A geometric approach to the Lagrangian and Hamiltonian formalism of electrodynamics

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ABSTRACT

In solving field problems, for example problems of electrodynamics, we commonly use the Lagrangian and Hamiltonian formalisms. Hamiltonian formalism of field theory has the advantage over the Lagrangian, which inherently contains a gauge condition. While the gauge condition is introduced ad hoc from some external reasons in the Lagrangian formalism. However, the use of the Hamiltonian formalism in the field theory is difficult due to the non-regularity of the field Lagrangian. We must use such variant of the Lagrangian and the Hamiltonian formalism, which would allow us to work with the field models, in particular, to solve the problem of electrodynamics. We suggest to use the modern differential geometry and the algebraic topology, in particular the theory of fiber bundles, as a mathematical apparatus. This apparatus leads to greater clarity in the understanding of mathematical structures, associated with physical and technical models. The usage the fiber bundles theory allows us to deepen and expand both the Lagrangian and the Hamiltonian formalism. We can detect a wide range of these formalisms. Also we can select the most appropriate formalism. Actually just using the fiber bundles formalism we can adequately solve the problems of the field theory, in particular the problems of electrodynamics.

Keywords: fiber bundles, connection, Lagrangian formalism, Hamiltonian formalism, Yang–Mills theory

1. INTRODUCTION

Lagrangian and Hamiltonian formalisms used in mechanics and field theory. However, the usually used classic mathematical apparatus does not allow to take full advantages of the possibilities of these formalisms. Moreover, this apparatus is often used mechanically, which does not allow to recognize the strengths and weaknesses of the used formalisms and apply them to the precarious situation. For example, it causes difficulty in the use of the Hamiltonian approach to the field problem. The authors propose to use more advanced mathematical technique, namely, the bundle theory.[1,2] This apparatus helps to understand the Lagrangian and Hamiltonian approaches[3] better and allows to use them in new areas. For example, we may more effectively use the Hamiltonian formalism in problems of field theory.[4,5] This work is considered by the authors as a brief summary of the methods of the theory of fiber bundles. To illustrate these methods, we use electrodynamics.

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2. LAGRANGIAN FORMALISM

We consider the bundle:

$$\pi : Y \to X,$$

with coordinates (x^λ, y^i).

Then the Lagrangian density (the first order Lagrangian) will defined as:

$$L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda)\omega,$$

where $\omega := \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$.

In this case the Lagrangian density is the horizontal density on the jet bundle of the first order:

$$L : J^1 Y \to \overset{n}{\wedge} T^* X, \tag{1}$$

the jet bundle $J^1 Y$ plays the role of the configuration space.

For the Lagrangian L we can write the second order differential operator (the Euler–Lagrange operator):

$$\mathcal{E}_L : J^2 Y \underset{Y}{\to} \overset{n+1}{\wedge} T^* Y. \tag{2}$$

Also, we may define the first order Euler–Lagrange operator:

$$\mathcal{E}'_L = \left[\partial_i - \left(\partial_\lambda + y^i_\lambda \partial_i + y^i_{\mu\lambda}\right)\partial^\lambda_i\right]\mathcal{L}\,\mathrm{d}y^i \wedge \omega. \tag{3}$$

It is a first order differential operator on the jet bundle $J^1 Y \to X$:

$$\mathcal{E}'_L : J^2 Y \underset{J^1 Y}{\to} \overset{n+1}{\wedge} T^* Y.$$

We can distinguish three types of equations in the Lagrangian formalism (the Euler–Lagrange equations).

- The Euler–Lagrange algebraic equations for sections of the repeated jet bundle:

$$J^1 J^1 Y \to J^1 Y.$$

- The first order Euler–Lagrange equations for sections of the fibred jet manifold:

$$J^1 Y \to X.$$

- The second order Euler–Lagrange equations or sections of the fibred manifold:

$$Y \to X.$$

We define the Lagrangian connection on the bundle $J^1 Y \to X$: $\bar{\Gamma}^i_{\mu\lambda} = \bar{\Gamma}^i_{\mu\lambda}(x^\nu, y^j, y^j_\nu)$:

$$\bar{\Gamma} = \mathrm{d}x^\lambda \otimes \left(\partial_\lambda + y^i_\lambda \partial_i + \bar{\Gamma}^i_{\mu\lambda}\partial^\mu_i\right). \tag{4}$$

The Lagrangian connectivity takes values in the kernel of the first order Euler–Lagrange operator (3):

$$\mathcal{E}'_L \circ \bar{\Gamma} = 0.$$

Then the Euler–Lagrange algebraic equations will have the form:

$$\partial_i \mathcal{L} - \left(\partial_\lambda + y^j_\lambda \partial_j + \bar{\Gamma}^j_{\mu\lambda}\partial^\mu_j\right)\partial^\lambda_i \mathcal{L} = 0.$$

We write this equation for the Lagrangian connection components.

Let $\bar{s}$ is the integral section of connection bundle (4) ($J^1Y \to X$). It takes a value in the kernel of operator (3) $\ker \mathcal{E}'_L$:

$$\mathcal{E}'_L \circ J^1\bar{s} = 0.$$

Then we obtain the first order Euler–Lagrange equations:

$$\partial_i \mathcal{L} - \Big(\partial_\lambda + \bar{s}^j_\lambda \partial_j + \partial_\lambda \bar{s}^j_\mu \partial^\mu_j\Big)\partial^\lambda_i \mathcal{L} = 0, \tag{5}$$

where $\bar{s}^i_\lambda := \partial_\lambda \bar{s}^i$. We write equations (5) for $\bar{s} = \bar{s}(x^\nu, y^j)$.

Consider instead of the $\bar{s}$ section s of the bundle $Y \to X$. His second jet continuation J^2s takes values in the kernel of operator (2) $\ker \mathcal{E}_L$:

$$\mathcal{E}_L \circ J^2 s = 0.$$

Then the second order Euler–Lagrange equations will look like:

$$\partial_i \mathcal{L} - \big(\partial_\lambda + \partial_\lambda s^j \partial_j + \partial_\lambda \partial_\mu s^j \partial^\mu_j\big)\partial^\lambda_i \mathcal{L} = 0. \tag{6}$$

We write equations (6) for $s = s(x^\nu)$. Equation (5) and (6) are equivalent.

3. HAMILTONIAN FORMALISM

There are several variants of the Hamiltonian formalism.

- Symplectic Hamiltonian formalism.
- When applied to the field problem the symplectic formalism becomes the Dirac–Bergman formalism for systems with constraints.[6,7]
- Hamilton–De Donder Hamiltonian formalism.
- Multimomentum Hamiltonian formalism. This formalism is the polisymplectic generalization of the symplectic Hamiltonian.[4] formalism.[3–5]

For the field theory the multimomentum Hamiltonian formalism is the most convenient.

In order to pass to the Hamiltonian formalism we don't use the Lagrangian density, but the first order Lepagean equivalent. In the Hamilton–De Donder formalism and in multimomentum formalism we use Poincare–Cartan form:

$$\Xi_L = \pi^\lambda_i \, \mathrm{d}y^i \wedge \omega_\lambda - \pi^\lambda_i y^i_\lambda \omega + \mathcal{L}\omega,$$

where $\pi^\lambda_i := \partial^\lambda_i \mathcal{L}$, $\omega_\lambda := \partial_\lambda \lrcorner \omega$.

For bundle $Y \to X$ we introduce the Legendre bundle with coordinates $(x^\lambda, y^i, p^\lambda_i)$:

$$\Pi = \overset{n}{\wedge} T^*X \underset{Y}{\otimes} TX \underset{Y}{\otimes} V^*Y.$$

We have the associated Legendre morphism:

$$\hat{L} = J^1Y \underset{Y}{\to} \Pi, \quad p^\lambda_i \circ \hat{L} = \pi^\lambda_i.$$

The Legendre bundle Π is the phase space of the multimomentum Hamiltonian formalism.

The Legendre bundle $\Pi \to Y$ carries the generalized Liouville form

$$\theta = -p^\lambda_i \, \mathrm{d}y^i \wedge \omega \otimes \partial_\lambda.$$

We have the corresponding multisymplectic form:

$$\Omega = \mathrm{d}p_i^\lambda \wedge \mathrm{d}y^i \wedge \omega \otimes \partial_\lambda.$$

In case of bundle $X = \mathbb{R}$ we have the standard Liouville form and the standard symplectic form.

Multimomentum Hamiltonian can be represented as the external form:

$$\mathrm{d}H = \gamma \lrcorner \Omega,$$

where γ is some Hamiltonian connection (for Hamiltonian connection the exterior form $\gamma \lrcorner \Omega$ is closed.)

Multimomentum Hamiltonian can be represented in the form

$$H = p_i^\lambda \, \mathrm{d}y^i \wedge \omega_\lambda - p_i^\lambda \Gamma_\lambda^i \omega - \widetilde{\mathcal{H}}_\Gamma \omega = p_i^\lambda \, \mathrm{d}y^i \wedge \omega_\lambda - \mathcal{H}\omega,$$

whereΓ is the connection on the bundle $Y \to X$.

Then we can write the system of equations:

$$\partial_\lambda r^i(x) = \partial_\lambda^i \mathcal{H},$$
$$\partial_\lambda r_i^\lambda(x) = -\partial_i \mathcal{H}.$$

Where r is the section of the bundle $\Pi \to X$.

4. GAUGE THEORY APPROACH

We use the gauge theory approach to describe the physical fields within a geometric field theory. Physical fields are considered as sections of the bundle $Y \to X$. On this bundle we introduce connectivity and covariant derivatives ∇. On the jet bundle we built Lagrangian (1).

We consider the principal bundle with structure group of internal symmetries G:

$$\pi_P : P \to X.$$

The gauge potentials correspond to internal symmetry group G. We identify them with connections on the principal bundle P. The gauge potentials are represented by global sections A on bundle connections $C = J^1P/G \to X$:

$$A_\mu^m := \big(k_\mu^m \circ A\big)(x).$$

On the C we define coordinates $C = C(x^\mu, k_\mu^m)$.

The jet bundle J^1C on the principal connection bundle C is named as the configuration space of gauge potentials with coordinates $(x^\mu, k_\mu^m, k_{\mu\lambda}^m)$. For J^1C the splitting exists:

$$J^1C = C_+ \underset{C}{\oplus} C_-.$$

Where

$$C_- = C \times \overset{2}{\wedge} T^*X \underset{X}{\otimes} V^G P,$$

and $V^G P := VP/G$.

The bundle $C_+ \to C$ is an affine bundle modelled on the vector bundle:

$$\bar{C}_+ = \overset{2}{\vee} T^*X \underset{C}{\otimes} V^G P.$$

The coordinates of this splitting are as follows:

$$k_{\mu\lambda}^m = \frac{1}{2}\big(k_{\mu\lambda}^m + k_{\lambda\mu}^m + c_{nl}^m k_\lambda^n k_\mu^l\big) + \frac{1}{2}\big(k_{\mu\lambda}^m - k_{\lambda\mu}^m - c_{nl}^m k_\lambda^n k_\mu^l\big),$$

where c^k_{mn} are the structure constants of the Lie algebra g_r of group G in the basis $\{I_m\}$.

Let us write the canonical surjection $\mathcal{S} = pr_1 : J^1C \to C_+$:

$$\mathcal{S}^m_{\lambda\mu} = k^m_{\mu\lambda} + k^m_{\lambda\mu} + c^m_{nl} k^n_\lambda k^l_\mu.$$

Let us write the canonical surjection $\mathcal{F} = pr_2 : J^1C \to C_-$:

$$\mathcal{F} = \frac{1}{2}\mathcal{F}^m_{\lambda\mu}\, \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \otimes I_m, \quad \mathcal{F}^m_{\lambda\mu} = k^m_{\mu\lambda} - k^m_{\lambda\mu} - c^m_{nl} k^n_\lambda k^l_\mu.$$

We introduce the value $F := \mathcal{F} \circ J^1 A$. Then we obtain:

$$F^m_{\lambda\mu} = \partial_\lambda A^m_\mu - \partial_\mu A^m_\lambda - c^m_{nl} A^n_\lambda A^l_\mu. \tag{7}$$

For the configuration space J^1C we define the Yang–Mills Lagrangian:

$$L_{YM} = \frac{1}{4\varepsilon^2}\, {}^G a_{mn} g^{\lambda\mu} g^{\beta\nu} \mathcal{F}^m_{\lambda\beta} \mathcal{F}^n_{\mu\nu} \sqrt{|g|}\omega, \tag{8}$$

where ${}^G a_{mn}$ is a nondegenerate G-invariant metric on the Lie algebra g_r, ε is the interaction constant, $g_{\lambda\mu}$ and $g^{\lambda\mu}$ is the metric in the tangent (TX) and cotangent (T^*X) bundles on X, $g := \det\{g_{\lambda\mu}\}$.

5. ELECTROMAGNETIC FIELD

Let us describe the structure of gauge approach for the electromagnetic field and consider the electromagnetic field on the manifold X^4. The group $U(1)$ is the internal symmetry group G.

The principal bundle:

$$\pi_P : P \to X^4.$$

The conjugate of a principal connection:

$$V^G P = X^4 \times \mathbb{R}.$$

The connections bundle is isomorphic to affine cotangent bundle:

$$C \cong T^* X^4.$$

From (7) we can write the electromagnetic tensor:

$$F_{\mu\lambda} = \mathcal{F}_{\mu\lambda} \circ J^1 A = \partial_\lambda A_\mu - \partial_\mu A_\lambda.$$

For the Minkowski space $(X^4 = M^4)$ with metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ from (8) we can obtain the lagrangian of sourceless electromagnetic field:

$$L = -\frac{1}{16\pi c}\eta^{\lambda\mu}\eta^{\beta\nu}\mathcal{F}_{\lambda\beta}\mathcal{F}_{\mu\nu}\omega.$$

The Legendre bundle with coordinates $(X^\lambda, k_\mu, p^{\mu\lambda})$ will be the phase space:

$$\Pi = \left(\overset{4}{\wedge} T^*X \otimes TX \otimes TX\right) \underset{X}{\times} C.$$

Associated with the Lagrangian the Legendre morphism can be written as follows:

$$p^{(\mu\lambda)} \circ \hat{L} = 0;$$

$$p^{[\mu\lambda]} \circ \hat{L} = -\frac{1}{4\pi c}\eta^{\lambda\alpha}\eta^{\mu\beta}\mathcal{F}_{\alpha\beta}.$$

The multimomentum Hamiltonian takes the following form:

$$H = p^{\mu\lambda}\,\mathrm{d}k_\mu \wedge \omega_\lambda - p^{\mu\lambda}S_{\mu\lambda}\omega - \tilde{H}\omega,$$
$$S_{\mu\lambda} = \frac{1}{2}(\partial_\mu A_\lambda - \partial_\lambda A_\mu),$$
$$\tilde{H} = -\pi\eta_{\mu\alpha}\eta_{\lambda\beta}p^{[\mu\lambda]}p^{\alpha\beta}.$$

Thus, we can write Hamilton equations:

$$\partial_\lambda r^{\mu\lambda} = 0,$$
$$\partial_\lambda r_\mu + \partial_\mu r^\lambda = \partial_\lambda A_\mu + \partial_\mu A_\lambda.$$

6. CONCLUSIONS

The authors provides an essential overview of the basic materials on geometrical approach to the Lagrangian and Hamiltonian formalism based on fiber bundles. Application of this approach is demonstrated by the the electromagnetic field in the the Yang–Mills presentation.

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REFERENCES

[1] Saunders, D. J., [*The Geometry of Jet Bundles*] (1989).

[2] Vargas, J. G., [*Differential geometry for physicists and mathematicians*], World Scientific Publishing Company (2014).

[3] Giachetta, G., Mangiarotti, L., and Sardanashvily, G. A., [*Advanced Classical field theory*], World Scientific Publishing Company, Singapore (2009).

[4] Sardanashvily, G. A., "Generalized Hamiltonian formalism for field theory," viii+155 (1995).

[5] Giachetta, G., Mangiarotti, L., and Sardanashvily, G. A., "Covariant Hamilton equations for field theory," *Journal of Physics A: Mathematical and General* **32**, 6629–6642 (sep 1999).

[6] Dirac, P. A. M., [*Lectures on Quantum Mechanics*], Courier Corporation (2001).

[7] Newman, E. and Bergmann, P. G., "Lagrangians Linear in the "Velocities"," *Physical Review* **99**, 587–592 (jul 1955).