ISSN 0361-7688, Programming and Computer Software, 2016, Vol. 42, No. 2, pp. 77–83. © Pleiades Publishing, Ltd., 2016.
Original Russian Text © D.S. Kulyabov, 2016, published in Programmirovanie, 2016, Vol. 42, No. 2.

Using Two Types of Computer Algebra Systems to Solve Maxwell Optics Problems

D. S. Kulyabov

Department of Applied Probability and Informatics, Peoples' Friendship University of Russia, ul. Miklukho-Maklaya 6, Moscow, 117198 Russia

Laboratory of Information Technologies, Joint Institute for Nuclear Research, ul. Zholio-Kyuri 6, Dubna, 141980 Russia

E-mail: yamadharma@gmail.com

Received September 12, 2015

Abstract—To synthesize Maxwell optics systems, the mathematical apparatus of tensor and vector analysis is generally employed. This mathematical apparatus implies executing a great number of simple stereotyped operations, which are adequately supported by computer algebra systems. In this paper, we distinguish between two stages of working with a mathematical model: model development and model usage. Each of these stages implies its own computer algebra system. As a model problem, we consider the problem of geometrization of Maxwell's equations. Two computer algebra systems—Cadabra and FORM—are selected for use at different stages of investigation.

DOI: 10.1134/S0361768816020043

1. INTRODUCTION

This paper considers the application of computer algebra systems to designing a Maxwell optics system described by Maxwell's equations in arbitrary locally orthogonal curvilinear coordinates. To describe this problem mathematically using the apparatus of tensor and vector analysis, computer algebra systems that support tensor calculus are required. An additional requirement is the freewareness of such computer algebra systems, since, at the current stage of investigation, this problem is of scientific interest and involves no commercial benefit. Among freeware computer algebra systems that implement the apparatus of tensor and vector analysis, two different systems are selected for use at two different stages of investigation [1].

The first stage consists in writing a prototype program; for this purpose, the Cadabra system can be successfully used. The second stage is of particular importance for the domain experts who run the constructed program. At this stage, a great deal of trial computations with different parameters, which are arbitrary or systematically varied, are carried out to find the optimum solution among feasible ones or even a new solution of the problem under consideration. The FORM system seems to be the best choice for this stage.

Our attempts to find a "silver bullet" [2] have not been successful, and searching for such a universal system has led us to the conclusion that each problem requires its own specific remedy.

This paper is organized as follows. Section 2 introduces basic notations and conventions. Types of computer algebra systems are considered in Section 3. Section 4 describes the formalism of geometrization of Maxwell's equations. Section 5 illustrates the use of this formalism by an example of designing and calculating three-dimensional waveguide objects of Maxwell optics.

2. NOTATION AND CONVENTIONS

1. In this paper, the abstract index notation is used [3], in which a tensor, as a complete object, is denoted simply by an index (for example, x^i), while its components are denoted by an underlined index (for example, $x^{\underline{i}}$).

2. We adhere to the following conventions. Greek indices (α and β) refer to a four-dimensional space and, in a component form, have the following notation: $\underline{\alpha} = \overline{0,3}$. Latin indices from the middle of the alphabet (i, j, and k) refer to a three-dimensional space and, in a component form, have the following notation: $\underline{i} = \overline{1,3}$.

3. In the index, a comma denotes a partial derivative with respect to the corresponding coordinate ($f_{,i} := \partial_i f$), while a semicolon denotes a covariant derivative ($f_{;i} := \nabla_i f$).

4. To write electrodynamics equations, the symmetric CGS system is used.

3. TYPES OF COMPUTER ALGEBRA SYSTEMS

Computer algebra systems can be classified based on various criteria. Here, we confine ourselves to the criterion of interactivity. Originally, batch processing prevailed in computing systems. However, when the power of computers had made it possible to reduce the response time to an acceptable level, interactivity became a leading paradigm. Each paradigm has its own area of application. In software development, interactivity was associated with prototyping tools, while the classical approach (programming, compilation, debugging, etc.) was used to create software products. In symbolic computation systems, interactivity became a ruling principle. Indeed, computer algebra systems were designed to increase the labor productivity of scientists and served as a kind of a smart notebook. Little by little, the problems involving noninteractive computations were pushed to the sidelines together with the corresponding computer algebra systems.

The architecture of many modern computer algebra systems became an obstacle in the way of increasing the computational efficiency. Therefore, the approach that implies using different computer algebra systems to solve different problems seems promising.

In this paper, we consider two extreme poles. On the one side is the Cadabra system, which is used to manipulate abstract objects. In this case, the human–computer interaction is mandatory. On the other side is a completely noninteractive (even batch) system, which resembles (in terms of development cycle) classical programming languages rather than common computer algebra systems.

3.1. Cadabra

The Cadabra is a special-purpose computer algebra system (more information is available at http://cadabra.phi-sci.com). It is mainly oriented to solving field theory problems. Since complex tensor computations are an integral part of the field theory, it is no wonder that this system support tensor computations to a high standard [4–9]. The Cadabra system extensively uses the notation of the TEX typesetting system.[1]

Presently, the Cadabra system implements only operations with abstract indices. Component computations are not supported. Yet, to implement component computations, it is required to supplement the Cadabra system with general-purpose tools of computer algebra.

[1] The deep integration between the Cadabra and TEX resulted in the fact that, upon installing TEX Live 2015 (http://www.tug.org/texlive), the Cadabra terminated with an error: `Undefined control sequence: \int_eval:w`. It was found that the error was caused by the conflict between the `breqn` and `expl3` packages of TEX Live 2015. The problem was resolved by updating the breqn package from the CTAN repository (http://ctan.org).

3.2. FORM

The FORM computer algebra system stands out quite markedly against the background of the other like systems: it is oriented to batch processing rather than to user interaction (more information is available at http://www.nikhef.nl/form) [10–12]. Hence, this system does not suffer from certain inherent drawbacks of common computer algebra systems, such as high resource usage, restrictions on the amount of computations, and slow speed. The FORM system supports various technologies of parallel and distributed computing [13], like, for example, multithreading and several implementations of the message passing interface (MPI). The system has an interface for interacting with external programs [14]. Thus, the FORM is often used as a backend to other (mostly, interactive) computer algebra systems. It finds an especially wide application in quantum field computations [15–19].

The FORM system had been developed in 1984, but it was made open-source only in 2010.

Basic features of FORM are as follows:

• arbitrary long mathematical expressions (limited only by the disk space);

• multithreaded execution and parallelization (MPI);

• fast trace calculation (of γ matrices);

• output into various formats (text, Fortran, etc.);

• interface for communicating with external programs.

3.3. Comparison between Cadabra and FORM

Below, the features of both computer algebra systems that are of interest in this work are outlined.

Cadabra

• The main feature is natural operations on tensors. Covariant and contravariant indices are supported by default.

• The system is effective for writing new formulas and relations in an interactive mode.

FORM

• The support of covariant and contravariant indices is implemented in an artificial way.

• The system is effective for final computations with already known formulas.

Thus, the Cadabra system is more appropriate for writing new formulas (see Section 4), while the FORM system is more suitable for direct computations with already derived formulas (see Section 5).

4. GEOMETRIZATION OF MAXWELL'S EQUATIONS

The two computer algebra systems described above are used for computer modeling and synthesis of Maxwell optics elements in terms of tensor calculus and curvilinear coordinates. This problem is naturally solved in two stages. First, the Cadabra system is used, since it is designed to manipulate tensor objects. All manipulations are performed with abstract indices only. Using the computer algebra system at this stage makes it possible to get rid of paper and a pen when carrying out the theoretical work. At the second stage, the FORM system is used to create a software complex for batch solution of stereotyped problems. This system is oriented to vector and tensor analysis and uses the noninteractive (batch) approach and parallelization of computations and external memory.

4.1. Idea of Geometrization

The employed methodology of modeling and synthesis of Maxwell optics elements uses an effective geometric paradigm [20], in which certain field parameters are translated into geometric parameters. In this case, macroscopic parameters of a medium are geometrized. Thus, direct and inverse problems can be solved: trajectories of electromagnetic wave propagation are found from known macroscopic parameters or these parameters are restored based on certain trajectories.

I.E. Tamm was first to try using methods of differential geometry in electrodynamics [21–23]. In 1960, J. Plebanski proposed a method for geometrizing constitutive equations of an electromagnetic field [24–27]; this method became classical.

The basic idea of a naive geometrisation of Maxwell's equations consists in the following.

1. Write Maxwell's equations in a medium in the Minkowski space.

2. Write Maxwell's equations in a vacuum in the effective Riemannian space.

3. Equate the corresponding terms of the equations.

Thus, we obtain the expression of dielectric permittivity and magnetic permeability via the metric of the corresponding effective space.

In more detail, this technique is described in [4, 28–30].

4.2. General Relations

Recall some basic facts about Maxwell's equations.

Let us write Maxwell's equation in terms of the electromagnetic field tensors $F_{\alpha\beta}$ and $H_{\alpha\beta}$ [31, 32]:

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} F_{[\alpha\beta;\gamma]} = 0, \qquad (1)$$

$$\nabla_\alpha H^{\alpha\beta} = \frac{4\pi}{c} J^\beta. \qquad (2)$$

Here, the tensors $F_{\alpha\beta}$ and $H^{\alpha\beta}$ have the following components:

$$F_{\underline{\alpha\beta}} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B^3 & B^2 \\ -E_2 & B^3 & 0 & -B^1 \\ -E_3 & -B^2 & B^1 & 0 \end{pmatrix},$$

$$H^{\underline{\alpha\beta}} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H_3 & H_2 \\ D^2 & H_3 & 0 & -H_1 \\ D^3 & -H_2 & H_1 & 0 \end{pmatrix}.$$

To take into account the medium, we introduce some macroscopic equations:

$$D^i = \varepsilon^{ij} E_j, \quad B^i = \mu^{ij} H_j, \qquad (3)$$

$$H^{\alpha\beta} = \lambda^{\alpha\beta}_{\gamma\delta} F^{\gamma\delta}, \quad \lambda^{\alpha\beta}_{\gamma\delta} = \lambda^{[\alpha\beta]}_{[\gamma\delta]}. \qquad (4)$$

Obviously, in the vacuum (when there is no medium), relations (3) and (4) take the following form:

$$\varepsilon^{ij} := \delta^{ij}, \quad \mu^{ij} := \delta^{ij}.$$

$$D^i = E^i, \quad B^i = H^i, \quad H^{\alpha\beta} = F^{\alpha\beta}.$$

4.3. Geometrization of Maxwell's Equations in Cartesian Coordinates

Note that, in the differential Bianchi identity (1), covariant derivatives can be replaced by partial derivatives:

$$\begin{gathered} \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = F_{[\alpha\beta;\gamma]} = 0, \\ \Downarrow \\ \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = F_{[\alpha\beta,\gamma]} = 0. \end{gathered} \qquad (5)$$

Indeed,

$$\begin{aligned} \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} &= \partial_\alpha F_{\beta\gamma} - \Gamma^\delta_{\alpha\beta} F_{\delta\gamma} \\ - \Gamma^\delta_{\alpha\gamma} F_{\beta\delta} + \partial_\beta F_{\gamma\alpha} &- \Gamma^\delta_{\beta\gamma} F_{\delta\alpha} - \Gamma^\delta_{\beta\alpha} F_{\gamma\delta} \\ +\partial_\gamma F_{\alpha\beta} &- \Gamma^\delta_{\gamma\alpha} F_{\delta\beta} - \Gamma^\delta_{\gamma\beta} F_{\alpha\delta}. \end{aligned} \qquad (6)$$

Let us demonstrate the evaluation of expression (6) in the Cadabra system.[2]

Set a list of indices:

```
{\alpha,\beta,\gamma,\delta}::Indices(vector).
```

Introduce partial and covariant derivatives:[3]

[2] A point at the end of an instruction suppresses the output of the result, while a semicolon allows printing the result.

[3] The symbol # is a pattern for any expression.

```
\partial_{#}::PartialDerivative.
\nabla_{#}::Derivative.
```

When deriving expression (6), it is important to take into account the symmetry of Christoffel symbols, which are defined using a Young diagram [33]:

```
\Gamma^{\alpha}_{\beta \gamma}::
TableauSymmetry(shape={2},
indices={1,2}).
```

For the tensor $F_{\alpha\beta}$, there is no need in the Young diagram, it is sufficient to indicate that this tensor is antisymmetric:

```
F_{\alpha \beta}::AntiSymmetric.
```

Let us write the expression for the covariant derivative as the substitution:[4]

```
nabla:=\nabla_{\gamma} A?_{\alpha
\beta} ->
\partial_{\gamma}{A?_{\alpha \beta}} -
A?_{\alpha \delta}
\Gamma^{\delta}_{\beta \gamma} -
A?_{\nabla \beta}
\Gamma^{\delta}_{\alpha \gamma};
```

$$nabla := \nabla_{\gamma} A?_{\alpha\beta} \to (\partial_{\gamma} A?_{\alpha\beta} - A?_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} - A?_{\delta\beta}\Gamma^{\delta}_{\alpha\gamma});$$

We use the postfix "?" to convert the preceding letter into a pattern (otherwise, the substitution can be used only with a fixed variable). Note that it is not necessary to apply the modifier "?" to indices.

Now, we rewrite equation (1) as follows:

```
maxwell1:= \nabla_{\alpha}F_{\beta
\gamma} +
\nabla_{\beta} F_{\nabla \alpha} +
\nabla_{\gamma} F_{\alpha \beta};
```

$$maxwell1 := \nabla_{\alpha} F_{\beta\gamma} + \nabla_{\beta} F_{\gamma\alpha} + \nabla_{\gamma} F_{\alpha\beta}.$$

Let us substitute |`nabla`| into expression |maxwell1|:

```
@substitute!(maxwell1)(@(nabla));
```

$$\begin{aligned} maxwell1 := {} & \partial_{\alpha} F_{\beta\gamma} - F_{\beta\delta}\Gamma^{\delta}_{\gamma\alpha} - F_{\delta\gamma}\Gamma^{\delta}_{\beta\alpha} \\ & + \partial_{\beta} F_{\gamma\alpha} - F_{\gamma\delta}\Gamma^{\delta}_{\alpha\beta} - F_{\delta\alpha}\Gamma^{\delta}_{\gamma\beta} \\ & + \partial_{\gamma} F_{\alpha\beta} - F_{\alpha\delta}\Gamma^{\delta}_{\beta\gamma} - F_{\delta\beta}\Gamma^{\delta}_{\alpha\gamma}. \end{aligned}$$

Let us collect like terms in expression |maxwell1|:

```
@canonicalise!(%);
@collect_terms!(%);
```

$$maxwell1 := \partial_{\alpha} F_{\beta\gamma} - \partial_{\beta} F_{\alpha\gamma} + \partial_{\gamma} F_{\alpha\beta};$$

Thus, we obtain exactly equation (5).

[4] The symbol := is used to set a label.

Now, we write Maxwell's equations (1) and (2) in a medium in Cartesian coordinates with a metric tensor $\eta_{\underline{\alpha\beta}} = \mathrm{diag}(1, -1, -1, -1)$:

$$\partial_{\alpha} F_{\beta\gamma} + \partial_{\beta} F_{\gamma\alpha} + \partial_{\gamma} F_{\alpha\beta} = 0,$$
$$\partial_{\alpha} H^{\alpha\beta} = \frac{4\pi}{c} j^{\beta}.$$

Similarly, we write Maxwell's equations in a vacuum in the effective Riemannian space with a metric tensor $g_{\alpha\beta}$:

$$\begin{gathered} \partial_{\alpha} f_{\beta\gamma} + \partial_{\beta} f_{\gamma\alpha} + \partial_{\gamma} f_{\alpha\beta} = 0, \\ \frac{1}{\sqrt{-g}} \partial_{\alpha}\left(\sqrt{-g} h^{\alpha\beta}\right) = \frac{4\pi}{c} \tilde{j}^{\beta}. \end{gathered} \tag{7}$$

Here, $f_{\alpha\beta}$ and $h^{\alpha\beta}$ are the Maxwell and Minkowski tensors in the effective Riemannian space.[5]

For this purpose, in the Cadabra system, covariant divergence is introduced:

```
div:=\nabla_{\alpha}A?^{\alpha \beta} ->
  1/\sqrt{-g}\partial_{\alpha}
  (\sqrt{-g} A?^{\alpha \beta});
```

$$div := \nabla_{\alpha} A?^{\alpha\beta} \to \frac{1}{\sqrt{-g}} \partial_{\alpha}(\sqrt{-g} A?^{\alpha\beta}).$$

Here, again, the postfix "?" converts the preceding symbol into a pattern.

Equation (2) is written in the effective Riemannian space:

```
riman:=\nabla_{\alpha}        h^{\alpha
\beta}=
j^{\beta} 4 \pi/c;
```

$$riman := \nabla_{\alpha} h^{\alpha\beta} = 4 j^{\beta} \frac{\pi}{c}.$$

Substitution of the covariant divergence into equation |`riman`| yields equation (7):

```
@substitute!(riman)(@(div));
```

$$riman := \frac{1}{\sqrt{-g}} \partial_{\alpha}(\sqrt{-g} h^{\alpha\beta}) = 4 j^{\beta} \frac{\pi}{c}.$$

Since the relation

$$f_{\alpha\beta} = h_{\alpha\beta}$$

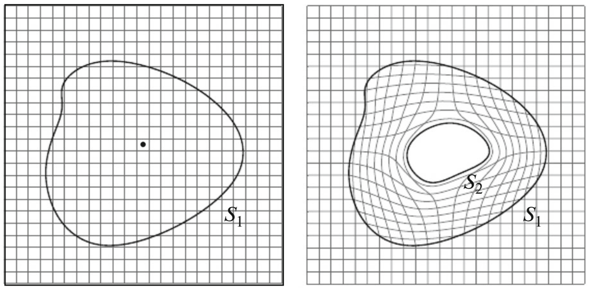
holds for the vacuum, by raising the indices, we obtain

$$f^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} h_{\gamma\delta}.$$

For this purpose, in the Cadabra, the corresponding substitution is performed:

```
fh:=h^{\alpha \beta} ->
  g^{\alpha \gamma}g^{\beta \delta}
  f_{\gamma \delta};
```

[5] Uppercase letters denote quantities in the Minkowski space, while lowercase letters denote quantities in the Riemannian space.



Two-dimensional projection of the coordinate transformation for the invisibility cap.

$$fh := h^{\alpha\beta} \to g^{\alpha\gamma} g^{\beta\delta} f_{\gamma\delta};$$

```
@substitute!(riman)(@(fh));
```

$$riman := \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} g^{\alpha\delta} g^{\beta\gamma} f_{\delta\gamma}) = 4 j^\beta \frac{\pi}{c};$$

From term by term comparison, we have

$$F_{\alpha\beta} = f_{\alpha\beta}, \quad j^\alpha \sqrt{-g} j^\alpha,$$
$$H^{\alpha\beta} = \sqrt{-g} g^{\alpha\gamma} g^{\beta\delta} F_{\gamma\delta}.$$

Therefore, we obtain the relation

$$F_{\alpha\beta} \frac{1}{\sqrt{-g}} g_{\alpha\gamma} g_{\beta\delta} H^{\gamma\delta}. \tag{8}$$

Based on relation (8), an electric displacement vector is expressed explicitly:

$$D^i = -\frac{\sqrt{-g}}{g_{00}} g^{ij} E_j + \frac{1}{g_{00}} \varepsilon^{ijk} g_{j0} H_k.$$

In this case, the geometrized dielectric permittivity is written as

$$\varepsilon^{ij} = -\frac{\sqrt{-g}}{g_{00}} g^{ij}. \tag{9}$$

Similar manipulations are performed for a magnetic displacement vector.

Thus, we obtain the magnetic displacement vector

$$B^i = -\frac{\sqrt{-g}}{g_{00}} g^{ij} H_j - \frac{1}{g_{00}} \varepsilon^{ijk} g_{j0} E_k$$

and the geometrized magnetic permittivity

$$\mu^{ij} = -\frac{\sqrt{-g}}{g_{00}} g^{ij}. \tag{10}$$

5. CASE STUDY

The symbolic manipulations performed in the Cadabra system yield the result that can be directly used for computer modeling and designing Maxwell (tensor-vector) optics elements, which can be represented in curvilinear coordinates. To carry out such computations, we employ the FORM system.

As an example, we consider the popular application of transformation optics: invisibility cap (or invisibility cloak) [27].

5.1. Invisibility Cap

Here, we solve the following inverse problem: given coordinates related to the configuration of the system to be designed, find parameters of the medium. Let the object to be hidden be inside a region S_1 (the invisibility cloak being designed) and be denoted by a point (see figure). Originally, we have a Cartesian coordinate system $x^{i'}$. Surround the object by a boundary S_2 and deform the coordinates in the region between S_1 and S_2 by converting $x^{i'}$ into x^i. Geodetic data outside S_1 remain unchanged. Moreover, light does not enter the region S_2 (any object inside the region S_2 causes no changes outside the region S_1). Hence, this structure actually acts as an invisibility cap.

For simplicity, we design a cylindrical invisibility cap. The original flat[6] coordinate system is denoted by (r', φ', z'). In these coordinates, the boundary S_1 is defined as $r' = b$. By transforming the coordinates $(r', \varphi', z') \to (r, \varphi, z)$, we obtain

$$r' \frac{b-a}{a} r + a, \quad r' \le br, \quad r' > b,$$
$$\varphi' = \varphi, \quad z' = z.$$

Thus, the region $r' \le b$ is contracted into the region $a \le r \le b$, while the metric tensor in the curvilinear coordinates takes the form $g_{ij} = \mathrm{diag}[b^2/(b-a)^2, 1, 1]$.

Below, we illustrate the calculation of the determinant for the metric tensor g in the FORM system.

Let us disable the additional information about the computational process (resources, time, etc.). To fit the calculation results in the paper, we decrease the output width up to 40 symbols (by default, it is 80):

```
Off statistic;
Format 40;
```

Let us specify explicitly that the four-dimensional space is used. Then, we define basic elements, i.e., indices (i, j, k, and l),[7] tensors ($g_{\alpha\beta}$), and objects with no additional semantics (a, b):

```
Dimension 4;
Indices i, j, k, l;
Tensors g;
```

[6] A flat manifold is that of zero curvature. Accordingly, the coordinate system superimposed on such a manifold is also called flat.

[7] For compactness, we use Latin letters (instead of Greek letters) as FORM indices.

```
Symbols a,b;
```

Let us define a formula for the determinant (since there is no such a function in the FORM). For this purpose, the Levi–Civita symbol is used:

$$\det g_{\alpha\beta} =: g = \varepsilon^{0123}\varepsilon^{\alpha\beta\gamma\delta} g_{0\alpha} g_{1\beta} g_{2\gamma} g_{3\delta}.$$

```
Local detG = e_(1,2,3)*e_(i,j,k,l)
  *g(0,i)*g(1,j)*g(2,k)*g(3,l);
  contact;
  Print;
. sort
```

The result is represented in the following form:

$$\begin{gathered}\det G = g(0,0)*g(1,1)*g(2,2)*g(3,3)\\ - g(0,0)*g(1,1)*g(2,3)*g(3,2) - g(0,0)\\ *g(1,2)*g(2,1)*g(3,3) + g(0,0)*g(1,2)*g(2,3)*g(3,1)\\ + g(0,0)*g(1,3)*g(2,1)*g(3,2) - g(0,0)*g(1,3)*\\ g(2,2)*g(3,1) - g(0,1)*g(1,0)*g(2,2)*g(3,3)\\ + g(0,1)*g(1,0)*g(2,3)*g(3,2)\\ + g(0,1)*g(1,2)*g(2,0)*g(3,3)\\ - g(0,1)*g(1,2)*g(2,3)*g(3,0)\\ - g(0,1)*g(1,3)*g(2,0)*g(3,2)\\ + g(0,1)*g(1,3)*g(2,2)*g(3,0)\\ + g(0,2)*g(1,0)*g(2,1)*g(3,3)\\ - g(0,2)*g(1,0)*g(2,3)*g(3,1) - g(0,2)\\ *g(1,1)*g(2,0)*g(3,3) + g(0,2)*g(1,1)*g(2,3)*g(3,0)\\ + g(0,2)*g(1,3)*g(2,0)*g(3,1) - g(0,2)*g(1,3)*\\ g(2,1)*g(3,0) - g(0,3)*g(1,0)*g(2,1)*g(3,2)\\ + g(0,3)*g(1,0)*g(2,2)*g(3,1)\\ + g(0,3)*g(1,1)*g(2,0)*g(3,2)\\ - g(0,3)*g(1,1)*g(2,2)*g(3,0)\\ - g(0,3)*g(1,2)*g(2,0)*g(3,1)\\ + g(0,3)*g(1,2)*g(2,1)*g(3,0).\end{gathered}$$

In the FORM system, particular values of tensor components are given by pattern substitutions (in this case, $g_{\alpha\beta} = \mathrm{diag}[1, -b^2/(b-a)^2, -1, -1]$). The postfix modifier "?" converts the preceding symbol into a pattern:

```
  id g(0,0) = 1;
  id g(1,1) = -b^2/(b-a)^2;
  id g(i?,i?) = -1;
  id g(i?,j?) = 0.
  Print;
  .sort
  .end
```

Thus, we obtain the value of the determinant:

$$\det g_{\alpha\beta} =: g = -\frac{b^2}{(b-a)^2}.$$

```
 detG = -1/(b^2 - 2*a*b + a^2)*b^2;
```

Now, based on (9) and (10), we find the parameters of the medium:

$$\varepsilon_r = \mu_r = \frac{r-a}{r},$$

$$\varepsilon_\varphi = \mu_\varphi = \frac{r}{r-a},$$

$$\varepsilon_z = \mu_z = \left(\frac{b}{b-a}\right)^2 \frac{r-a}{r}.$$

The medium parameters can be similarly geometrized for the invisibility cap that has other symmetry characteristics (or even has no explicit symmetry). For example, for a spherical invisibility cap, the medium parameters are as follows:

$$\varepsilon_r = \mu_r = \left(\frac{r-a}{r}\right)^2 \frac{b}{b-a},$$

$$\varepsilon_\vartheta = \mu_\vartheta = \frac{b}{b-a},$$

$$\varepsilon_\varphi = \mu_\varphi = \frac{b}{b-a}.$$

CONCLUSIONS

There are many applied scientific problems that require symbolic computations of two types: for prototype development of a new software product and for serial numerical and symbolic computations on the already debugged software product. Therefore, dividing computer algebra problems into interactive and noninteractive (batch) ones seems quite reasonable.

In this paper, as a model problem, the geometrization of material Maxwell's equations is considered. Our approach is based on the idea of adopting the most promising mathematical and conceptual frameworks from other scientific fields. In this case, the geometric paradigm is used in the framework of the field theory.

Since differential geometry forms the mathematical basis of the geometric paradigm [20], we select the computer algebra systems that are oriented to the tensor and vector analysis and support manipulations with abstract tensors. Thus, the Cadabra system is used for interactive operations, while the FORM system is employed for batch computations.

We hope that the selected examples clearly demonstrate the capabilities of this approach for solving such problems.

ACKNOWLEDGMENTS

The author thanks Jos Vermaseren for drawing his attention to the FORM computer algebra system.

This work was supported in part by the Russian Foundation for Basic Research (project nos. 14-01-00628 and 15-07-08795).

The computations were carried out on the Felix computational cluster (Peoples' Friendship University of Russia) and on the HybriLIT heterogeneous cluster (Multifunctional center for data storage, processing, and analysis at the Joint Institute for Nuclear Research).

REFERENCES

1. Hadamard, J.S., *Psychology of Invention in the Mathematical Field,* Dover, 1954, 2nd ed.
2. Brooks, F.P.J., No silver bullet-essence and accidents of software engineering, *Proc. IFIP Tenth World Computing Conference,* 1986, pp. 1069–1076.
3. Penrose, R. and Rindler, W., *Spinors and Space-Time: Two-Spinor Calculus and Relativistic Fields,* Cambridge: Cambridge Univ. Press, 1987, vol. 1.
4. Korol'kova, A.V., Kulyabov, D.S., and Sevast'yanov, L.A., Tensor computations in computer algebra systems, *Program. Comput. Software,* 2013, vol. 39, no. 3, pp. 135–142.
5. Sevastianov, L.A., Kulyabov, D.S., and Kokotchikova, M.G., An application of computer algebra system Cadabra to scientific problems of physics, *Phys. Part. Nucl. Lett.,* 2009, vol. 6, no. 7, pp. 530–534.
6. Peeters, K., Cadabra: A field-theory motivated symbolic computer algebra system, *Comput. Phys. Commun.,* 2007, vol. 176, no. 8, pp. 550–558.
7. Peeters, K., Introducing Cadabra: A symbolic computer algebra system for field theory problems. http://arxiv.org/abs/hep-th/0701238.
8. Peeters, K., Symbolic field theory with Cadabra, *Computeralgebra-Rundbrief,* 2007, no. 41, pp. 16–19.
9. Brewin, L., A brief introduction to Cadabra: A tool for tensor computations in general relativity, *Comput. Phys. Commun.,* 2010, vol. 181, no. 3, pp. 489–498.
10. Tung, M.M., FORM matters: Fast symbolic computation under UNIX, *Comput. Math. Appl.,* 2005, vol. 49, pp. 1127–1137.
11. Vermaseren, J.A.M., Kuipers, J., Tentyukov, M., et al., *FORM version 4.1 Reference Material,* 2013.
12. Heck, A.J.P. and Vermaseren, J.A.M., *FORM for Pedestrians,* Amsterdam, 2000.
13. Fliegner, D., Retey, A., and Vermaseren, J.A.M., Parallelizing the symbolic manipulation program FORM. http://arxiv.org/abs/hep-ph/9906426.
14. Tentyukov, M. and Vermaseren, J.A.M., Extension of the functionality of the symbolic program FORM by external software, *Comput. Phys. Commun.,* 2007, vol. 176, no. 6, pp. 385–405.
15. Boos, E.E. and Dubinin, M.N., Problems of automatic computations for physics on colliders, *Usp. Fiz. Nauk,* 2010, vol. 180, no. 10, pp. 1081–1094
16. Bunichev, V., Kryukov, A., and Vologdin, A., Using FORM for symbolic evaluation of Feynman diagrams in CompHEP package, *Nucl. Instrum. Methods Phys. Res., Sect. A,* 2003, vol. 502, pp. 564–566.
17. Hahn, T., Generating and calculating one-loop Feynman diagrams with FeynArts, FormCalc, and LoopTools. http://arxiv.org/abs/hep-ph/9905354.
18. Hahn, T., Automatic loop calculations with FeynArts, FormCalc, and LoopTools. http://arxiv.org/abs/hep-ph/0005029.
19. Hahn, T. and Lang, P., FeynEdit: A tool for drawing Feynman diagrams. http://arxiv.org/abs/0711.1345.
20. Wheeler, J.A., *Neutrinos, Gravitation, and Geometry,* Bologna, 1960.
21. Tamm, I.E., Electrodynamics of an anisotropic medium in the special relativity theory, *Zh. Russ. Fiz.-Khim. O-va., Chast Fiz.,* 1924, vol. 56, nos. 2–3, pp. 248–262.
22. Tamm, I.E., Crystal optics of the relativity theory in connection with the geometry of a biquadratic form, *Zh. Russ. Fiz.-Khim. O-va., Chast Fiz.,* 1925, vol. 57, nos. 3–4, pp. 209–240.
23. Tamm, I.E. and Mandelstam, L.I., Elektrodynamik der anisotropen Medien in der speziellen Relativitatstheorie, *Mathematische Annalen,* 1925, vol. 95, no. 1, pp. 154–160.
24. Plebanski, J., Electromagnetic waves in gravitational fields, *Phys. Rev.,* 1960, vol. 118, no. 5, pp. 1396–1408.
25. Felice, F., On the gravitational field acting as an optical medium, *Gen. Relativ. Gravitation,* 1971, vol. **2**, no. 4, pp. 347–357.
26. Leonhardt, U., Philbin, T.G., and Haugh, N., *General Relativity in Electrical Engineering,* 2008, pp. 1–19.
27. Leonhardt, U. and Philbin, T.G., Transformation optics and the geometry of light, *Prog. Opt.,* 2009, vol. **53**, pp. 69–152.
28. Kulyabov, D.S., Korolkova, A.V., and Korolkov, V.I., Maxwell's equations in arbitrary coordinate system, *Bulletin of Peoples' Friendship University of Russia, Series "Mathematics. Information Sciences. Physics,"* 2012, no. 1, pp. 96–106.
29. Kulyabov, D.S., Geometrization of electromagnetic waves, *Proc. Int. Conf. Mathematical Modeling and Computational Physics (MMCP),* Dubna, 2013, p. 120.
30. Kulyabov, D.S. and Nemchaninova, N.A., Maxwell's equations in curvilinear coordinates, *Bulletin of Peoples' Friendship University of Russia, Series "Mathematics. Information Sciences. Physics,"* 2011, no. 2, pp. 172–179.
31. Minkowski, H., Die grundlagen fur die electromagnetischen vorgange in bewegten korpern, *Nachr. Ges. Wiss. Goettingen, Math.-Phys. Kl.,* 1908, no. 68, pp. 53–111.
32. Stratton, J.A., *Electromagnetic Theory,* Wiley, 2007.
33. Fulton W., *Young Tableaux: With Applications to Representation Theory and Geometry,* Cambridge: Cambridge Univ. Press, 1997.

Translated by Yu. Kornienko