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Abstract. Spinors are more special objects than tensor. Therefore possess more properties than the more generic objects such as tensors. Thus, the group of Lorentz two-spinors is the covering group of the Lorentz group. Since the Lorentz group is a symmetry group of Maxwell's equations, it is assumed to reasonable to use when writing the Maxwell equations Lorentz two-spinors and not tensors. We describe in detail the representation of the Maxwell's equations in the form of Lorentz two-spinors. This representation of Maxwell's equations can be of considerable theoretical interest.

1. Introduction

Maxwell's equations have a large number of representations [1–4]. The principle of the introduction of the following: every representation must simplify the concrete theoretical and practical study. In this paper, we consistently describe the Lorentz two-spinor [5] representation of Maxwell's equations. It is supposed that this form will be interested in theoretical studies [6–8].

The structure of the article is as follows. In the section 2 basic notations and conventions are introduced. Section 3 gives a brief description of the Maxwell equations. Section 4 gives the spinors of the electromagnetic field. Further, section 5 gives the Lorenz two-spinor representation of Maxwell's equations.

2. Notations and conventions

(i) The abstract indices notation [9] is used in this work. Under this notation a tensor as a whole object is denoted just as an index (e.g., x^i), components are denoted by underlined index (e.g., $x^{\underline{i}}$).

(ii) We will adhere to the following agreements . Greek indices (α, β) will refer to the four-dimensional space , in component form it looks like: $\underline{\alpha} = \overline{0,3}$. Latin indices from the middle of the alphabet (i, j, k) will refer to the three-dimensional space , in the component form it looks like: $\underline{i} = \overline{1,3}$.

(iii) The comma in the index denotes partial derivative with respect to corresponding coordinate ($f_{,i} := \partial_i f$); semicolon denotes covariant derivative ($f_{;i} := \nabla_i f$).

(iv) To write the equations of electrodynamics in the article is used CGS symmetrical system.

3. Maxwell's Equations

Maxwell's equations in 3-dimensional form are as follows:

$$\begin{gathered}
\nabla_0 B^i = -e^{ijk}\nabla_j E_k;\\
\nabla_i D^i = 4\pi\rho;\\
\nabla_0 D^i = e^{ijk}\nabla_j H_k - \frac{4\pi}{c} j^i;\\
\nabla_i B^i = 0.
\end{gathered} \tag{1}$$

where $e^{\underline{i}\,\underline{j}\,\underline{k}}$ is the alternating tensor expressed by Levi-Civita simbol $\varepsilon^{\underline{i}\,\underline{j}\,\underline{k}}$:

$$e_{\underline{i}\,\underline{j}\,\underline{k}} = \sqrt{^3 g}\,\varepsilon_{\underline{i}\,\underline{j}\,\underline{k}}, \quad e^{\underline{i}\,\underline{j}\,\underline{k}} = \frac{1}{\sqrt{^3 g}}\varepsilon^{\underline{i}\,\underline{j}\,\underline{k}}.$$

Let's rewrite (1) with the help of electromagnetic field tensors $F_{\alpha\beta}$ and $G_{\alpha\beta}$ [10]:

$$\begin{gathered}
\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = F_{[\alpha\beta;\gamma]} = 0,\\
\nabla_\alpha G^{\alpha\beta} = \frac{4\pi}{c} j^\beta,
\end{gathered} \tag{2}$$

where

$$F_{\underline{\alpha}\,\underline{\beta}} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B^3 & B^2 \\ -E_2 & B^3 & 0 & -B^1 \\ -E_3 & -B^2 & B^1 & 0 \end{pmatrix}, \quad F^{\underline{\alpha}\,\underline{\beta}} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B_3 & B_2 \\ E^2 & B_3 & 0 & -B_1 \\ E^3 & -B_2 & B_1 & 0 \end{pmatrix},$$

$$G^{\underline{\alpha}\,\underline{\beta}} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H_3 & H_2 \\ D^2 & H_3 & 0 & -H_1 \\ D^3 & -H_2 & H_1 & 0 \end{pmatrix}, \quad G_{\underline{\alpha}\,\underline{\beta}} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & -H^3 & H^2 \\ -D_2 & H^3 & 0 & -H^1 \\ -D_3 & -H^2 & H^1 & 0 \end{pmatrix},$$

$E_{\underline{i}}$, $H^{\underline{i}}$ are components of electric and magnetic fields intensity vectors; $D_{\underline{i}}$, $B^{\underline{i}}$ are components of vectors of electric and magnetic induction.

4. Spinors of electromagnetic field

Spinors are used in physics quite extensively. The following spinors are mainly used: Dirac four-spinors; Pauli three-spinors; quaternions. If Dirac four-spinors are used, the main difficulty is γ-matrices. The essence of these objects is that they serve to connect the spinor and tensor spaces and therefore have two types of indices: spinor and tensor ones. It would be logical to perform calculations in one of these spaces only. In this paper we use semispinors of Dirac spinors, Lorentz two-spinors.

The tensor of electromagnetic field $F_{\alpha\beta}$ and its components $F_{\underline{\alpha}\,\underline{\beta}}$, $\underline{\alpha}$ may be considered in spinor form (and similarly for $G_{\alpha\beta}$):

$$\begin{gathered}
F_{\alpha\beta} = F_{A\dot{A}B\dot{B}};\\
F_{\underline{\alpha}\,\underline{\beta}} = F_{\underline{A}\,\underline{\dot{A}}\,\underline{B}\,\underline{\dot{B}}}\, g_{\underline{\alpha}}{}^{\underline{A}\,\underline{\dot{A}}}\, g_{\underline{\beta}}{}^{\underline{B}\,\underline{\dot{B}}},
\end{gathered}$$

where $g_{\underline{\alpha}}{}^{\underline{A}\underline{\dot{A}}}$ are Infeld–van der Waerden symbols defined in real spinor basis $\varepsilon_{\underline{A}\,\underline{B}}$ in the following way [9]:

$$g_{\underline{\alpha}}{}^{\underline{A}\underline{\dot{A}}} := g_{\underline{\alpha}}{}^{\alpha}\varepsilon_{A}{}^{\underline{A}}\varepsilon_{\dot{A}}{}^{\underline{\dot{A}}}, \quad g_{\underline{A}\,\underline{\dot{A}}}{}^{\underline{\alpha}} := g^{\underline{\alpha}}{}_{\alpha}\varepsilon^{A}{}_{\underline{A}}\varepsilon^{\dot{A}}{}_{\underline{\dot{A}}}, \tag{3}$$

$$\varepsilon_{\underline{A}\,\underline{B}} = \varepsilon_{\underline{\dot{A}}\,\underline{\dot{B}}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon^{\underline{A}\,\underline{B}} = \varepsilon^{\underline{\dot{A}}\,\underline{\dot{B}}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \varepsilon_{\underline{A}}{}^{A}\varepsilon_{A}{}^{\underline{B}} = \varepsilon_{\underline{A}}{}^{\underline{B}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

Let's $g_{\underline{\alpha\beta}} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski space metric. We use (4) as spinor space metric. Then the Infeld–van der Waerden symbols will have the following coordinate representation:

$$g_{0}^{\underline{A}\,\underline{\dot{A}}} = g^{0}_{\underline{A}\,\underline{\dot{A}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_{1}^{\underline{A}\,\underline{\dot{A}}} = g^{1}_{\underline{A}\,\underline{\dot{A}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$g_{2}^{\underline{A}\,\underline{\dot{A}}} = -g^{2}_{\underline{A}\,\underline{\dot{A}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad g_{3}^{\underline{A}\,\underline{\dot{A}}} = g^{3}_{\underline{A}\,\underline{\dot{A}}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The tensor $F_{\alpha\beta}$ is real and antisymmetric, it can be represented in the form

$$F_{\alpha\beta} = \varphi_{AB}\varepsilon_{\dot{A}\dot{B}} + \varepsilon_{AB}\bar{\varphi}_{\dot{A}\dot{B}},$$
$${}^{*}F^{\alpha\beta} = -\mathrm{i}\varphi^{AB}\varepsilon^{\dot{A}\dot{B}} + \mathrm{i}\varepsilon^{AB}\bar{\varphi}^{\dot{A}\dot{B}}. \tag{5}$$

where φ_{AB} is a spinor of electromagnetic field:

$$\varphi_{AB} := \frac{1}{2}F_{ABC'}{}^{C'} = \frac{1}{2}F_{A\dot{A}B\dot{B}}\varepsilon^{\dot{A}\dot{B}} = \frac{1}{2}F_{\alpha\beta}\varepsilon^{\dot{A}\dot{B}}.$$

The components of electromagnetic field spinor:

$$\varphi_{\underline{A}\,\underline{B}} = \frac{1}{2}F_{\underline{\alpha}\,\underline{\beta}}\varepsilon^{\underline{\dot{A}}\,\underline{\dot{B}}}g^{\underline{\alpha}}{}_{\underline{A}\,\underline{\dot{A}}}g^{\underline{\beta}}{}_{\underline{B}\,\underline{\dot{B}}}.$$

Using the equations (3), (4) and notation $F_i = E_i - \mathrm{i}B^i$, we will get:

$$\varphi_{00} = \frac{1}{2}(F_{31} + F_{01} - \mathrm{i}F_{32} - \mathrm{i}F_{02}) = \frac{1}{2}(F_1 - \mathrm{i}F_2),$$
$$\varphi_{01} = \varphi_{10} = \frac{1}{2}(-F_{03} - \mathrm{i}F_{12}) = -\frac{1}{2}F_3,$$
$$\varphi_{11} = \frac{1}{2}(F_{31} - F_{01} + \mathrm{i}F_{32} - \mathrm{i}F_{02}) = -\frac{1}{2}(F_1 + \mathrm{i}F_2).$$

Similarly

$$G^{\alpha\beta} = \eta^{AB}\varepsilon^{\dot{A}\dot{B}} + \varepsilon^{AB}\bar{\eta}^{\dot{A}\dot{B}},$$
$${}^{*}G_{\alpha\beta} = -\mathrm{i}\eta_{AB}\varepsilon_{\dot{A}\dot{B}} + \mathrm{i}\varepsilon_{AB}\bar{\eta}_{\dot{A}\dot{B}}. \tag{6}$$

where η^{AB} is a Minkowski spinor:

$$\eta^{AB} := \frac{1}{2}G^{ABC'}{}_{C'} = \frac{1}{2}G^{A\dot{A}B\dot{B}}\varepsilon_{\dot{A}\dot{B}} = \frac{1}{2}G^{\alpha\beta}\varepsilon_{\dot{A}\dot{B}}.$$

The components of the spinor η^{AB}:

$$\eta^{\underline{A}\,\underline{B}} = \frac{1}{2}G^{\underline{\alpha}\,\underline{\beta}}\varepsilon_{\underline{\dot{A}}\,\underline{\dot{B}}}g_{\underline{\alpha}}{}^{\underline{A}\,\underline{\dot{A}}}g_{\underline{\beta}}{}^{\underline{B}\,\underline{\dot{B}}}.$$

Using the equations (3), (4) and notation $G^i = D^i - \mathrm{i}H_i$, we will get:

$$\eta^{00} = \frac{1}{2}\big(G^{31} + G^{01} + \mathrm{i}G^{32} + \mathrm{i}G^{02}\big) = \frac{1}{2}\left(G^1 - \mathrm{i}G^2\right),$$
$$\eta^{01} = \eta^{10} = \frac{1}{2}\big(-G^{03} + \mathrm{i}G^{12}\big) = -\frac{1}{2}G^3,$$
$$\eta^{11} = \frac{1}{2}\big(G^{31} - G^{01} - \mathrm{i}G^{32} + \mathrm{i}G^{02}\big) = -\frac{1}{2}\left(G^1 + \mathrm{i}G^2\right).$$

5. Spinor Form of Maxwell's Equations

Let's write Maxwell's equations using the spinors.

Replacing in (2) abstract indices α by $A\dot{A}$ and β by $B\dot{B}$, we can write:

$$\nabla_{A\dot{A}} G^{A\dot{A}B\dot{B}} = \frac{4\pi}{c} j^{B\dot{B}}.$$

Using (6) we will get

$$\nabla^{A\dot{B}} \eta_A^B + \nabla^{B\dot{A}} \bar{\eta}_{\dot{A}}^{\dot{B}} = \frac{4\pi}{c} j^{B\dot{B}}.$$

Similarly, from (5) it follows

$$\nabla^{\dot{A}B} \varphi_B^A - \nabla^{A\dot{B}} \bar{\varphi}_{\dot{B}}^{\dot{A}} = 0.$$

In so doing the system of Maxwells equations can be written as

$$\begin{aligned} \nabla^{\dot{A}B} \varphi_B^A - \nabla^{A\dot{B}} \bar{\varphi}_{\dot{B}}^{\dot{A}} &= 0, \\ \nabla^{A\dot{B}} \eta_A^B + \nabla^{B\dot{A}} \bar{\eta}_{\dot{A}}^{\dot{B}} &= \frac{4\pi}{c} j^{B\dot{B}}. \end{aligned} \tag{7}$$

In the vacuum case (no medium), we can put ${\eta^A}_B = {\varphi^A}_B$. Then we can write the equations (7) as follows:

$$\nabla^{\dot{A}B} \varphi_B^A = \nabla^{A\dot{B}} \bar{\varphi}_{\dot{B}}^{\dot{A}},$$
$$\nabla^{A\dot{B}} \varphi_A^B + \nabla^{B\dot{A}} \bar{\varphi}_{\dot{A}}^{\dot{B}} = \frac{4\pi}{c} j^{B\dot{B}}.$$

Thus, the spinor form of Maxwell's equations system in vacuum can be written in the form of one equation:

$$\nabla^{A\dot{B}} \varphi_A^B = \frac{2\pi}{c} j^{B\dot{B}}.$$

6. Conclusions

Thus, in the article, we have proposed a representation of Maxwell's equations in the form of Lorentz 2-spinors. We consider that the given representation might be interested in in theoretical studies.

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