EPJ Web of Conferences **173**, 02003 (2018) https://doi.org/10.1051/epjconf/201817302003
Mathematical Modeling and Computational Physics 2017

Functional Integral Approach to the Solution of a System of Stochastic Differential Equations

Edik Ayryan[1,2,⋆], *Alexander* Egorov[3,⋆⋆], *Dmitri* Kulyabov[1,2,⋆⋆⋆], *Victor* Malyutin[3,⋆⋆⋆⋆], and *Leonid* Sevastianov[2,4,†]

[1] *Laboratory of Information Technologies, Joint Institute for Nuclear Research, Dubna, Russia*
[2] *Peoples' Friendship University of Russia (RUDN University), Moscow, Russia*
[3] *Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus*
[4] *Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Russia*

Abstract. A new method for the evaluation of the characteristics of the solution of a system of stochastic differential equations is presented. This method is based on the representation of a probability density function p through a functional integral. The functional integral representation is obtained by means of the Onsager-Machlup functional technique for a special case when the diffusion matrix for the SDE system defines a Riemannian space with zero curvature.

1 Introduction

Stochastic differential equations (SDE) are very popular in physics, chemistry, biology and etc. [1, 2]. There are many publications devoted to different aspects of SDE theory [1–3]. Except for some very special cases, SDE cannot be solved analytically and numerical methods are needed. The most common approach to the numerical solution of SDE is based on the discretization of a time interval and numerical simulation of SDE solutions at discrete times [4–7]. In this work a new method for evaluating of the characteristics of a SDE solution is presented. This method is based on the representation of the probability density function (PDF) p through a functional integral. The Onsager-Machlup functional technique is used to get the functional integral representation [8–11]. For one dimension case this approach was considered in [12]. In this work this approach is given for a system of SDE. Here we consider the Onsager-Machlup functional technique only for the flat space when the diffusion matrix for a SDE system defines a Riemannian space with zero curvature.

In section 2 the representation of PDF p through a functional integral is discussed. In section 3 the method of approximate evaluation of functional integrals is presented. Using the minimal action principle [13], we can distinguish among all trajectories the classical trajectory for which the action S takes the extremal value. The classical trajectory is found as the solution of the multidimensional Euler-Lagrange equation. Further, we use the expansion of the action with respect to the classical

⋆e-mail: ayrjan@jinr.ru
⋆⋆e-mail: egorov@im.bas-net.by
⋆⋆⋆e-mail: kulyabov_ds@rudn.university
⋆⋆⋆⋆e-mail: malyutin@im.bas-net.by
†e-mail: sevastianov_la@rudn.university

trajectory to compute the integral. In section 4 approximate and exact values for the expectations of some functionals of solution of specific system of SDE are presented.

2 Representation of characteristics through functional integral

We consider the SDE system (the Ito interpretation)

$$\begin{cases} dx_1(t) = a_1(\vec{x},t)\,dt + \sum_{j=1}^{d} g_{1j}(\vec{x},t)\,dw_j(t), \\ \dots \\ dx_d(t) = a_d(\vec{x},t)\,dt + \sum_{j=1}^{d} g_{dj}(\vec{x},t)\,dw_j(t), \end{cases} \tag{1}$$

where the solution $\vec{x}(t)$ satisfies the initial condition $\vec{x}(t_0) = \vec{x}_0$, $\vec{w}(t)$ is a d-dimensional Wiener process.

By means of the Onsager-Machlup functional technique the PDF can be represented in a form of the functional integral

$$p(\vec{x},t,\vec{x}_0,t_0) = \int D[\vec{x}]\exp\Big\{-\int_{t_0}^{t} L_0(\vec{x}(\tau),\dot{\vec{x}}(\tau))\,d\tau\Big\}, \tag{2}$$

$$D[\vec{x}] = \lim_{N\to\infty}\prod_{j=1}^{N-1}\prod_{k=1}^{d} dx_{kj}\prod_{j=1}^{N}\frac{1}{(2\pi\Delta t)^{d/2}\sqrt{\det G(\vec{x}_{j-1},t_{j-1})}}, \qquad x_{kN} = x_k(t),$$

$$L_0(\vec{x}(\tau),\dot{\vec{x}}(\tau)) = \frac{1}{2}\sum_{k,j=1}^{d} G_{kj}^{-1}(\vec{x}(\tau),\tau)\times[\dot{x}_k - A_k(\vec{x}(\tau),\tau)][\dot{x}_j - A_j(\vec{x}(\tau),\tau)],$$

$$A_k(\vec{x}(\tau),\tau) = a_k(\vec{x}(\tau),\tau) - \frac{1}{2}\sum_{i,j=1}^{d} g_{ij}(\vec{x}(\tau),\tau)\frac{\partial}{\partial x_i}g_{kj}(\vec{x}(\tau),\tau),$$

where G denotes a matrix of elements

$$G_{ij}(\vec{x}(\tau),\tau) = \sum_{k=1}^{d} g_{ik}(\vec{x}(\tau),\tau)g_{jk}(\vec{x}(\tau),\tau).$$

3 Evaluation of functional integrals

The quantity $L_0(\vec{x}(\tau),\dot{\vec{x}}(\tau))$ in (2) can be considered as the Lagrangian of the system, and the quantity $S = \int_{t_0}^{t} L_0(\vec{x}(\tau),\dot{\vec{x}}(\tau))\,d\tau$ can be considered as its action. Using the minimal action principle [13], we can distinguish among all trajectories the classical trajectory $\vec{x}_{cl}$ for which the action S takes the extremal value. The classical trajectory is found as the solution of the multidimensional Euler-Lagrange equation

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L_0}{\partial \dot{x}_1}\right) - \dfrac{\partial L_0}{\partial x_1} = 0, \\ \qquad\vdots \\ \dfrac{d}{dt}\left(\dfrac{\partial L_0}{\partial \dot{x}_d}\right) - \dfrac{\partial L_0}{\partial x_d} = 0. \end{cases} \tag{3}$$

Further, to compute the integral, we can use the decomposition of the action with respect to the classical trajectory $\vec{x}_{cl}$

$$S[\vec{x}(\tau)] \approx S[\vec{x}_{\rm cl}(\tau)] + \frac{1}{2}\delta^2 S[\vec{x}_{\rm cl}(\tau)].$$

The second order variation $\delta^2 S[\vec{x}_{\rm cl}(\tau)]$ can be written in the form

$$\delta^2 S[\vec{x}_{\rm cl}(\tau)] = \int_{t_0}^{t} \sum_{i,j=1}^{d} \delta x_i \Lambda_{ij} \delta x_j \, d\tau,$$

where $\vec{x} = \vec{x}_{\rm cl} + \vec{\delta x}$,

$$\Lambda_{ij} = \left[\frac{\partial^2 L}{\partial x_i \partial x_j}\right]_{x_{\rm cl}} + \left[\frac{\partial^2 L}{\partial x_i \partial \dot{x}_j}\right]_{x_{\rm cl}} \frac{d}{dt} - \frac{d}{dt}\left[\frac{\partial^2 L}{\partial \dot{x}_i \partial x_j}\right]_{x_{\rm cl}} - \frac{d}{dt}\left[\frac{\partial^2 L}{\partial \dot{x}_i \partial \dot{x}_j}\right]_{x_{\rm cl}} \frac{d}{dt}.$$

After these transformations the formula (2) is written in the form

$$p(\vec{x}, t, \vec{x}_0, t_0) = \exp\{-S[\vec{x}_{\rm cl}(\tau)]\} \int D[\vec{\delta x}] \exp\Big\{-\frac{1}{2}\int_{t_0}^{t} \sum_{i,j=1}^{d} \delta x_i \Lambda_{ij} \delta x_j \, d\tau\Big\}.$$

By evaluating the functional integral from the exponential function with the quadratic form in exponent we obtain result for the PDF p.

4 Numerical results

In this section we consider an application of the proposed method for the evaluation of the expectation of some functionals of the SDE system solutions

$$\begin{cases} dx_1(t) = (a_1 x_1 + c_1\sqrt{x_1 x_2} + b_1)\,dt + \sigma_1\sqrt{x_1}\,dw_1(t), \\ dx_2(t) = (a_2 x_2 + c_2\sqrt{x_1 x_2} + b_2)\,dt + \sigma_2\sqrt{x_2}\,dw_2(t), \end{cases} \tag{4}$$

$$x_1(t_0) = x_{10}, \qquad x_2(t_0) = x_{20}.$$

The approximate and exact values of the expectation of some functionals of the SDE system solution (4) for concrete coefficients and initial conditions are presented in Tables 1 and 2. The form of the functionals is chosen in such way that we can calculate the exact values of the expectations. To compute the function $\vec{x}_{\rm cl}$ approximately we use the partitioning of the interval $[t_0, t]$ into n parts.

Table 1. Approximate and exact values for expectations for $a_1 = a_2 = 2$, $c_1 = c_2 = -2$, $\sigma_1 = \sigma_2 = 1$, $b_1 = b_2 = 1/4$, $t_0 = 0$, $t = 1$, $x_{10} = x_{20} = 1/4$

	$E[\sqrt{x_1}]$	$E[\sqrt{x_2}]$	$E[(x_1 - x_2)]$
Approximate values $n = 40$	0.4651	0.4651	0.00003
Approximate values $n = 120$	0.4837	0.4837	0.00002
Exact values	0.5	0.5	0

5 Conclusion

A new method for evaluating of the characteristics of a SDE solution is proposed. From results presented in Tables 1 and 2 one can see that the increase of the parameter n leads to the decreasing differences between exact and approximate values.

Table 2. Approximate and exact values for expectations for $a_1 = a_2 = 2$, $c_1 = c_2 = -2$, $\sigma_1 = 9$, $\sigma_2 = 10$, $b_1 = 81/4$, $b_2 = 100/4$, $t_0 = 0$, $t = 1$, $x_{10} = x_{20} = 4$

	$E[\sqrt{x_1}]$	$E[\sqrt{x_2}]$	$E[(x_1 - x_2)]$
Approximate values $n = 40$	6.037	10.72	−11.17
Approximate values $n = 120$	5.962	11.51	−13.53
Exact values	5.806	13.19	−15.17

Acknowledgements

This work was financially supported by the Ministry of Education and Science of the Russian Federation (the Agreement number 02.a03.21.0008) and by the Belarussian Republic's Fund for Basic Research (project F16D-002).

References

[1] C.W. Gardiner, *Handbook of Stochastic Methods: For Physics, Chemistry, and the Natural Sciences* (Springer-Verlag; 2 Sub edition, January 1986)

[2] N.G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 2007)

[3] I.I. Gikhman and A. V. Skorokhod, *Stochastic Differential Equations* ("Naukova Dumka", Kiev, 1968)

[4] D. F. Kuznetsov, *Numerical Integration of Stochastic Differential Equations* (Polytechnical University Publishing House, Saint Petersburg, 2001)

[5] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations* (Springer, Berlin, 1999)

[6] P. E. Kloeden, E. Platen, and H. Schurz, *Numerical Solution of Stochastic Differential Equations Through Computer Experiments* (Springer, Berlin, 1994)

[7] G. N. Milstein, *The Numerical Integration Solution of Stochastic Differential Equations*, (Kluwer Academic, Dordrecht, 1995)

[8] L. Onsager and S. Machlup, Phys. Rev. **91**, 1505 (1953)

[9] F. Langouche, D. Roekaerts, and E. Tirapegui, *Functional Integration and Semi-Classical Expansions* (D. Reidel, Dordrecht, 1982)

[10] H.S. Wio, *Application of Path Integration to Stochastic Process: An Introduction* (World Scientific, Singapore, 2013)

[11] E. Bennati, M. Rosa-Clot, and S. Taddei, Int. J. Theor. Appl. Finan. **02**, 381 (1999)

[12] E. A. Ayryan, A. D. Egorov, D. S. Kulyabov, V. B. Malyutin, and L. A. Sevastianov, Matematicheskoe Modelirovanie **28**, 11, 113–125, (2016)

[13] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965)