Kink-like Configurations of Interacting Scalar, Electromagnetic, and Gravitational Fields

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We have obtained exact kink-like static plane-symmetric solutions to the self-consistent system of electromagnetic, scalar, and gravitational field equations. It was shown that under certain choice of the interaction Lagrangian the solutions are regular and have localized energy. The linearized instability of corresponding solutions was established both for the case of flat space-time and that of interaction with the proper gravitational field.

Introduction

There exist many nonlinear field models, exploiting one self-interacting field or several interacting fields, that admit the existence of localized configurations of soliton type. These configurations are often considered as images of extended elementary particles.[1] Among such systems the interacting scalar, electromagnetic, and gravitational fields are the most widely encountered in nature.

In this paper we consider the massles scalar, electromagnetic, and gravitational fields for the choice of interaction Lagrangian explicitly depending on the electromagnetic potentials. For the latter case, after excluding from the system of equations the scalar field function, we obtain the electromagnetic field equation with induced nonlinearity containing the electromagnetic potentials.[2] Such an interaction appears to be interesting due to the fact that in the static spherically-symmetric case the corresponding system of equations admit regular localized solutions of soliton type both in flat space-time and when taking into account of the proper gravitational field. In addition at space infinity the solutions to the electromagnetic field equations become those of the linear electromagnetism. Therefore the dependence on elecromagnetic potentials emerges only locally, that is in the region of particle localization where the electromagnetic field seems to be unobservable.[3] Explicit inclusion of the potentials in the equations of motion is motivated by the non-existence of regular solutions to the gauge-invariant nonlinear electromagnetic field equations.[4] We show that the model suggested admits the existence of kink-like solutions with localized energy and charge. The configuration of this kind describes highly polarized system in which positive and negative charges are localized in the neighbour regions. It will be shown that such a system is linearly unstable.

1 Structure of localized solutions

Lagrangian density for the system of interacting scalar, electromagnetic, and gravitational fields is choosen as follows:

$$L = \frac{R}{2\varkappa} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha}\Psi(I), \tag{1}$$

where R is the scalar curvature, $\varkappa$ is the Einstein gravitational constant, $I = A_\alpha A^\alpha$, $\Psi = 1 + \lambda\Phi(I)$, λ is an interaction parameter, Φ is an arbitrary function. For $\lambda = 0$ we come to the system of scalar and electromagnetic fields with the minimal coupling.

The static plane-symmetric metric is taken in the form:

$$ds^2 = e^{2\gamma(x)}dt^2 - e^{2\alpha(x)}dx^2 - e^{2\beta(x)}\left(dy^2 + dz^2\right), \tag{2}$$

where the velocity of light $c = 1$, and the functions α, β, γ depend only on x. In the following we use the harmonic coordinate condition:

$$\alpha = 2\beta + \gamma. \tag{3}$$

The Einstein's equations for the metric (2) under the condition (3) read:

$$G_0^0 = e^{-2\alpha}\left(2\beta'' - 2\gamma'\beta' - \beta'^2\right) = -\varkappa T_0^0, \tag{4}$$

$$G_1^1 = e^{-2\alpha}\left(2\gamma'\beta' + \beta'^2\right) = -\varkappa T_1^1, \tag{5}$$

$$G_2^2 = e^{-2\alpha}\left(\beta'' + \gamma'' - 2\gamma'\beta' - \beta'^2\right) = -\varkappa T_2^2, \tag{6}$$

$$G_2^2 = G_3^3, \qquad T_2^2 = T_3^3. \tag{7}$$

Let us write down the material field equations:

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}g^{\nu\mu}\varphi_{,\mu}\Psi\right) = 0, \tag{8}$$

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}F^{\nu\mu}\right) - \varphi_{,\alpha}\varphi^{,\alpha}\Psi_I A^\nu = 0, \tag{9}$$

where $\Psi_I = d\Psi/dI$.

The energy-momentum tensor of the interacting fields takes the form

$$T_\mu^\nu = \varphi_{,\mu}\varphi^{,\nu}\Psi(I) - F_{\mu\alpha}F^{\nu\alpha} + \varphi_{,\alpha}\varphi^{,\alpha}\Psi_I A^\nu A_\mu - \\ - \delta_\mu^\nu\left[-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{2}\left(\varphi_{,\alpha}\varphi^{,\alpha}\right)\Psi(I)\right]. \tag{10}$$

The field functions depending only on x, the electromagnetic field is determined by the single component of the vector-potential $A_0 = A(x)$, being given by the component of the field tensor $F_{10} = dA/dx$. In this case $I = A_\alpha A^\alpha = g^{00}A_0^2 = e^{-2\gamma}A^2(x)$.

Scalar field equation (8) is written in metric (2) as follows:

$$\frac{d}{dx}(\varphi'\Psi) = 0. \tag{11}$$

Equation (11) has the solution

$$\varphi'(x) = CP(I), \tag{12}$$

where $P(I) = 1/\Psi(I)$, C is the integration constant.

Components of the energy-momentum tensor are:

$$T_0^0 = \frac{1}{2}e^{-2\alpha}\left[C^2P(I) + e^{-2\gamma}A'^2 + 2C^2P_I(I)e^{-2\gamma}A^2\right], \tag{13}$$

$$T_1^1 = -T_2^2 = -T_3^3 = \frac{1}{2}e^{-2\alpha}\left[C^2P(I) + e^{2\gamma}A'^2\right], \tag{14}$$

where $P_I = dP/dI$, $A' = dA/dx$.

Electromagnetic field equation (9) in metric (2) in view of (12) reads

$$\left(e^{-2\gamma}A'\right)' - C^2P_Ie^{-2\gamma}A = 0. \tag{15}$$

Sum of the Enstein's equations (5) and (6) with taking into account of (14) leads to the equation

$$\beta'' + \gamma'' = 0. \tag{16}$$

Equation (16) has the solution

$$\beta(x) = -\gamma(x) + Ex, \qquad E = \text{const.} \tag{17}$$

Sum of the Einstein's equations (4) and (5) implies

$$\beta'' = -\frac{\varkappa e^{-2\gamma}}{2}\left[A'^2 + C^2 P_I A^2\right]. \tag{18}$$

By substituting $AP_I e^{-2\gamma}$ from (15) into (18) we get the equation

$$\beta'' = -\frac{\varkappa}{2}\left(AA' e^{-2\gamma}\right)'. \tag{19}$$

We find for (16) due to $\gamma(x)$ the analogous equation:

$$\gamma'' = \frac{\varkappa}{2}\left(AA' e^{-2\gamma}\right)'. \tag{20}$$

The first integral for the equation (20) is

$$\gamma' = \frac{\varkappa}{2} AA' e^{-2\gamma} + k, \tag{21}$$

with k being constant.

Let us now consider the special case when $k = 0$. Then equation (21) is integrated to:

$$e^{2\gamma} = \frac{\varkappa}{2} A^2 + H, \qquad H = \text{const.} \tag{22}$$

By substituting (22) into (15) we get:

$$\left(\frac{A'}{\varkappa A^2/2 + H}\right)' - C^2 P_I \frac{A}{\varkappa A^2/2 + H} = 0. \tag{23}$$

Since $dI/dx = 2HAA'/(\varkappa A^2/2+H)^2$, then multiplying (23) by $A'/(\varkappa A^2/2+H)$, we obtain the equation which has the first integral

$$\left(\frac{A'}{\varkappa A^2/2 + H}\right)^2 - \frac{C^2}{H} P(I) = C_1, \qquad C_1 = \text{const.} \tag{24}$$

Let us consider the solution of the equation (24) with $C_1 = 0$. Then we have finally:

$$\int \frac{dA}{(\varkappa A^2/2 + H)\sqrt{P(I)}} = \pm \frac{C(x + x_1)}{\sqrt{H}}, \qquad x_1 = \text{const.} \tag{25}$$

The energy length density of the interacting electromagnetic and scalar fields appears to be

$$E_f = \int T_0^0 \sqrt{-^3 g}\, dx =$$
$$= \frac{1}{2}\int \left[C^2 P(I) + e^{-2\gamma} A'^2 + 2C^2 P_I I\right] e^{-\alpha + 2\beta} dx. \tag{26}$$

After substituting the equality

$$e^{-2\gamma}A'^2 = \frac{C^2}{H}P(I)e^{2\gamma}, \tag{27}$$

emerging from (24), into (26)
and using

$$dx = \frac{\sqrt{H}}{C}\frac{e^{-2\gamma}}{\sqrt{P}}\,dA, \tag{28}$$

which we obtain from (27), equality (26) becomes

$$E_f = \frac{C\sqrt{H}}{2}\int\left[\sqrt{P}\left(1+\frac{e^{2\gamma}}{H}\right)+\frac{2IP_I}{\sqrt{P}}\right]e^{-3\gamma}dA. \tag{29}$$

Since $I = e^{-2\gamma}A^2 = A^2/\left(\varkappa A^2/2 + H\right)$, then

$$e^{-3\gamma}dA = \frac{d\sqrt{I}}{H}. \tag{30}$$

Substituting (30) into (29) we get the following expression for the field energy length density:

$$E_f = \frac{C}{2\sqrt{H}}\int\left[\sqrt{P}\left(1+\frac{e^{2\gamma}}{H}\right)+\frac{2IP_I}{\sqrt{P}}\right]d\sqrt{I}., \tag{31}$$

and the invariant energy density:

$$T_0^0\sqrt{-^3g} = \frac{1}{2}\left[C^2P(I)+e^{-2\gamma}A'^2+2C^2P_II\right]e^{-\gamma}. \tag{32}$$

Let us find the density of the electric charge ρ_e and the total charge Q. The charge distribution we take from (9):

$$j^\alpha = -\varphi_{,\gamma}\varphi^{,\gamma}\Psi_IA^\alpha. \tag{33}$$

In the case of the static electric field we get from (33):

$$j^0 = -C^2e^{-2(\alpha+\gamma)}P_IA. \tag{34}$$

Chronometricaly-invariant charge density is defined as

$$\rho_e = \frac{j^0}{\sqrt{g^{00}}} = -C^2e^{-2\alpha-\gamma}P_IA. \tag{35}$$

The total charge Q is

$$Q = \int\rho_e\sqrt{-^3g}\,dx. \tag{36}$$

Let us choose the function $P(I)$ as

$$P(I) = (1-\lambda I)^2, \tag{37}$$

where λ is the selfcoupling parameter. Substituting (37) into (15) we obtain the equation

$$\left(e^{-2\gamma}A'\right)' + 2\lambda C^2 e^{-2\gamma}A - 2\lambda^2 C^2 e^{-4\gamma}A^3 = 0. \tag{38}$$

Inserting (37) into (25), one gets the following solution of the equation (38):

$$A(x) = \sqrt{\frac{H}{\lambda - \varkappa/2}}\,\mathrm{th}\, bx, \tag{39}$$

where $b = \sqrt{C^2(\lambda - \varkappa/2)}$, $-\infty \leqslant x \leqslant \infty$. The potential $A(x)$ in (39) is the regular function taking finite values $-\varkappa/2$ as $x \to \pm\infty$.

Under substitution of (39) into (32) we get the expression for $e^{2\gamma}$:

$$e^{2\gamma} = \frac{H\lambda}{\lambda - \varkappa/2}\left(1 - \frac{\varkappa}{2\lambda}\frac{1}{\mathrm{ch}^2\, bx}\right). \tag{40}$$

As follows from (40), under the restriction

$$H\lambda = \lambda - \frac{\varkappa}{2} > 0, \tag{41}$$

$e^{2\gamma} \to 1$ as $x \to \pm\infty$. Thus $e^{2\gamma}$ is a regular function for $x \in [-\infty, \infty]$.

The Einstein equation (5) generates, the first integral of the system of equations (4) and (6):

$$-\gamma'^2 + E^2 = -\frac{\varkappa}{2}\left[-C^2 P(I) + e^{-2\gamma}A'^2\right]. \tag{42}$$

Under sustituting into (42) the function $P(I)$ from (37) and also $A(x)$ from (39) and $e^{2\gamma}$ from (40) we get the identity if $E = 0$. One concludes, in view of (17), that

$$\beta(x) \equiv -\gamma(x). \tag{43}$$

Let us consider the energy distribution, that is the energy density $T_0^0\sqrt{-{}^3g}$:

$$\begin{aligned} T_0^0\sqrt{-{}^3g} &= \\ &= \frac{1}{2}\left[C^2(1-\lambda I)^2 + e^{-2\gamma}A'^2 - 4C^2(1-\lambda I)\lambda I\right] e^{-\gamma} = \\ &= \frac{1}{2}\frac{C^2(1-\sigma^2)}{\mathrm{ch}^2\, bx - \sigma}\left[\frac{1-\sigma^2}{\mathrm{ch}^2\, bx - \sigma^2} + \frac{1}{\mathrm{ch}^2\, bx} - \frac{4\,\mathrm{sh}^2\, bx}{\mathrm{ch}^2\, bx - \sigma^2}\right] e^{-\gamma}, \end{aligned} \tag{44}$$

where $\sigma^2 = \varkappa/2\lambda < 1$.

It follows from (44) that the energy density is localized:

$$T_0^0\sqrt{-{}^3g} \to 0, \qquad x \to \pm\infty. \tag{45}$$

The energy E_f of the interacting A_μ and φ fields is given by the following expression:

$$\begin{aligned} E_f &= \int_{-\infty}^{\infty} T_0^0\sqrt{-{}^3g}\,dx = \\ &= \frac{\lambda C}{2(\varkappa/2)^{3/2}}\left[\frac{\sigma}{\sqrt{1-\sigma^2}} - \frac{\sqrt{1-\sigma^2}}{2}\ln\left(\frac{1+\sigma}{1-\sigma}\right)\right]. \end{aligned} \tag{46}$$

From (46) we deduce that under $0 < \sigma < 1$, E_f is a positive bounded quantity.

Let us consider the distribution of charge density ρ_e for the solution found. From (35) we get

$$\rho_e = \frac{2C^2\sqrt{\lambda}(1-\sigma^2)\,\mathrm{sh}\,bx}{\mathrm{ch}^2\,bx\sqrt{\mathrm{ch}^2\,bx-\sigma^2}}. \tag{47}$$

It follows from (47) that $\rho_e = 0$ at $x = 0$, $x = \pm\infty$.

Since ρ_e is an odd function, one finds that the charge Q calculated according to (36) vanishes.

2 Stability analysis

Let us study the influence of the gravitational field on the stability of the solutions obtained. For this purpose we compare two cases, with the gravitation being involved or not.

It should be first noticed that the second variation of the total energy of the system in question turns out to be sign-indefinite quadratic functional. To check this important property it is sufficient to consider the case of flat space-time. The modes A_2 and A_3 being separated one can omit them without loss of generality.

Let us introduce the notations for the perturbations:

$$a_0 := \delta A_0, \quad a_1 := \delta A_1, \quad \xi := \delta\varphi.$$

After tedeons calculations we get for the second variation of the energy the following expression:

$$\delta^2 E = \int_{-\infty}^{\infty}\Big[\frac{1}{2}(a_0' - \dot a_1)^2 + \frac{1}{2}(\dot\xi^2 + \xi'^2)\frac{1}{(1-\lambda A_0^2)^2} - \\ - a_0^2\frac{\lambda\varphi'^2}{(1-\lambda A_0^2)^4}(1-5\lambda A_0^2) - \frac{\lambda\varphi'^2}{(1-\lambda A_0^2)^3}a_1^2\Big]\,dx. \tag{48}$$

Sign-alternating character of the functional (48) makes very probable the conclusion about the instability of the configuration in question. In order to prove it let us consider the linearized equations for the perturbations. In particular let us select the equation for the perturbation of the A_1 mode.

$$(\dot a_1 - a_0')^{\cdot} = \frac{2\lambda\varphi'^2}{(1-\lambda A_0^2)^3}a_1. \tag{49}$$

If we consider $\dot a_0'$ as a source then the solution of the equation (49) can be written in the form

$$a_1 = a_1^{(0)} + a_1^{(1)},$$

where $a_1^{(1)}$ is a particular solution of the nonhomogeneous equation and $a_1^{(0)}$ satisfies the homogeneous one. It is easy to see that $a_1^{(0)}$ increases exponentially with time thus configuring the instability of the unpertiorbed configuration.

Let us now write down the equation of motion for the a_1 with the gravitation included perturbation :

$$(\dot{a}_1 - a_0')^{\cdot} = \frac{2\lambda {\varphi'}^2}{(1-\lambda A_0^2)^3} a_1 - 4A_0'\delta\alpha. \tag{50}$$

Repeating all the previous arguments and representing $\dot{a}_0' - 4A_0'\delta\alpha$ as a source in the equation (50), we confirm the instability of the self-gravitating configuration.

References

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