\beta\delta} - g^{\alpha\delta} g^{\beta\gamma}\right) F_{\gamma\delta}.$$

For clarity, we will write out this equation in components:

$$\begin{aligned} G^{0\underline{i}} &= \sqrt{-g}\left(g^{00} g^{\underline{i}\,\underline{j}} g^{0\underline{i}} g^{0\underline{j}}\right) F_{0\underline{j}} + \sqrt{-g}\left(g^{0\underline{j}} g^{\underline{i}\,\underline{k}} g^{0\underline{k}} g^{\underline{i}\,\underline{j}}\right) F_{\underline{j}\,\underline{k}}, \\ G^{\underline{i}\,\underline{j}} &= \sqrt{-g}\left(g^{\underline{i}\,0} g^{\underline{j}\,\underline{k}} g^{0\underline{j}} g^{\underline{i}\,\underline{k}}\right) F_{0\underline{k}} + \sqrt{-g}\left(g^{\underline{i}\,\underline{k}} g^{\underline{j}\,\underline{l}} g^{\underline{i}\,\underline{l}} g^{\underline{j}\,\underline{k}}\right) F_{\underline{k}\,\underline{l}}. \end{aligned} \tag{8}$$

Let us express equations (8) through the field vectors E_i, B^i, D^i, H_i:

$$D^{\underline{i}} = -\sqrt{-g}\left(g^{00} g^{\underline{i}\,\underline{j}} g^{0\underline{i}} g^{0\underline{j}}\right) E_{\underline{j}} + \sqrt{-g}\varepsilon_{\underline{k}\,\underline{l}\,\underline{j}} g^{0\underline{k}} g^{\underline{i}\,\underline{l}} B^{\underline{j}}, \tag{9}$$

$$H_{\underline{i}} = \sqrt{-g}\varepsilon_{\underline{m}\,\underline{n}\,\underline{i}} \varepsilon_{\underline{k}\,\underline{l}\,\underline{j}} g^{\underline{n}\,\underline{k}} g^{\underline{m}\,\underline{l}} B^{\underline{j}} + \sqrt{-g}\varepsilon^{\underline{k}\,\underline{l}\,\underline{j}} g_{0\underline{k}} g_{\underline{i}\,\underline{l}} E_{\underline{j}}. \tag{10}$$

From (9) one can formally write the expression for the permittivity ε^{ij}:

$$\varepsilon^{\underline{i}\,\underline{j}} = -\sqrt{-g}\left(g^{00} g^{\underline{i}\,\underline{j}} g^{0\underline{i}} g^{0\underline{j}}\right).$$

From (10) one can formally write the expression for the permeability μ^{ij}:

$$(\mu^{-1})_{\underline{i}\,\underline{j}} = \sqrt{-g}\varepsilon_{\underline{m}\,\underline{n}\,\underline{i}}\varepsilon_{\underline{k}\,\underline{l}\,\underline{j}}\, g^{\underline{n}\,\underline{k}} g^{\underline{m}\,\underline{l}}.$$

Thus, the geometrized constitutive equations in the components have the following form:

$$\begin{gathered}
D^i = \varepsilon^{ij}E_j + {}^{(1)}\gamma^i_j B^j, \\
H_i = (\mu^{-1})_{ij}B^j + {}^{(2)}\gamma^j_i E_j, \\
\varepsilon^{\underline{i}\,\underline{j}} = -\sqrt{-g}\big(g^{00} g^{\underline{i}\,\underline{j}} g^{0\underline{i}} g^{0\underline{j}}\big), \\
(\mu^{-1})_{\underline{i}\,\underline{j}} = \sqrt{-g}\varepsilon_{\underline{m}\,\underline{n}\,\underline{i}}\varepsilon_{\underline{k}\,\underline{l}\,\underline{j}}\, g^{\underline{n}\,\underline{k}} g^{\underline{m}\,\underline{l}}, \\
{}^{(1)}\gamma^i_j = {}^{(2)}\gamma^i_j = \sqrt{-g}\varepsilon_{\underline{k}\,\underline{l}\,\underline{j}}\, g^{0\underline{k}} g^{\underline{i}\,\underline{l}}.
\end{gathered}$$

Since this geometrization and Plebanski geometrization[5,18] are done on the basis of Riemannian geometry, it is possible to demonstrate their similarity. Indeed, it is easy to show that for Tamm's geometrization approach the following equation is valid $\varepsilon^{\underline{i}\,\underline{j}} = \mu^{\underline{i}\,\underline{j}}$ under the condition $g^{0\underline{i}} = 0$. This means that the geometrization of Maxwell's constitutive equations on the basis of a quadratic metric imposes a restriction on the impedance:

$$Z = \sqrt{\frac{\mu}{\varepsilon}} = 1.$$

This result is a consequence of the insufficient number of components of the Riemannian metric tensor $g_{\alpha\beta}$ (10 components), even for tensor ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$ (20 components), not to mention the total tensor $\lambda^{\alpha\beta\gamma\delta}$ (36 components). Even the usage of the geometrization of Riemannian geometry with torsion and nonmetricity[19,20] does not change the situation. Actually, when geometriztion is based on the Riemannian geometry, we the refractive index n_{ij}, but not the permittivity $varepsilon_{ij}$ and the permeability μ_{ij}.

The authors suggest that in order to solve the problem of the geometrization of the Maxwell's equations one need to rely on Finsler geometry. We propose to consider the equation (3) as the basis for the geometrization. As a metric, we propose to use the Berwald-Moor metric[21,22] with interval as follows:

$$\mathrm{d}s^4 = g_{\alpha\beta\gamma\delta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta\,\mathrm{d}x^\gamma\,\mathrm{d}x^\delta\,.$$

With this choice of metric, it is possible to obtain the full number of components for the tensor $\lambda_{\alpha\beta\gamma\delta}$. This approach is expected to be implemented in further research.

5. CONCLUSION

The authors demonstrated that the geometrization on the basis of the quadratic geometry can not adequately describe the Maxwell's equations, since it does not allow us to investigate the general case. That is why, according to the authors, this direction for a long time could not find an adequate application in practice. In fact, it found an application only within the framework of transformation optics for the calculation of metamaterials, since it was required to obtain exactly the refractive index for impedance matched materials. As an option for solving this problem, the authors propose to use the Finsler geometry, namely the Berwald-Moor space.

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