An application of two-spinors calculus to quantum field and quantum mechanics problems

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General Notion of Spinors

Clifford-Dirac equation:

$$\gamma_{(a}\gamma_{b)} = -g_{ab}\mathbf{I}. \tag{1}$$

If spinor indexes not suppresed

$$\gamma^{\sigma}_{a\rho}\gamma^{\tau}_{b\sigma} + \gamma^{\sigma}_{b\rho}\gamma^{\tau}_{a\sigma} = -2g_{ab}\delta^{\tau}_{\rho}. \tag{2}$$

$$\begin{cases} N = 2^{n/2}, & \text{even } n; \\ N = 2^{n/2-1/2}, & \text{odd } n. \end{cases} \tag{3}$$

Quaternions and Two-Spinors I

Let

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}. \tag{4}$$

Then matrix representation of quaternions is

$$\mathbf{A} = \mathbf{I}a + \mathbf{i}b + \mathbf{j}c + \mathbf{j}d = \begin{pmatrix} a + \mathrm{i}d & -c + \mathrm{i}b \\ c + \mathrm{i}b & a - \mathrm{i}d \end{pmatrix}, \tag{5}$$

where $a, b, c, d \in \mathbb{R}$.

$$\mathbf{A}^* = \mathbf{I}a - (\mathbf{i}b + \mathbf{j}c + \mathbf{k}d). \tag{6}$$

Quaternions and Two-Spinors II

A (5) is spin-matrix if **A** unimodular and unitary:

$$\det \mathbf{A} = a^2 + b^2 + c^2 + d^2 = 1, \tag{7}$$

$$\mathbf{A}\mathbf{A}^* = \mathbf{I}(a^2 + b^2 + c^2 + d^2) = \mathbf{I}. \tag{8}$$

This equivalent to:

$$\|\mathbf{A}\| := a^2 + b^2 + c^2 + d^2 = 1. \tag{9}$$

From Spinors to Vectors

Infeld–van der Verden symbols:

$$
\begin{aligned}
g_{\mathbf{a}}{}^{\mathbf{AA'}} &:= g_{\mathbf{a}}{}^{a}\varepsilon_{A}{}^{\mathbf{A}}\varepsilon_{A'}{}^{\mathbf{A'}}, \\
g^{\mathbf{a}}{}_{\mathbf{AA'}} &:= g_{a}{}^{\mathbf{a}}\varepsilon_{\mathbf{A}}{}^{A}\varepsilon_{\mathbf{A'}}{}^{A'},
\end{aligned} \tag{10}
$$

For spin reference frame and Minkovski space:

$$
\begin{aligned}
g_0{}^{\mathbf{AB'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g^0_{\mathbf{AB'}}, \qquad & g_1{}^{\mathbf{AB'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g^1_{\mathbf{AB'}}, \\
g_2{}^{\mathbf{AB'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = -g^2_{\mathbf{AB'}}, \qquad & g_3{}^{\mathbf{AB'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = g^3_{\mathbf{AB'}}.
\end{aligned} \tag{11}
$$

Dirac Four-Spinors and Lorentz Two-Spinors I

Let Dirac four-spinor is

$$\psi^{\alpha} = \begin{pmatrix} \varphi^{A} \\ \pi^{A'} \end{pmatrix}, \tag{12}$$

where φ^{A} and $\pi^{A'}$ are Lorentz two-spinors.
Spinor conjugation:

$$\overline{\psi^{\alpha}} = \bar{\psi}_{\alpha} = \left(\bar{\pi}_{A}, \bar{\varphi}_{A'}\right). \tag{13}$$

Reflex operator:

$$\hat{P} = \begin{pmatrix} \varphi^{A} \\ \pi^{A'} \end{pmatrix} \mapsto \begin{pmatrix} \pi^{A'} \\ \varphi^{A} \end{pmatrix}. \tag{14}$$

Dirac Four-Spinors and Lorentz Two-Spinors II

γ-matrix are:

$$\gamma_{a\rho}{}^{\sigma} = \sqrt{2}\begin{pmatrix} 0 & \varepsilon_{A'R'}\varepsilon_{A}{}^{S} \\ \varepsilon_{AR}\varepsilon_{A'}{}^{S'} & 0 \end{pmatrix}, \qquad \eta_{\rho}{}^{\sigma} = \begin{pmatrix} -\mathrm{i}\varepsilon_{R}{}^{S} & 0 \\ 0 & \mathrm{i}\varepsilon_{R'}{}^{S'} \end{pmatrix}, \tag{15}$$

and

$$\gamma_{ab\rho}{}^{\sigma} = \begin{pmatrix} \varepsilon_{A'B'}\varepsilon_{R(A}\varepsilon_{B)}{}^{S} & 0 \\ 0 & \varepsilon_{AB}\varepsilon_{R'(A'}\varepsilon_{B')}{}^{S'} \end{pmatrix}. \tag{16}$$

Let $\gamma_5 := \mathrm{i}\eta$.

Invariants

We use

- Dirac four-spinors (ψ and $\bar{\psi}$).
- Lorentz two-spinors (φ^A, $\bar{\varphi}_{A'}$, $\pi^{A'}$ and $\bar{\pi}_A$).

Scalars

Scalar s and pseudoscalar p:

$$s = \bar{\pi}_A \varphi^A + \bar{\varphi}_{A'} \pi^{A'} = \bar{\psi}_\alpha \psi^\alpha, \tag{17}$$

$$p = \mathrm{i}(\bar{\pi}_A \varphi^A - \bar{\varphi}_{A'} \pi^{A'}) = \mathrm{i}\bar{\psi}_\alpha {\gamma_5}_\beta{}^\alpha \psi^\beta. \tag{18}$$

Vectors

Vector j^a and pseudovector $\tilde{j}^a$:

$$j^a = \sqrt{2}(\bar{\pi}^A \pi^{A'} + \varphi^A \bar{\varphi}^{A'}) = \bar{\psi}_\alpha \gamma^{a\alpha}_\beta \psi^\beta, \tag{19}$$

$$\tilde{j}^a = \sqrt{2}(\bar{\pi}^A \pi^{A'} - \varphi^A \bar{\varphi}^{A'}) = \bar{\psi}_\alpha \gamma^{a\alpha}_\beta {\gamma_{5\delta}}^\beta \psi^\delta. \tag{20}$$

Tensors

Real antisymmetric tensor:

$$a^{ab} = \mathrm{i}(\varphi^{(A}\bar{\pi}^{B)}\varepsilon^{A'B'} - \bar{\varphi}^{(A'}\pi^{B')}\varepsilon^{AB}) = \bar{\psi}_{\alpha}\sigma^{ab}{}_{\beta}{}^{\alpha}\psi^{\beta}. \tag{21}$$

Matrix Elements I

Let

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}, \tag{22}$$

where sign depend on sign of $1 \pm \gamma_5$.
And γ-matrix are

$$\gamma_a = \begin{pmatrix} 0 & \gamma_{a+} \\ \gamma_{a-} & 0 \end{pmatrix}, \qquad \hat{p} = \begin{pmatrix} 0 & \hat{p}_+ \\ \hat{p}_- & 0 \end{pmatrix}, \tag{23}$$

where $\hat{p} := p^a \gamma_a$.

Matrix Elements II

When we have projectors $(1 \pm \gamma_5)$:

$$\bar{\psi}_f \gamma^{a_1} \hat{p}_{(a)} \gamma^{a_2} \hat{p}_{(b)} \cdots \gamma^{a_n} \left[\frac{1}{2}(1 \pm \gamma_5) \right] \psi_i. \tag{24}$$

$$\begin{cases} \bar{\psi}_{f\pm} \gamma^{a_1}_{\mp} \hat{p}_{(a)\pm} \gamma^{a_2}_{\mp} \hat{p}_{(b)\pm} \cdots \gamma^{a_n}_{\pm} \psi_{i\pm}, & \text{odd number of } \gamma\text{-matrix}, \\ \bar{\psi}_{f\mp} \gamma^{a_1}_{\pm} \hat{p}_{(a)\mp} \gamma^{a_2}_{\pm} \hat{p}_{(b)\mp} \cdots \gamma^{a_n}_{\mp} \psi_{i\pm}, & \text{even number of } \gamma\text{-matrix}. \end{cases} \tag{25}$$

Matrix Elements III

$$\gamma_{a\alpha}{}^{\beta}_{+}\gamma^{a\delta}_{\gamma-} = 2\delta_{\alpha}{}^{\delta}\delta_{\gamma}{}^{\beta}, \tag{26a}$$

$$\gamma_{a\alpha}{}^{\beta}_{\pm}\gamma^{a\delta}_{\gamma\pm} = 2\left(\delta_{\alpha}{}^{\beta}\delta_{\gamma}{}^{\delta} - \delta_{\alpha}{}^{\delta}\delta_{\gamma}{}^{\beta}\right). \tag{26b}$$

We may prove (for example) (26a):

$$\varepsilon_{C'A'}\varepsilon_{C}{}^{B}\varepsilon^{C}{}_{G}\varepsilon^{C'D'} = 2\varepsilon_{A'}{}^{D'}\varepsilon_{G}{}^{B}, \tag{27}$$

granting

$$\alpha \leftrightarrow A', \quad \beta \leftrightarrow B, \quad \gamma \leftrightarrow G, \quad \delta \leftrightarrow D',$$

Matrix Elements IV

As result we have

$$u^{\dagger}_{f\pm}\hat{p}_{(a)\mp}\hat{p}_{(b)\pm}\dots\hat{e}_{+\text{ or }-}\dots u_{i\pm}, \tag{28}$$

where e is polarization.

Matrix Elements V

In case of plane wave (longitudinal polarization)

$$u_\pm = \left(\sqrt{E+\varepsilon m} \pm \varepsilon s \sqrt{E-\varepsilon m}\right) \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\varphi/2}\sqrt{1+s\cos\theta} \\ \mathrm{e}^{\mathrm{i}\varphi/2}\sqrt{1-s\cos\theta} \end{pmatrix}, \quad (29)$$

where s — spirality, ε — energy sign.

Example I

Let we have reaction

$$\nu + n \to p + e^- . \tag{30}$$

Matrix element is

$$M = \frac{G_F}{\sqrt{2}} \left(\bar{\psi}_e \gamma_a (1 + \gamma_5) \psi_\nu\right) \left(\bar{\psi}_p \gamma^a (g_V + g_A \gamma_5) \psi_n\right) . \tag{31}$$

Example II

Based on (25) and (26) we write (31) as:

$$M = \frac{2G_F}{\sqrt{2}}(u^{\dagger}_{e\alpha+}\gamma_{a\beta-}{}^{\alpha}u^{\beta}_{\nu+})\Big[(g_A - g_V)(u^{\dagger}_{p\gamma+}\gamma^{a}_{\delta-}{}^{\gamma}u^{\delta}_{n+} + u^{\dagger}_{p\gamma-}\gamma^{a}_{\delta+}{}^{\gamma}u^{\delta}_{n-}) +$$

$$+2g_A u^{\dagger}_{p\gamma+}\gamma^{a}_{\delta-}{}^{\gamma}u^{\delta}_{n+}\Big] = \frac{2G_F}{\sqrt{2}}\Big[(g_V - g_A)u^{\dagger}_{e\alpha+}\gamma_{a\beta-}{}^{\alpha}u^{\beta}_{\nu+}u^{\dagger}_{p\gamma-}\gamma^{a}_{\delta+}{}^{\gamma}u^{\delta}_{n-} +$$

$$+(g_V + g_A)u^{\dagger}_{e\alpha+}\gamma_{a\beta-}{}^{\alpha}u^{\beta}_{\nu+}u^{\dagger}_{p\gamma+}\gamma^{a}_{\delta-}{}^{\gamma}u^{\delta}_{n+}\Big] =$$

$$= \frac{4G_F}{\sqrt{2}}\Big[(g_V - g_A)u^{\dagger}_{e\alpha\,+}u^{\alpha}_{n\,-}u^{\dagger}_{p\beta\,-}u^{\beta}_{\nu\,+} +$$

$$+(g_V + g_A)\left(u^{\dagger}_{e\alpha+}u^{\alpha}_{\nu+}u^{\dagger}_{p\beta+}u^{\beta}_{n+} - u^{\dagger}_{e\alpha+}u^{\alpha}_{n+}u^{\dagger}_{p\beta+}u^{\beta}_{\nu+}\right)\Big]. \quad (32)$$

Example III

Let $\varphi_\nu = \varphi_n = \varphi_p = \varphi_e = 0$, $\theta_\nu = \theta_n = \pi/2$, θ_e and θ_p — voluntary.
Let

$$\begin{gathered}
|s_0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |s_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \\
|s_2\rangle = \begin{pmatrix} \cos\theta_p/2 \\ \sin\theta_p/2 \end{pmatrix}, \quad |s_3\rangle = \begin{pmatrix} -\sin\theta_e/2 \\ \cos\theta_e/2 \end{pmatrix}.
\end{gathered} \tag{33}$$

Example IV

Then [see (29)]:

$$u_{\nu\pm} = \frac{1}{\sqrt{2}}\left(\sqrt{E_\nu + m_\nu} \pm s_\nu \sqrt{E_\nu - m_\nu}\right)|s_0\rangle, \tag{34}$$

$$u_{n\pm} = \frac{1}{\sqrt{2}}\left(\sqrt{E_n + m_n} \pm s_n \sqrt{E_n - m_n}\right)|s_1\rangle, \tag{35}$$

$$u_{p\pm} = \frac{1}{\sqrt{2}}\left(\sqrt{E_p + m_p} \pm s_p \sqrt{E_p - m_p}\right)|s_2\rangle, \tag{36}$$

$$u_{e\pm} = \frac{1}{\sqrt{2}}\left(\sqrt{E_e + m_e} \pm s_e \sqrt{E_e - m_e}\right)|s_3\rangle. \tag{37}$$

Example V

Based on (32):

$$
\begin{aligned}
M = {} & \frac{G_F}{\sqrt{2}} \left(\sqrt{E_e + m_e} + s_e \sqrt{E_e - m_e} \right) \left(\sqrt{E_\nu + m_\nu} + s_\nu \sqrt{E_\nu - m_\nu} \right) \times \\
& \times \Big[(g_V - g_A) \left(\sqrt{E_n + m_n} - s_n \sqrt{E_n - m_n} \right) \left(\sqrt{E_p + m_p} - s_p \sqrt{E_p - m_p} \right) \langle s_3 | s_1 \rangle \langle s_2 | s_0 \rangle + \\
& + (g_V + g_A) \left(\sqrt{E_n + m_n} + s_n \sqrt{E_n - m_n} \right) \left(\sqrt{E_p + m_p} + s_p \sqrt{E_p - m_p} \right) \Big(\langle s_3 | s_0 \rangle \langle s_2 | s_1 \rangle - \langle s_3 | s_1 \rangle \langle s_2 | s_0 \rangle \Big) \Big] = \\
= {} & \frac{G_F}{\sqrt{2}} \left(\sqrt{E_e + m_e} + s_e \sqrt{E_e - m_e} \right) \left(\sqrt{E_\nu + m_\nu} + s_\nu \sqrt{E_\nu - m_\nu} \right) \times \\
& \times \Big[(g_V - g_A) \left(\sqrt{E_n + m_n} - s_n \sqrt{E_n - m_n} \right) \left(\sqrt{E_p + m_p} - s_p \sqrt{E_p - m_p} \right) c \cos\theta_e/2 \cos\theta_p/2 - \\
& - (g_V + g_A) \left(\sqrt{E_n + m_n} + s_n \sqrt{E_n - m_n} \right) \left(\sqrt{E_p + m_p} + s_p \sqrt{E_p - m_p} \right) \times \\
& \times \Big(\sin\theta_e/2 \cos\theta_p/2 + \cos\theta_e/2 \cos\theta_p/2 \Big) \Big] . \qquad (38)
\end{aligned}
$$