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ABSTRACT

When the Maxwell equations are geometrized, the Maxwell Lagrangian is usually reduced to the Yang-Mills Lagrangian. In this case, the effective quadratic metric, usually corresponding to the Riemannian metric of our space, is considered. However, it is more reasonable to use Finsler approach to Maxwell's equations. In the paper the Finsler representation of the geometrized Maxwell equations is considered. The comparison with the Riemannian approach also is made.

Keywords: Maxwell's equations, Riemannian geometry, Finsler geometry, transformation optics

1. INTRODUCTION

The ideology of transformational optics is based on the following trick.[1–4] The Maxwell's equations for medium are usually written for flat Minkowski space. Also Maxwell's equations may be written in vacuum, but for an arbitrary Riemannian space. By equating the relevant terms in these systems, we obtain the geometrization of the medium parameters. However,the given method has certain drawbacks. The permeability (permittivity) tensor is constructed from the metric tensor,[5,6] but the metric tensor in Riemannian space has only 10 independent components. This not enough to describe the full permittivity tensor. So it seems justified to use a richer geometric structure such as Finsler geometry.[7]

The paper presents the basic elements of Finsler geometry. On this basis, we use differential forms to write Maxwell's equations both for Riemannian and Finsler geometries.[8,9]

2. NOTATIONS AND CONVENTIONS

1. We will adhere to the following agreements. Greek indices (α, β) will refer to the four-dimensional space. Latin indices will refer to the space of arbitrary dimension.
2. The comma in the index denotes a partial derivative with respect to corresponding coordinate ($f_{,i} := \partial_i f$); the semicolon denotes a covariant derivative ($f_{;i} := \nabla_i f$).

3. ELEMENTS OF FINSLER GEOMETRY

Finsler geometry can be considered as geometry without restriction by quadratic metric.[10] In the general case the Finsler metric depends not only on the coordinates, but also on the velocities.[11,12]

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3.1 General definitions

Let M be a smooth n-dimensional manifold, TM is a tangent bundle over M, (x^i) are local coordinates on M, (x^i, y^i) are natural local coordinates on TM. The Finsler structure on M is determined by scalar function $L(x, y)$ on TM which satisfies the following conditions:

- $L(x, y)$ — positive homogeneous function of the first degree with respect of the tangent vector coordinates:

$$L(x, \lambda y) = \lambda(x, y), \quad \lambda > 0;$$

- $L(x, y)$ is positive if $y \neq 0$:

$$L(x, y) > 0, \quad y \neq 0;$$

- the quadratic form $\varphi(\xi) = g_{ij}(x, y)\xi^i\xi^j$ is positive definite, that is $\varphi(\xi) > 0$ for all $\xi \neq 0$, where functions

$$g_{ij} := F_{\cdot i \cdot j}$$

are components of the non-degenerate tensor field g (the metric tensor of the space). Here

$$F_{\cdot i} := \frac{\partial F}{\partial y^i}.$$

Manifold M with given Finsler structure L is called the Finsler space F^n. The function

$$F = L^2$$

is called the metric function of the Finsler space F^n. If x and $x + \mathrm{d}x$ are two infinitely close points of space F^n, then the distance $\mathrm{d}s$ between them is the value of function L in point $(x, \mathrm{d}x)$:

$$\mathrm{d}s = L(x, \mathrm{d}x).$$

The length s of the curve $c : x = x(t)$ that connects the points $x_1 = x(t_1)$ and $x_2 = x(t_2)$ is determined by the integral

$$s = \int_{t_1}^{t_2} L(x(t), \dot{x}(t))\, \mathrm{d}t\,,$$

where $\dot{x}(t) = \frac{\mathrm{d}x(t)}{\mathrm{d}t}$ is the velocity vector of the curve. The fundamental function F is a homogeneous function of the second degree from the coordinates of the tangent vector. Therefore, by the Euler theorem we get:

$$y^i F_{\cdot i} = 2F.$$

Differentiating this expression by y^I, we obtain:

$$y^i F_{\cdot i \cdot j} = F_{\cdot j},$$

from where, from convolution with y^j, we get:

$$g_{ij} y^i y^j = F$$

and

$$\mathrm{d}s^2 = g_{ij}(x, \mathrm{d}x)\, \mathrm{d}x^i\, \mathrm{d}x^j\,.$$

In the last expression the functions $g_{ij}(x, y)$ are homogeneous functions of y:

$$y^k g_{ij \cdot k} = 0.$$

In Finsler geometry the characteristic element is the tensor

$$C_{ijk} = \frac{1}{2} g_{ij \cdot k},$$

the vanishing of which is a necessary and sufficient condition for the space F^n to be Riemannian. This tensor is symmetric by all indices.

In the tangent space $T_x M$ we will consider the equation

$$L(x^i, y^i) = 1.$$

By identifying $T_x M$ with centroaffine space, the center o of which ($y^i = 0$) is the tangent point $x \in M$, we can consider that this equation defines in $T_x M$ a hypersurface which is called as the indicatrix. It turns out that points y of $T_x M$ satisfying the inequality

$$L(x^i, y^i) \leqslant 1,$$

are internal or boundary points of a convex body whose boundary is given by the indicatrix equation. This is the consequence from the conditions of the basic definition. Now we can define the length of the vector y of the centroaffine space $T_x m$ by using equality

$$|y| = L(x^i, y^i).$$

Thus defined metric in the vector space $T_x M$ will be the Minkowski metric. Therefore, the Finsler space can be defined as a smooth manifold M and in its tangent spaces $T_x M$ one can define the Minkowski metric which depends on $x \in M$ as smooth function.

The manifold M is called Finsler space F^n with alternating metric if in its tangent bundle TM the metric is given by homogeneous function $F(x, y)$ of second degree on the coordinates of the tangent vector:

$$F(x, \lambda y) = \lambda^2 F(x, y), \quad \lambda \neq 0.$$

Moreover, it is a non-degenerate function:

$$\det \| F_{\cdot i \cdot j} \| \neq 0.$$

3.2 Berwald connection

In Finsler geometry there are two main connections: the Berwald and the Cartan ones.

The characteristic feature of the Berwald connection is the coincidence of its geodesic with the extremals of the functional:

$$F(x, \dot{x}) \, \mathrm{d}t \, .$$

The Euler–Lagrange equations of this functional can be led to the canonical form due the nondegeneracy of the metric function F:

$$\ddot{x}^k + 2G^k(x, \dot{x}) = 0,$$

where

$$G^k := \frac{1}{4} g^{ik} (\partial_j F_{\cdot i} \dot{x}^j - \partial_i F).$$

Here g^{ik} are contravariant components of the metric tensor $g_{ik} g^{kj} = \delta_i^j$. The function G^k is changed according to the following law:

$$G^{k'} = \partial_k x^{k'} G^k - \frac{1}{2} \partial^2_{ij} x^{k'} y^i y^j.$$

Herewith $G^k_{ij} := G^k_{\cdot i \cdot j}$ are converted as connection coefficients:

$$G^{k'}_{i'j'} = G^k_{ij} \partial_{i'} x^i \partial_{j'} x^j \partial_k x^{k'} + \partial_p x^{k'} \partial^2_{i'j'} x^p,$$

which determine the Berwald connection. Since $G^k(x,y)$ are homogeneous functions of the second degree by y, then

$$G^k_{ij} y^i y^j = 2G^k.$$

The extremal equations coincide with the geodesic equations in Berwald connections:

$$\ddot{x}^k + G^k_{ij}\dot{x}^i\dot{x}^j = 0.$$

We define the vector field on the tangent bundle TM of the Finsler space F^n

$$X = y^i \frac{\partial}{\partial x^i} - 2G^k \frac{\partial}{\partial y^i}.$$

The integral curves $(x(t), \dot{x}(t))$ determine the geodesic of the Finsler space:

$$\dot{x}^k = y^k, \quad \dot{y}^k = -2G^k(x, \dot{x}).$$

Berwald connection generates the infinitesimal connection, that is, the distribution $H : z \to Hz$ on tangent bundle TM of the basis manifold M. At each point $z \in TM$ vectors

$$\delta_i = \partial_i - N^k_i \dot{\partial}_k$$

form the basis of the horizontal distribution, where $N^k_i = G^k_{\cdot i} := G^k_{ij} y^j$ are the coefficients of the infinitesimal connectivities, and $\dot{\partial}_k := \frac{\partial}{\partial y^k}$ is the basis of the vertical distribution $V : z \to Vz$ formed by all vectors tangent to the layer at z. The dual basis is $\delta x^i = \mathrm{d}x^i$, δy^i, where $\delta y^i = \mathrm{d}y^i + N^I_k\, \mathrm{d}x^k$.

3.3 Cartan connection

The metric tensor of the Finsler space is not covariantly constant in Berwald's connectivity. One can specify the connectivity that is consistent with the metric (when the metric tensor is covariantly constant). This is the Cartan connection.

Suppose that the infinitesimal connection is given on TM.

The connection coefficients (F^k_{ij}, C^k_{ij}) of the ∇ are defined by expansions

$$\nabla_i \partial_j = \nabla_{\delta_i} \partial j = F^k_{ij}\partial_k, \quad \dot{\nabla}_i \partial_j = \nabla_{\dot{\delta}_i} \partial j = C^k_{ij}\partial_k.$$

We require consistency with the metric:

$$F^k_{ij} = F^k_{ji}, \quad C^k_{ij} = C^k_{ji}, \quad \nabla_{\mathcal{A}} g_{ij} = 0.$$

Then

$$F^k_{ij} = \frac{1}{2} g^{ks}(\delta_i g_{sj} + \delta_j g_{is} - \delta_s g_{ij}),$$
$$C^k_{ij} = \frac{1}{2} g^{ks}\left(\dot{\partial}_i g_{sj} + \dot{\partial}_j g_{is} - \dot{\partial}_s g_{ij}\right).$$

The Finsler connection is called the Cartan connection, if $N^k_i = F^k_{ij} y^j$. The coefficients of the Cartan connection are denoted as Γ^{*k}_{ij}:

$$\Gamma^{*k}_{ij} = \left\{ \begin{matrix} k \\ ij \end{matrix} \right\} - \frac{1}{2} g^{kp}\left(g_{pi\cdot s} G^s_{\cdot j} + g_{jp\cdot s} G^s_{\cdot i} - g_{ij\cdot s} G^s_{\cdot p}\right).$$

The tensor part of the connection ∇ takes the form

$$C^k_{ij} = \frac{1}{2} g^{ks} g_{ij\cdot s}.$$

4. MAXWELL EQUATIONS

We will write Maxwell equations with use of differential forms.[13,14]

$$\mathrm{d}F = 0, \tag{1}$$

$$\mathrm{d}^* F = \frac{4\pi}{c}{}^* j. \tag{2}$$

The 2-form of the electromagnetic field F is expressed in 1-form field potential a as follows:

$$F = \mathrm{d}A\,. \tag{3}$$

In this case, the vector potential A makes sense of connectivity, and the Maxwell tensor F makes sense curvature.

4.1 Maxwell equation in Riemann geometry

1-form of the potential of the field is written as follows:

$$A = A_\alpha \,\mathrm{d}x^\alpha\,.$$

Assuming that $a = A(x^i)$, we may write the 2-form of the electromagnetic field (3):

$$F = \frac{1}{2} F_{\alpha\beta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta\,.$$

The tensor $F_{\alpha\beta}$ has the form:

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha = A_{\beta;\alpha} - A_{\alpha;\beta} = A_{\beta,\alpha} - A_{\alpha,\beta}.$$

The equation (1) takes the form:

$$\mathrm{d}F = \mathrm{d}\,\mathrm{d}A = \frac{1}{2}\nabla_\gamma F_{\alpha\beta}\,\mathrm{d}x^\gamma \wedge \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta = \frac{1}{6}(\nabla_\gamma F_{\alpha\beta} + \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha})\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma = 0$$

or

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0.$$

The second Maxwell equation (2) takes the form:

$$\nabla_\alpha F^{\alpha\beta} = \frac{4\pi}{c} j^\beta$$

or in partial derivatives:[15]

$$\frac{1}{\sqrt{-g}}\left[\partial_\alpha\left(\sqrt{-g}F^{\alpha\beta}\right)\right] = \frac{4\pi}{c} j^\beta.$$

4.2 Maxwell equation in Finsler geometry

In the case of Finsler geometry, the natural local coordinates on TM have the form (x^{α}, y^{α}). Then for the metric tensor the following may be written:

$$g_{\alpha\beta} = g_{\alpha\beta}(x^{\delta}, y^{\delta}).$$

Accordingly, for the potential vector we have:

$$A_{\alpha} = A_{\alpha}(x^{\delta}, y^{\delta}).$$

Let us write down the 2-form of the electromagnetic field (3):

$$F = \frac{1}{2} F_{\alpha\beta}\, \mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{\beta} + F_{\alpha\bar{\beta}}\, \mathrm{d}x^{\alpha} \wedge \mathrm{d}y^{\bar{\beta}}\,.$$

We expand the last expression in terms of vector potential:

$$F_{\alpha\beta} = \delta_{\alpha} A_{\beta} - \delta_{\beta} A_{\alpha} = A_{\beta;\alpha} - A_{\alpha;\beta},$$
$$F_{\alpha\bar{\beta}} = -\partial_{\bar{\beta}} A_{\alpha} = -A_{\alpha\cdot\bar{\beta}}.$$

We assume that the connection is consistent with the metric. That is, we will use the Cartan connection. Then the first Maxwell equation (1) takes the form:

$$F_{\bar{\alpha}\beta;\gamma} + F_{\gamma\bar{\alpha};\beta} + F_{\beta\gamma\cdot\bar{\alpha}} = 0.$$

The second Maxwell equation (2) takes the form:

$$\frac{1}{\sqrt{-g}} \left[\delta_{\beta} \big(\sqrt{-g} F^{\alpha\beta} \big) + \partial_{\bar{\beta}} \Big(\sqrt{-g} F^{\alpha\bar{\beta}} \Big) \right] = -\frac{4\pi}{c} j^{\alpha}.$$

We have written down the Maxwell equations in the case of Finsler geometry. Obviously, the correspondence principle is fulfilled for the written formulas. That is, when there is a dependence on only one coordinate x^{i}, we get the case of Riemannian geometry.

5. CONCLUSION

In this paper we make the next step in construction of geometrized Maxwell's equations based on Finsler geometry but not on Riemann geometry. From the viewpoint of authors this will allow not only to construct a complete permeability tensor, but also to describe an anisotropic medium. As a whole this will allow us to develop an adequate method for solving the inverse problem of optics.

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