Geometrization of Maxwell's equations in the construction of optical devices

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ABSTRACT

The paper considers the technics of construction of optical devices based on the method of geometrization of Maxwell's equations. The method is based on representation of material equations in the form of an effective space-time geometry. Thus we get a problem similar to that of some bimetric theory of gravity. That allows to use a well-developed apparatus of differential geometry. On this basis, we can examine the propagation of the electromagnetic field on the given parameters of the medium. It is also possible to find the parameters of the medium by a given law of propagation of electromagnetic fields.

Keywords: Maxwell's equations, constitutive equations, Maxwell's equations geometrization, Riemann geometry, curvilinear coordinates

1. INTRODUCTION

Differential geometry was an important language of physics of XX-th century. Basic elements of differential geometry were developed within the general relativity theory. There is a desire to use its power in other areas of physics, in particular in the optics.

The first attempts to apply the methods of differential geometry in electrodynamics should be attributed to publications of I. E. Tamm.[1–3] In 1960 E. Plebanski proposed method of geometrization for the constitutive equations of the electromagnetic field, and this method became classic.[4–7] All subsequent works, either used it or tried to correct a little, without changing the ideology.[8] Unfortunately Plebanski[4] gives no deriving formulas. Ideology is not expressed explicitly too. In addition, the Plebanski's technique looks more like a clever trick.

The authors have tried to perform the geometrization of the Maxwell's equations more formal.

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Saratov Fall Meeting 2016: Laser Physics and Photonics XVII; and Computational Biophysics and Analysis of Biomedical Data III, edited by Vladimir L. Derbov, Dmitry E. Postnov, Proc. of SPIE Vol. 10337, 103370K · © 2017 SPIE · CCC code: 1605-7422/17/$18 · doi: 10.1117/12.2267959

2. NOTATIONS AND CONVENTIONS

1. We will use the notation of abstract indices.[9] In this notation tensor as a complete object is denoted merely by an index (e.g., x^i). Its components are designated by underlined indices (e.g., $x^{\underline{i}}$).

2. We will adhere to the following agreements. Greek indices (α, β) will refer to the four-dimensional space, in the component form it looks like: $\underline{\alpha} = \overline{0,3}$. Latin indices from the middle of the alphabet (i, j, k) will refer to the three-dimensional space, in the component form it looks like: $\underline{i} = \overline{1,3}$.

3. The comma in the index denotes a partial derivative with respect to corresponding coordinate ($f_{,i} := \partial_i f$); semicolon denotes a covariant derivative ($f_{;i} := \nabla_i f$).

4. The CGS symmetrical system[10] is used for notation of the equations of electrodynamics.

3. MAXWELL'S EQUATIONS REPRESENTATIONS

Here the basics of Maxwell's equations in curvilinear coordinates are presented. A more detailed description is given in articles.[11–13]

Maxwell's equations in 3-dimensional form are as follows:

$$\begin{cases} e^{ijk}\nabla_j E_k &= -\dfrac{1}{c}\partial_t B^i, \\ \nabla_i D^i &= 4\pi\rho, \\ e^{ijk}\nabla_j H_k &= \dfrac{1}{c}\partial_t D^i + \dfrac{4\pi}{c} j^i, \\ \nabla_i B^i &= 0. \end{cases}$$

where $e^{\underline{i}\underline{j}\underline{k}}$ is the alternating tensor.

Let's write the Maxwell equations with the help of electromagnetic field tensors $F_{\alpha\beta}$ and $G_{\alpha\beta}$:[14,15]

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = F_{[\alpha\beta;\gamma]} = 0, \tag{1}$$

$$\nabla_\alpha G^{\alpha\beta} = \frac{4\pi}{c} j^\beta, \tag{2}$$

where the tensors $F_{\alpha\beta}$ and $G^{\alpha\beta}$ have the following components

$$F_{\underline{\alpha}\,\underline{\beta}} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B^3 & B^2 \\ -E_2 & B^3 & 0 & -B^1 \\ -E_3 & -B^2 & B^1 & 0 \end{pmatrix}, \tag{3}$$

$$G^{\underline{\alpha}\,\underline{\beta}} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H_3 & H_2 \\ D^2 & H_3 & 0 & -H_1 \\ D^3 & -H_2 & H_1 & 0 \end{pmatrix}. \tag{4}$$

Where $E_{\underline{i}}$, $H_{\underline{i}}$ are the components of electric and magnetic fields intensity vectors; $D^{\underline{i}}$, $B^{\underline{i}}$ are the components of vectors of electric and magnetic induction.

Let us write the equation (1) and (2) through differential forms by using the fiber bundles formalism.

Let us consider the bundle

$$Y \to X,$$

where $X = M^4$ is four-dimensional space. In this case we do not make the assumption about the metric of this space.

Let us define F (two-form), G (bivector) and j (vector):

$$F = \frac{1}{2} F_{\underline{\alpha}\,\underline{\beta}}\, \mathrm{d}x^{\underline{\alpha}} \wedge \mathrm{d}x^{\underline{\beta}}\,, \quad F_{\alpha\beta} = \frac{1}{2} F_{\underline{\alpha}\,\underline{\beta}}\, \mathrm{d}x^{\alpha}_{\underline{\alpha}} \wedge \mathrm{d}x^{\beta}_{\underline{\beta}}\,, \quad F \in \Lambda^2,$$
$$G = \frac{1}{2} G^{\underline{\alpha}\,\underline{\beta}} \partial_{\underline{\alpha}} \wedge \partial_{\underline{\beta}}, \quad G_{\alpha\beta} = \frac{1}{2} G^{\underline{\alpha}\,\underline{\beta}} \partial^{\alpha}_{\underline{\alpha}} \wedge \partial^{\beta}_{\underline{\beta}}, \quad G \in \Lambda_2,$$
$$j = j^{\underline{\alpha}} \partial_{\underline{\alpha}} \quad j^{\alpha} = j^{\underline{\alpha}} \partial^{\alpha}_{\underline{\alpha}}, \quad j \in \Lambda_1.$$

Then the equation (1) and (2) will take the form:

$$\mathrm{d}F = 0,$$
$$\delta G = \frac{4\pi}{c} j. \tag{5}$$

Here $\delta = \#^{-1}\,\mathrm{d}\,\#$ is the divergence, $\# : \Lambda_k \to \Lambda^{n-k}$ defines the Poincare duality.

In this case $F_{\alpha\beta}$ and $G^{\alpha\beta}$ make sense of the curvature in the cotangent (T^*X) and the tangent (TX) bundles. The relationship between these values is defined as follows:

$$G^{\alpha\beta} = \lambda(F_{\gamma\delta}). \tag{6}$$

Then the equation (5) takes the form:

$$\delta\lambda = \frac{4\pi}{c} j.$$

In the linear case the relation (6) can be defined as

$$G^{\alpha\beta} = \lambda^{\alpha\beta\gamma\delta} F_{\gamma\delta}.$$

In this case, the equation (5) has the following form:

$$\delta\big(\lambda^{\alpha\beta\gamma\delta} F_{\gamma\delta}\big) = \frac{4\pi}{c} j.$$

Let us assume that λ defines a certain effective metric on X.

4. CONSTITUTIVE TENSOR

Let us assume that the mapping $\lambda : \Lambda^2 \to \Lambda_2$ is a linear and a local one. Then it can be represented as follows:

$$G^{\alpha\beta} = \lambda^{\alpha\beta\gamma\delta} F_{\gamma\delta}, \tag{7}$$

here constitutive tensor $\lambda^{\alpha\beta\gamma\delta}$ contains information on the permittivity and permeability, as well as electromagnetic coupling.[1]

Equation (7) shows that the $\lambda^{\alpha\beta\gamma\delta}$ has the following symmetry:

$$\lambda^{\alpha\beta\gamma\delta} = \lambda^{[\alpha\beta][\gamma\delta]}$$

To clarify the symmetry of the tensor $\lambda^{\alpha\beta\gamma\delta}$ we can represent it as follows:[16–19]

$$\lambda^{\alpha\beta\gamma\delta} = {}^{(1)}\lambda^{\alpha\beta\gamma\delta} + {}^{(2)}\lambda^{\alpha\beta\gamma\delta} + {}^{(3)}\lambda^{\alpha\beta\gamma\delta},$$
$${}^{(1)}\lambda^{\alpha\beta\gamma\delta} = {}^{(1)}\lambda^{([\alpha\beta][\gamma\delta])},$$
$${}^{(2)}\lambda^{\alpha\beta\gamma\delta} = {}^{(2)}\lambda^{[[\alpha\beta][\gamma\delta]]},$$
$${}^{(3)}\lambda^{\alpha\beta\gamma\delta} = {}^{(3)}\lambda^{[\alpha\beta\gamma\delta]}.$$

It's obvious that $\lambda^{\alpha\beta\gamma\delta}$ has 36 independent components, ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$ has 20 independent components, ${}^{(2)}\lambda^{\alpha\beta\gamma\delta}$ has 15 independent components, ${}^{(3)}\lambda^{\alpha\beta\gamma\delta}$ has only one independent component.

From now we will consider only the term ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$.

Let us write the constitutive relations:

$$\begin{aligned} D^i &= \varepsilon^{ij} E_j + {}^{(1)}\gamma^i_j B^j, \\ H_i &= \mu^{-1}_{ij} B^j + {}^{(2)}\gamma^j_i E_j, \end{aligned} \tag{8}$$

where ε^{ij} and μ^{ij} are the permittivity and permeability tensors. ${}^{(1)}\gamma^i_j$ and ${}^{(2)}\gamma^i_j$ are coupling terms.

In consideration of the tensor $F_{\alpha\beta}$ (3) and $G^{\alpha\beta}$ (4) structure and of the constitutive relations (8), we can write

$$\begin{gathered} F_{\underline{0}\,\underline{i}} = E_{\underline{i}}\,, \quad G^{\underline{0}\,\underline{i}} = -D^{\underline{i}}\,, \\ G^{\underline{i}\,\underline{j}} = -\varepsilon^{\underline{i}\,\underline{j}\,\underline{k}} H_{\underline{k}}\,, \quad F_{\underline{i}\,\underline{j}} = -\varepsilon_{\underline{i}\,\underline{j}\,\underline{k}} B^{\underline{k}}\,. \end{gathered} \tag{9}$$

From the (8) and (9) we can write out the structure of tensor $\lambda^{\alpha\beta\gamma\delta}$:

$$\lambda^{0\underline{i}\,0\underline{j}} = \varepsilon^{\underline{i}\,\underline{j}}/2, \quad \lambda^{\underline{i}\,\underline{j}\,\underline{m}\,\underline{n}} = \varepsilon^{\underline{i}\,\underline{j}\,\underline{k}}\varepsilon^{\underline{l}\,\underline{m}\,\underline{n}}(\mu^{-1})_{\underline{l}\,\underline{k}}/2.$$

5. LINEAR LOCAL GEOMETRIZATION OF MATERIAL MAXWELL'S EQUATIONS

Plebanski has offered the elementary geometrization of Maxwell's equations.[4, 5, 20, 21] Despite certain drawbacks, this method found some application, for example, in transformation optics, where the relation $\varepsilon_{ij} = \mu_{ij}$ is desirable.[7, 22, 23]

The basic ideas of Plebanski's geometrization are as follows:

1. to write out the Maxwell's equations in a medium in the Minkowski space;
2. to write the vacuum Maxwell's equations in the effective Riemann space;
3. to set equal the corresponding terms of both equations;
4. as a result, to obtain an expression of the permittivity and permeability in terms of geometric objects.

However, this approach to geometrization looks rather like a trick. The authors tried to perform these calculations more formally.

5.1 Auxiliary relations for the metric tensor

In addition, we will need some simple relations for the metric tensor.

$$g_{\alpha\delta} g^{\delta\beta} = \delta^\beta_\alpha \tag{10}$$

The relation (10) leads to the following special relations:

$$g_{0\delta} g^{\delta i} = g_{00} g^{0i} + g_{0k} g^{ki} = \delta^i_0 = 0, \tag{11}$$

$$g_{i\delta} g^{\delta j} = g_{i0} g^{0j} + g_{ik} g^{kj} = \delta^j_i. \tag{12}$$

The equation (11) can be rewritten in form

$$g^{0i} = -\frac{1}{g_{00}} g_{0k} g^{ki}. \tag{13}$$

Substituting the equation (13) in the equation (12), we obtain:

$$\left(g_{ik} - \frac{1}{g_{00}} g_{0i} g_{0k} \right) g^{kj} = \delta^j_i. \tag{14}$$

This relation will be used at a later time for the final equations simplication.

5.2 Constitutive relations for moving media

Minkowski derived equations for connection isotropic moving media[14,24] (Minkowski's equations for moving media). Suppose that u^{α} is a environments four-speed, then, assuming that the permittivity and permeability ε and μ are scalars, we can write

$$G^{\alpha\beta}u_{\beta} = \varepsilon F^{\alpha\beta}u_{\beta}, \quad {}^{*}F^{\alpha\beta}u_{\beta} = \mu\, {}^{*}G^{\alpha\beta}u_{\beta}. \tag{15}$$

In three-dimensional form the equations (15) take the following form:

$$\begin{aligned} D^{i} &= \varepsilon\left(E^{i} + \left[\frac{u_j}{c}, B_k\right]^{i}\right) - \left[\frac{u_j}{c}, H_k\right]^{i} = \varepsilon E^{i} + (\varepsilon\mu - 1)\left[\frac{u_j}{c}, H_k\right]^{i}, \\ B^{i} &= \mu\left(H^{i} - \left[\frac{u_j}{c}, D_k\right]^{i}\right) + \left[\frac{u_j}{c}, E_k\right]^{i} = \mu H^{i} - (\varepsilon\mu - 1)\left[\frac{u_j}{c}, E_k\right]^{i}. \end{aligned} \tag{16}$$

Tamm expanded equations (16) for the anisotropic case,[1,3] namely, assuming that the permittivity and permeability are of the form

$$\varepsilon^{i}_{\underline{j}} = \operatorname{diag}(\varepsilon^{1}_{1}, \varepsilon^{2}_{2}, \varepsilon^{3}_{3}), \quad \mu^{i}_{\underline{j}} = \operatorname{diag}(\mu^{1}_{1}, \mu^{2}_{2}, \mu^{3}_{3}),$$

and velocity vector u^{i} of frame of reference is parallel to one of the principal axes of anisotropy. Then the Minkowski's equations for moving media acquire the following form:

$$\begin{aligned} D^{i} &= \varepsilon^{i}_{l}\left(E^{l} + \left[\frac{u_j}{c}, B_k\right]^{l}\right) - \left[\frac{u_j}{c}, H_k\right]^{i}, \\ B^{i} &= \mu^{i}_{l}\left(H^{l} - \left[\frac{u_j}{c}, D_k\right]^{l}\right) + \left[\frac{u_j}{c}, E_k\right]^{i}. \end{aligned}$$

5.3 Generic geometrization

Let us write the Maxwell's equations geometrization.

Let us introduce an effective metric $g_{\alpha\beta}$ in X and construct the tensor $\lambda^{\alpha\beta\gamma\delta}$ in the following way:

$$\lambda^{\alpha\beta\gamma\delta} = \left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right).$$

Then the equation (7) takes the following form:

$$G^{\alpha\beta} = \left(g^{\alpha\gamma}g^{\beta\delta} - g^{\alpha\delta}g^{\beta\gamma}\right)F_{\gamma\delta}.$$

For clarity, let us write out the components:

$$\begin{aligned} G^{0\underline{i}} &= \left(g^{00}g^{\underline{i}\,\underline{j}}g^{0\underline{i}}g^{0\underline{j}}\right)F_{0\underline{j}} + \left(g^{0\underline{j}}g^{\underline{i}\,\underline{k}}g^{0\underline{k}}g^{\underline{i}\,\underline{j}}\right)F_{\underline{j}\,\underline{k}}, \\ G^{\underline{i}\,\underline{j}} &= \left(g^{\underline{i}\,0}g^{\underline{j}\,\underline{k}}g^{0\underline{j}}g^{\underline{i}\,\underline{k}}\right)F_{0\underline{k}} + \left(g^{\underline{i}\,\underline{k}}g^{\underline{j}\,\underline{l}}g^{\underline{i}\,\underline{l}}g^{\underline{j}\,\underline{k}}\right)F_{\underline{k}\,\underline{l}}. \end{aligned}$$

Based on equation (9) we obtain:

$$\begin{aligned} D^{i} &= -g^{00}\left(g^{\underline{i}\,\underline{j}}\frac{1}{g^{00}}g^{0\underline{i}}g^{0\underline{j}}\right)E_{\underline{j}} - \left(g^{0\underline{j}}g^{\underline{i}\,\underline{k}}g^{0\underline{k}}g^{\underline{i}\,\underline{j}}\right)\varepsilon_{\underline{j}\,\underline{k}\,\underline{l}}B^{\underline{l}}, \\ -\epsilon^{\underline{i}\,\underline{j}\,\underline{k}}H_{\underline{k}} &= \left(g^{\underline{i}\,0}g^{\underline{j}\,\underline{k}}g^{0\underline{j}}g^{\underline{i}\,\underline{k}}\right)E_{\underline{k}} - \left(g^{\underline{i}\,\underline{k}}g^{\underline{j}\,\underline{l}}g^{\underline{i}\,\underline{l}}g^{\underline{j}\,\underline{k}}\right)\varepsilon_{\underline{k}\,\underline{l}\,\underline{m}}B^{\underline{m}}. \end{aligned}$$

Based on equation (9) and taking into account the equation (14) we obtain:

$$D^{\underline{i}} = -g^{00}g^{\underline{i}\,\underline{j}}E_{\underline{j}} + \varepsilon_{\underline{k}\,\underline{l}\,\underline{j}}g^{0\underline{k}}g^{\underline{i}\,\underline{l}}B^{\underline{j}}, \tag{17}$$

$$H_{\underline{i}} = \varepsilon_{\underline{m}\,\underline{n}\,\underline{i}}\, \varepsilon_{\underline{k}\,\underline{l}\,\underline{j}}\, g^{\underline{n}\,\underline{k}} g^{\underline{m}\,\underline{l}} B^{\underline{j}} + \varepsilon^{\underline{k}\,\underline{l}\,\underline{j}}\, g_{0\underline{k}}\, g_{\underline{i}\,\underline{l}}\, E_{\underline{j}}\,, \tag{18}$$

From (17) we can formally deduce the expression for the permittivity:

$$\varepsilon^{\underline{i}\,\underline{j}} = -g^{00} g^{\underline{i}\,\underline{j}}. \tag{19}$$

In this case the geometrical meaning of the second term in (17) needs further clarification.

From (18) we can formally write out the expression for permeability:

$$(\mu^{-1})_{\underline{i}\,\underline{j}} = \varepsilon_{\underline{m}\,\underline{n}\,\underline{i}}\, \varepsilon_{\underline{k}\,\underline{l}\,\underline{j}}\, g^{\underline{n}\,\underline{k}} g^{\underline{m}\,\underline{l}}. \tag{20}$$

Thus geometrized constitutive relations of coordinates are as follows:

$$\begin{gathered} D^i = \varepsilon^{ij} E_j + {}^{(1)}\gamma^i_j B^j, \\ H_i = (\mu^{-1})_{ij} B^j + {}^{(2)}\gamma^j_i E_j, \\ \varepsilon^{\underline{i}\,\underline{j}} = -g^{00} g^{\underline{i}\,\underline{j}}, \quad (\mu^{-1})_{\underline{i}\,\underline{j}} = \varepsilon_{\underline{m}\,\underline{n}\,\underline{i}}\, \varepsilon_{\underline{k}\,\underline{l}\,\underline{j}}\, g^{\underline{n}\,\underline{k}} g^{\underline{m}\,\underline{l}}, \quad {}^{(1)}\gamma^i_j = {}^{(2)}\gamma^i_j = \epsilon_{\underline{k}\,\underline{l}\,\underline{j}}\, g^{0\underline{k}} g^{\underline{i}\,\underline{l}}. \end{gathered} \tag{21}$$

Leonhard proposed to interpret the coupling term in (21) as a speed of geometrized reference frame.[6] Indeed, on the basis of (16) the equations (21) can be rewritten as:

$$\begin{gathered} D^i = \varepsilon^{ij} E_j + \left[\frac{u_j}{c}, B_k\right]^i, \\ H_i = (\mu^{-1})_{ij} B^j + \left[\frac{u^j}{c}, E^k\right]_i, \end{gathered}$$

where u^i is three-dimensional velocity of frame of reference.

5.4 Geometrization for an isotropic medium

In the case of an isotropic medium, the expression (8) takes the form:

$$D^i = \varepsilon \delta^{ij} E_j, \quad H_i = \mu^{-1} \delta_{ij} B^j, \quad {}^{(1)}\gamma^i_j = {}^{(2)}\gamma^i_j = 0,$$

where ε and μ are the permittivity and permeability scalars.

Then, based on (19) and (20) we can obtain the following effective metric tensor:

$$\begin{gathered} g_{\underline{\alpha}\,\underline{\beta}} = \operatorname{diag}\left(\frac{1}{\varepsilon\sqrt{\mu}}, -\sqrt{\mu}, -\sqrt{\mu}, -\sqrt{\mu}\right), \\ g^{\underline{\alpha}\,\underline{\beta}} = \operatorname{diag}\left(\varepsilon\sqrt{\mu}, -\frac{1}{\sqrt{\mu}}, -\frac{1}{\sqrt{\mu}}, -\frac{1}{\sqrt{\mu}}\right). \end{gathered}$$

Similar relationships were offered by Tamm.[1,3]

6. CONCLUSIONS

The authors proposed a formal approach to the problem of geometrization of equations of electromagnetic field. As an illustration of this method we presented a local linear geometrization of Maxwell's equations. It should be noted that this method can not be considered fully satisfactory. Indeed, the total constitutive tensor $\lambda^{\alpha\beta\gamma\delta}$ has 36 independent components. And even its principal part, the tensor ${}^{(1)}\lambda^{\alpha\beta\gamma\delta}$ has 20 independent components. While the Riemannian metric tensor has only 10 independent components.

It should also be said that the proposed method is different in ideology and the results from the Plebanski's method.

ACKNOWLEDGMENTS

The work is partially supported by RFBR grants No's 14-01-00628, 15-07-08795, and 16-07-00556. Also the publication was supported by the Ministry of Education and Science of the Russian Federation (the Agreement No 02.a03.21.0008).

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