# Maxwell's equations instantaneous Hamiltonian

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## ABSTRACT

The Hamiltonian formalism is extremely elegant and convenient to mechanics problems. However, its application to the classical field theories is a difficult task. In fact, you can set one to one correspondence between the Lagrangian and Hamiltonian in the case of hyperregular Lagrangian. It is impossible to do the same in field theories. In the case of irregular Lagrangian the Dirac–Bergman Hamiltonian formalism with constraints is usually used, and this leads to a number of certain difficulties. The paper proposes a reformulation of the problem to the case of a field without sources. This allows to use a instantaneous (symplectic) Hamiltonian formalism.

**Keywords:** Maxwell's equations, curvilinear coordinates, symplectic manifold, Hamiltonian formalism, doubling of variables, Yang–Mills theory

## 1. INTRODUCTION

The Hamiltonian formalism[1] is well known and widely used in geometrical optics. As a Hamiltonian the Hamiltonian of material particle is used.

In the case of wave optics there is a number of difficulties in the construction of the Hamiltonian formalism. Since the Lagrangian of the electromagnetic field is not regular, then the construction of a symplectic Hamiltonian formalism is not possible. In this case the Dirac–Bergman formalism with constraints or multimomentum formalism is commonly used.

However, if we restrict ourselves to systems without sources, it is possible to build a standard symplectic Hamiltonian formalism.

Due the impossibility of constructing symplectic Hamiltonian for the general case of the electromagnetic field, in this article for the case of the sourceless field the general method for constructing symplectic Hamiltonian and the example of such constructing are presented.

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## 2. NOTATIONS AND CONVENTIONS

1. We will use the notation of abstract indices.[2] In this notation tensor as a complete object is denoted merely by an index (e.g., $x^i$). Its components are designated by underlined indices (e.g., $x^{\underline{i}}$).

2. We will adhere to the following agreements . Greek indices ($\alpha$, $\beta$) will refer to the four-dimensional space, in the component form it looks like: $\underline{\alpha} = \overline{0,3}$. Latin indices from the middle of the alphabet ($i$, $j$, $k$) will refer to the three-dimensional space, in the component form it looks like: $\underline{i} = \overline{1,3}$.

3. The comma in the index denotes a partial derivative with respect to corresponding coordinate ($f_{,i} := \partial_i f$); semicolon denotes a covariant derivative ($f_{;i} := \nabla_i f$).

4. The CGS symmetrical system[3] is used for notation of the equations of electrodynamics.

## 3. MAXWELL'S EQUATIONS

Maxwell's equations will be written both via field variables and in the gauge-invariant form (in the formalism of bundles).[4–6]

### 3.1 Maxwell's equations via field variables

The Maxwell's equations via the field variables in the tensor formalism in holonomic basis:

$$\begin{cases} \nabla_i D^i = 4\pi\rho, \\ e^{ijk}\nabla_j H_k - \dfrac{1}{c}\partial_t D^i = \dfrac{4\pi}{c} j^i, \\ \nabla^i B_i = 0, \\ e_{ijk}\nabla^j E^k + \dfrac{1}{c}\partial_t B_i = 0. \end{cases} \tag{1}$$

Here $e_{ijk}$ is the Levi–Civita tensor (alternating tensor):

$$e_{ijk} = \sqrt{^3 g}\,\varepsilon_{ijk}, \quad e^{ijk} = \frac{1}{\sqrt{^3 g}}\varepsilon^{ijk},$$
$$e_{\alpha\beta\gamma\delta} = \sqrt{-^4 g}\,\varepsilon_{\alpha\beta\gamma\delta}, \quad e^{\alpha\beta\gamma\delta} = -\frac{1}{\sqrt{-^4 g}}\varepsilon^{\alpha\beta\gamma\delta}.$$

Field functions $\vec{E}$ and $\vec{B}$ can be represented via field potentials $\varphi$ and $\vec{A}$:

$$\vec{B} = \operatorname{rot}\vec{A}, \quad \vec{E} = -\nabla\varphi - \frac{1}{c}\partial_t \vec{A},$$

or in the index notation:

$$\begin{cases} B^i = \left(\operatorname{rot}\vec{A}\right)^i = e^{ikl}\partial_k A_l, \\ E_i = -\partial_i\varphi - \partial_0 g_{ij} A^j. \end{cases} \tag{2}$$

### 3.2 The Lagrangian of the electromagnetic field

We consider the principal bundle $P \to X$ with the structure group $U(1)$. Where $X = M^4$ is a four-dimensional oriented manifold. The tangent and cotangent bundles the metric $g_{\alpha\beta}$ and $g^{\alpha\beta}$ with the signature $(+,-,-,-)$ is given.

The Lagrangian density is defined as the horizontal density:

$$L : J^1 P \to \overset{4}{\wedge} T^* X.$$

On the bundle $P \to X$ coordinates $(x^\lambda, y^i)$ are defined. On the jet bundle $J^1P$ coordinates $(x^\lambda, y^i, y^i_\lambda)$ are given. Then the Lagrangian density can be written as follows:

$$L = \mathcal{L}(x^\lambda, y^i, y^i_\lambda)\omega,$$

where $\omega := \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$.

For any section $A$ of cotangent bundle $T^*X$ we can introduce the value $F_{\alpha\beta}$ as the electromagnetic field tensor. It has the geometric meaning of curvature on the cotangent bundle:

$$F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha, \tag{3}$$

where the connection $A_\alpha$ means the 4-vector potential $A_\alpha = (\varphi, \vec{A})$.

We will write the tensor (3) by components taking into account the relation (2):

$$F_{0\underline{i}} = \partial_0 A_{\underline{i}} - \partial_{\underline{i}} A_0 = -\partial_0 A^{\underline{i}} - \partial_i A^0 = E_{\underline{i}},$$
$$F_{\underline{ik}} = \partial_{\underline{i}} A_{\underline{k}} - \partial_{\underline{k}} A_{\underline{i}} = -\varepsilon_{\underline{ikl}} B^{\underline{l}}.$$

Similarly, we will introduce the Minkowski's tensor (displacement tensor) $G^{\alpha\beta}$[7] on the tangent bundle $TX$. Usually the connection on the tangent bundle is not considered. Instead, we postulate the relation between curvatures on the tangent and cotangent bundles:

$$G^{\alpha\beta} = \lambda(F_{\gamma\delta}).$$

In the linear case we may set the constitutive tensor $\lambda^{\alpha\beta\gamma\delta}$ with the symmetry $\begin{array}{|c|c|}\hline \alpha & \gamma \\ \hline \beta & \delta \\ \hline \end{array}$:

$$G^{\alpha\beta} = \lambda^{\alpha\beta\gamma\delta} F_{\gamma\delta}.$$

The tensors $F_{\alpha\beta}$ and $G^{\alpha\beta}$ have the following components

$$F_{\underline{\alpha\beta}} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B^3 & B^2 \\ -E_2 & B^3 & 0 & -B^1 \\ -E_3 & -B^2 & B^1 & 0 \end{pmatrix},$$
$$G^{\underline{\alpha\beta}} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H_3 & H_2 \\ D^2 & H_3 & 0 & -H_1 \\ D^3 & -H_2 & H_1 & 0 \end{pmatrix}.$$

Here $E_{\underline{i}}$, $H_{\underline{i}}$ are components of electric and magnetic fields intensity vectors; $D^{\underline{i}}$, $B^{\underline{i}}$ are components of vectors of electric and magnetic induction.

Let's write the Maxwell equations with the help of electromagnetic field tensors $F_{\alpha\beta}$ and $G_{\alpha\beta}$:[8,9]

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = F_{[\alpha\beta;\gamma]} = 0, \tag{4}$$
$$\nabla_\alpha G^{\alpha\beta} = \frac{4\pi}{c} j^\beta. \tag{5}$$

Let's construct the Lagrangian $\mathcal{L}$ explicitly.

The equation (4) represents the second Bianchi identity, that is performed by the construction of (3). To construct the Lagrangian we only need to use the equations (5).

Then the Lagrangian has the form:

$$\mathcal{L}(x^\alpha, A_\beta, A_{\alpha,\beta}) = -\frac{1}{16\pi c} F_{\alpha\beta} G^{\alpha\beta} \sqrt{-{}^4g} - \frac{1}{c} A_\alpha j^\alpha \sqrt{-{}^4g}.$$

The form of Euler–Lagrange equation:

$$\nabla_\beta \frac{\delta \mathcal{L}}{\delta A_{\alpha;\beta}} - \frac{\delta \mathcal{L}}{\delta A_\alpha} = 0. \tag{6}$$

### 3.3 The problem of constructing the Hamiltonian of the electromagnetic field

There are several variants of the Hamiltonian formalism.

- The symplectic Hamiltonian formalism.
- The Dirac–Bergman Hamiltonian formalism for systems with constraints.[10,11]
- The Hamilton–De Donder Hamiltonian formalism.[12]
- The multimomentum Hamiltonian formalism.[12–14]

We will consider the symplectic Hamiltonian formalism due to its simplicity. In this case, the Lagrangian density is defined as

$$L : J^1 P \to \mathbb{R}.$$

To bundle $P \to X$ we introduce the Legendre bundle $\Pi$ with atlas coordinates $(x^\lambda, y^i, p_i)$.

The bundle morphism is defined by Lagrangian density:

$$\hat{L} = J^1 P \underset{P}{\to} \Pi, \quad p_i \circ \hat{L} = \pi_i.$$

The Legendre bundle $\Pi$ is the phase space of the Hamiltonian formalism.

The Liouville form is defined on the Legendre bundle $\Pi \to Y$:

$$\theta = -p_i \, \mathrm{d}y^i \wedge \mathrm{d}t \,.$$

The corresponding symplectic form is follow:

$$\Omega = \mathrm{d}p_i \wedge \mathrm{d}y^i \wedge \mathrm{d}t \,.$$

The Hamiltonian of the electromagnetic field is constructed by Lagrangian with the help of Legendre transformations:

$$\mathcal{H} := p_\alpha \dot{A}^\alpha - \mathcal{L}, \tag{7}$$

where $p_\alpha$ is momentum density, $\mathcal{L}$ is Lagrangian.

Because in Hamiltonian formalism all the equations are constructed with generalized coordinates and momenta, in order to write down the Hamiltonian density (7) and the corresponding Hamilton equations we need to express the generalized momentum $p_\alpha$ through velocities $\dot{A}^\alpha$ in the (6):

$$\begin{cases} \dot{A}^\alpha = \dfrac{\delta \mathcal{H}}{\delta p_\alpha}, \\ \dot{p}_\alpha = -\dfrac{\delta \mathcal{H}}{\delta A^\alpha}. \end{cases}$$

This requires that the determinant of the Hessian matrix is nonzero:

$$\det\{\mathbf{H}\}(\mathcal{L}) \neq 0,$$

where the elements of the Hessian matrix are:

$$\{\mathbf{H}(\mathcal{L})\}_{\underline{\alpha\beta}} = \frac{\partial^2 \mathcal{L}}{\partial \dot{A}^{\underline{\alpha}} \, \partial \dot{A}^{\underline{\beta}}}.$$

But $F_{00} = 0$ and $\{\mathbf{H}(\mathcal{L})\}_{00} = \dfrac{\partial^2 \mathcal{L}}{\partial (\dot{A}^0)^2} = 0$. Therefore, $\det\{\mathbf{H}\}(\mathcal{L}) = 0$. That means that the Lagrangian is irregular, and the construction of symplectic Hamiltonian formalism in such a manner is impossible.

# 4. THE SYMPLECTIC HAMILTONIAN CONSTRUCTION

It turns out that in the absence of sources ($j^{\alpha} = 0$) we can construct a symplectic Hamiltonian formalism making the desired change of variables. As one of the methods, the method of doubling variables is considered. This method is applicable when the system contains only generalized variables, and generalized momenta are absent.

## 4.1 Doubling of variables method

Consider the system of $s$ equations:

$$\dot{q}^{\underline{n}} = f^{\underline{n}}(q^n, q^n_{;i}, x^i, t), \quad \underline{n} = \overline{0, s}. \tag{8}$$

We define the space $\mathbb{R}^{2s}$ with the following coordinates:

$$\xi^{\underline{n}} := q^{\underline{n}}, \quad \xi^{\underline{n}+\underline{s}} := p_{\underline{n}}, \quad \xi^a \in \mathbb{R}^{2s}; \quad \underline{n} = \overline{0, s}, \quad \underline{a} = \overline{0, 2s}.$$

In this space we introduce the Poisson bracket:

$$\{A(\xi^c, t), B(\xi^c, t)\} = \Omega^{\underline{ab}} \frac{\partial A(\xi^c, t)}{\partial \xi^{\underline{a}}} \frac{\partial B(\xi^c, t)}{\partial \xi^{\underline{b}}},$$
$$\Omega^{\underline{ab}} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad \underline{a}, \underline{b}, \underline{c} = \overline{0, 2s}.$$

The Hamiltonian is defined as follows:

$$\mathcal{H}(q^n, p_n, x^i, t) = p_{\underline{n}} f^{\underline{n}}(q^n, q^n_{;i}, x^i, t). \tag{9}$$

Then the first group of the Hamilton equations will be the same as the original system (8), and the second group will be as follows:

$$\dot{p}_{\underline{n}} = -\frac{\delta \mathcal{H}}{\delta q^{\underline{n}}} = -p_{\underline{m}} \frac{\delta f^{\underline{m}}}{\delta q^{\underline{n}}}.$$

## 4.2 The applicability of the method of doubling the variables for the Maxwell equations

Let's establish that Maxwell's equations without sources satisfy the condition of applicability of the doubling of variables method. To do this, we rewrite the equations (1) in the form:

$$\begin{cases} \partial_t B_i = -c e_{ijk} \nabla^j E^k, \\ \partial_t D^i = c e^{ijk} \nabla_j H_k, \\ \nabla^i B_i = 0, \\ \nabla_i D^i = 0. \end{cases}$$

It is obvious that the second pair of equations violates the condition of method applicability. However, we can show that in the absence of sources these equations are linearly dependent on the rest of the Maxwell equations. Let's write first and second equations in component form:

$$\begin{aligned} \frac{1}{\sqrt{^3 g}} [E_{3,2} - E_{2,3}] &= -\frac{1}{c} \partial_t B^1, \\ \frac{1}{\sqrt{^3 g}} [E_{1,3} - E_{3,1}] &= -\frac{1}{c} \partial_t B^2, \\ \frac{1}{\sqrt{^3 g}} [E_{2,1} - E_{1,2}] &= -\frac{1}{c} \partial_t B^3. \end{aligned} \tag{10}$$

$$
\begin{aligned}
\frac{1}{\sqrt{^3g}}\left[H_{3,2}-H_{2,3}\right] &= \frac{1}{c}\partial_t D^1,\\
\frac{1}{\sqrt{^3g}}\left[H_{1,3}-H_{3,1}\right] &= \frac{1}{c}\partial_t D^2,\\
\frac{1}{\sqrt{^3g}}\left[H_{2,1}-H_{1,2}\right] &= \frac{1}{c}\partial_t D^3.
\end{aligned} \tag{11}
$$

The following two equations are written in the same way:

$$
\frac{1}{\sqrt{^3g}}\partial_i\left(\sqrt{^3g}B^i\right) = \frac{1}{\sqrt{^3g}}\left[\partial_1\left(\sqrt{^3g}B^1\right)+\partial_2\left(\sqrt{^3g}B^2\right)+\partial_3\left(\sqrt{^3g}B^3\right)\right] = 0, \tag{12}
$$

$$
\frac{1}{\sqrt{^3g}}\partial_i\left(\sqrt{^3g}D^i\right) = \frac{1}{\sqrt{^3g}}\left[\partial_1\left(\sqrt{^3g}D^1\right)+\partial_2\left(\sqrt{^3g}D^2\right)+\partial_3\left(\sqrt{^3g}D^3\right)\right] = 0. \tag{13}
$$

Differentiating both sides of equations (10) and (11) we get (assuming that $g$ is time independent):

$$
\begin{aligned}
E_{3,21}-E_{2,31} &= -\frac{1}{c}\partial_t\partial_1\left(\sqrt{^3g}B^1\right),\\
E_{1,32}-E_{3,12} &= -\frac{1}{c}\partial_t\partial_2\left(\sqrt{^3g}B^2\right),\\
E_{2,13}-E_{1,23} &= -\frac{1}{c}\partial_t\partial_3\left(\sqrt{^3g}B^3\right).
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
H_{3,21}-H_{2,31} &= \frac{1}{c}\partial_t\partial_1\left(\sqrt{^3g}D^1\right),\\
H_{1,32}-H_{3,12} &= \frac{1}{c}\partial_t\partial_2\left(\sqrt{^3g}D^2\right),\\
H_{2,13}-H_{1,23} &= \frac{1}{c}\partial_t\partial_3\left(\sqrt{^3g}D^3\right).
\end{aligned} \tag{15}
$$

Summing the equations (14) and (15) term by term we obtain (12) and (13) respectively.

Thus, the system of Maxwell's equations is transformed into the following reduced system which satisfies the condition of method:

$$
\begin{cases}
\partial_t B_i = -c e_{ijk}\nabla^j E^k,\\
\partial_t D^i = c e^{ijk}\nabla_j H_k.
\end{cases} \tag{16}
$$

### 4.3 Example of the Hamiltonian

We define the constitutive equations:

$$
D^i = \varepsilon^{ij}(x^k)E_j, \quad H_i = (\mu^{-1})_{ij}(x^k)B^j.
$$

We rewrite (16) as follows (assuming that the metric is time independent):

$$
\begin{cases}
\partial_t E^i = c(\varepsilon^{-1})^i_l \dfrac{1}{\sqrt{^3g}}\varepsilon^{ljk}H_{k,j},\\[2ex]
\partial_t H^i = -c(\mu^{-1})^i_l \dfrac{1}{\sqrt{^3g}}\varepsilon^{ljk}E_{k,j}.
\end{cases}
$$

We choose the generalized coordinates in the form:

$$
q^n = \left(E^1, E^2, E^3, H^1, H^2, H^3\right)^T, \quad \underline{n} = \overline{1,6}.
$$

The system (8) takes the following form:

$$\begin{cases} \dot{q}^{\underline{i}} = f^{\underline{i}}(q^n, q^n_{;i}, x^i, t) = c(\varepsilon^{-1})^{\underline{i}}_{\underline{l}} \dfrac{1}{\sqrt{^3 g}} \varepsilon^{\underline{ljk}} q_{\underline{k+3}, \underline{j}}\,, \\ \dot{q}^{\underline{i+3}} = f^{\underline{i+3}}(q^n, q^n_{;i}, x^i, t) = -c(\mu^{-1})^{\underline{i}}_{\underline{l}} \dfrac{1}{\sqrt{^3 g}} \varepsilon^{\underline{ljk}} q_{\underline{k}, \underline{j}}\,. \end{cases} \tag{17}$$

We write the Hamiltonian based on the (9) and (17):

$$\mathcal{H}(q^n, p_n, x^i, t) = p_{\underline{n}} f^{\underline{n}}(q^n, q^n_{;i}, x^i, t) = p_{\underline{i}}\, c(\varepsilon^{-1})^{\underline{i}}_{\underline{l}} \frac{1}{\sqrt{^3 g}} \varepsilon^{\underline{ljk}} q_{\underline{k+3}, \underline{j}} - p_{\underline{i+3}}\, c(\mu^{-1})^{\underline{i}}_{\underline{l}} \frac{1}{\sqrt{^3 g}} \varepsilon^{\underline{ljk}} q_{\underline{k}, \underline{j}}\,.$$

The corresponding system of Hamiltonian equations has the following form:

$$\begin{cases} \dot{q}^n = \dfrac{\delta \mathcal{H}}{\delta p_n} = f^n, \\ \dot{p}_n = -\dfrac{\delta \mathcal{H}}{\delta q^n} = -p_m \dfrac{\delta f^m}{\delta q^n} = \\ \quad = -p_m \dfrac{\partial f^m}{\partial q_n} + p_m \partial_i \dfrac{\partial f^m}{\partial q^n_{,i}} = p_m \partial_i \dfrac{\partial f^m}{\partial q^n_{,i}}. \end{cases}$$

### 4.4 The momentum representation

We make the change of variables, putting $j^{\underline{\alpha}} = 0$:

$$C^l = E^l + \mathrm{i} B^l,$$

and thus we obtain the Maxwell equations in the following form:

$$\begin{cases} \nabla_l C^l = 4\pi\rho = \dfrac{4\pi}{c} j^0 = 0, \\ [\nabla, C]^l - \mathrm{i}\dfrac{1}{c}\partial_t C^l = \dfrac{4\pi}{c}\mathrm{i} j^l = 0. \end{cases} \tag{18}$$

Let us expand the $C$ in a Fourier series:

$$C^l(x) = \frac{1}{\Omega} \sum_{\vec{k}} \mathrm{e}^{\mathrm{i} k_j x^j} a^l(\vec{k}), \quad \vec{k} = (0, 0, k). \tag{19}$$

From the Maxwell equations we may find $a^l(\vec{k})$.

Substituting (19) in the first equation of (18) we will obtain:

$$\partial_l \left( \frac{1}{\sqrt{\Omega}} \sum_{\vec{k}} \mathrm{e}^{\mathrm{i} k_j x^j} a^l(\vec{k}) \right) = 0 \quad \Rightarrow \quad \partial_l \left( \mathrm{e}^{\mathrm{i} k_j x^j} a^l(\vec{k}) \right) = 0,$$
$$\mathrm{i} k_j \mathrm{e}^{\mathrm{i} k_j x^j} a^l(\vec{k}) = 0 \quad \Rightarrow \quad \mathrm{i} k_j a^l(\vec{k}) = 0.$$

Since $\vec{k} = (0, 0, k)$, then $\mathrm{i} k_j a^3 = 0$, and therefore $a^3 = 0$.

From the second equation of (18) we get:

$$e^{ijk} \partial_j C_k - \mathrm{i}\frac{1}{c}\partial_t C^i = 0.$$

If $i = 1$:

$$e^{1jk}\partial_j C_k - \mathrm{i}\frac{1}{c}\partial_t C^1 = 0,$$

$$\sqrt{{}^3g}\,(\partial_2 C_3 - \partial_3 C_2) - i\frac{1}{c}\partial_t C^1 = 0.$$

When we make a substitution from (19), we obtain:

$$\sqrt{{}^3g}\,(\mathrm{i}k_2 a_3 - \mathrm{i}k_3 a_2)\,\mathrm{e}^{\mathrm{i}k_j x^j} - \mathrm{i}\frac{1}{c}\mathrm{e}^{\mathrm{i}k_j x^j}\,\partial_t a^1 = 0.$$

Since $\vec{k} = (0, 0, k)$ $(k_2 = 0)$, then

$$\dot{a}^1 = -c\sqrt{{}^3g}k_3 a_2.$$

When $i = 2$ we obtain:

$$e^{2jk}\partial_j C_k - \mathrm{i}\frac{1}{c}\partial_t C^2 = 0,$$

$$\sqrt{{}^3g}\,(\partial_3 C_1 - \partial_1 C_3) - \mathrm{i}\frac{1}{c}\partial_t C^2 = 0.$$

Substituting (19), we obtain:

$$\sqrt{{}^3g}\,(\mathrm{i}k_3 a_1 - \mathrm{i}k_1 a_3)\,\mathrm{e}^{\mathrm{i}k_j x^j} - \mathrm{i}\frac{1}{c}\mathrm{e}^{\mathrm{i}k_j x^j}\,\partial_t a^2 = 0.$$

Since $\vec{k} = (0, 0, k)$ $(k_1 = 0)$, then

$$\dot{a}^2 = c\sqrt{{}^3g}k_3 a_1.$$

When $i = 3$, we obtain $\dot{a}^3 = 0$.

After the index raising, we have:

$$\begin{cases} \dot{\hat{a}}^1 = -c\sqrt{{}^3g}k\hat{a}^2 g_{22}, \\ \dot{\hat{a}}_1^* = -c\sqrt{{}^3g}k\hat{a}_2^* g_{22}, \\ \dot{\hat{a}}^2 = c\sqrt{{}^3g}k\hat{a}^1 g_{11}, \\ \dot{\hat{a}}_2^* = c\sqrt{{}^3g}k\hat{a}_1^* g_{11}. \end{cases} \tag{20}$$

The Hamiltonian can be represented as follows:

$$\hat{\mathcal{H}} \propto \hat{a}_1^*\hat{a}^1 + \hat{a}_2^*\hat{a}^2.$$

$$q^1 = \hat{a}^2, \quad q^2 = \hat{a}_2^*, \quad p_1 = \hat{a}_1^*, \quad p_2 = \hat{a}^1.$$

We need to find the $a^1$. To do this, we will solve the follow equation:

$$\ddot{a}^1 + c^2 g k^2 g_{22} g_{11} a^1 = 0.$$

The characteristic equation is:

$$\lambda^2 + c^2 g k^2 g_{22} g_{11}\lambda = 0,$$

then $\lambda_{1,2} = \pm\mathrm{i}ck\sqrt{{}^3g}\sqrt{g_{22}g_{11}}$. Consequently, the solution has the form:

$$a^1 = C_1\cos\left(ck\sqrt{{}^3g}\sqrt{g_{22}g_{11}}\right) + C_2\sin\left(ck\sqrt{{}^3g}\sqrt{g_{22}g_{11}}\right),$$

where $C_1$ and $C_2$ are constant.

Substituting $a^1$ in (20):

$$\dot{a}^2 = c\sqrt{{}^3g}k\left[C_1\cos\left(ck\sqrt{{}^3g}\sqrt{g_{22}g_{11}}\right) + C_2\sin\left(ck\sqrt{{}^3g}\sqrt{g_{22}g_{11}}\right)\right]g_{11},$$

we may obtain $a^2$.

## 5. CONCLUSION

In this paper a formal method of obtaining symplectic Hamiltonian formalism for Maxwell's equations without sources was constructed. The authors also hope that represented examples sufficiently clarify the application of the proposed method.

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