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Application of Two-Spinor Calculus in Quantum Mechanical and Field Calculations

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Abstract—This paper describes Lorentz two-spinors and proposes using them in calculations with Dirac four-spinors and quaternions.

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1. INTRODUCTION

Spinors are used in physics quite extensively [1]. The following spinors are mainly used:

(i) Dirac four-spinors;

(ii) Pauli three-spinors;

(iii) quaternions.

If Dirac four-spinors are used, the main difficulty is γ matrices. The essence of these objects is that they serve to connect the spinor and tensor spaces and therefore have two types of indices: spinor and tensor ones. It would be logical to perform calculations in one of these spaces only.

We propose using semispinors of Dirac spinors, Lorentz two-spinors [2], as simpler objects.

2. GENERAL NOTION OF A SPINOR

Let us determine a spinor using the Clifford–Dirac equation,

$$\gamma_{(a}\gamma_{b)} = -g_{ab}\mathbf{I}. \tag{1}$$

or omitting the spinor indices,

$$\gamma^{\sigma}_{a\rho}\gamma^{\tau}_{b\sigma} + \gamma^{\sigma}_{b\rho}\gamma^{\tau}_{a\sigma} = -2g_{ab}\delta^{\tau}_{\rho}. \tag{2}$$

The dimension of the spinor space is:

$$\begin{cases} N = 2^{n/2}, & \text{for even } n; \\ N = 2^{n/2-1/2}, & \text{for odd } n. \end{cases} \tag{3}$$

3. CONNECTION OF TWO-SPINORS WITH QUATERNIONS

Let us assume that

$$\begin{aligned} \mathbf{I} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix}, \\ \mathbf{j} &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} \end{aligned} \tag{4}$$

A general quaternion is represented by the matrix

$$\mathbf{A} = \mathbf{I}a + \mathbf{i}b + \mathbf{j}c + \mathbf{j}d = \begin{pmatrix} a+\mathbf{i}d & -c+\mathbf{i}b \\ c+\mathbf{i}b & a-\mathbf{i}d \end{pmatrix}, \tag{5}$$

where $a, b, c, d \in \mathbb{R}$. The sum and the product of two quaternions are obtained as the matrix sum and matrix product. The adjoint quaternion $\mathbf{A}^*$ is determined by the matrix operation:

$$\mathbf{A}^* = \mathbf{I}a - (\mathbf{i}b + \mathbf{j}c + \mathbf{k}d). \tag{6}$$

If matrix $\mathbf{A}$ (5) is unimodular and unitary, it represents the unitary spin matrix. The following relations can be written:

$$\det\mathbf{A} = a^2 + b^2 + c^2 + d^2 = 1, \tag{7}$$

$$\mathbf{A}\mathbf{A}^* = \mathbf{I}(a^2 + b^2 + c^2 + d^2) = \mathbf{I}. \tag{8}$$

Thus, the quaternion should have the unitary norm,

$$\|\mathbf{A}\| := a^2 + b^2 + c^2 + d^2 = 1. \tag{9}$$

Therefore, the unitary quaternion can be represented as the unitary spin matrix.

In spite of the fact that unitary spin matrices and unitary quaternions represent in essence the same thing, in the general case, no close connection exists between quaternions and spin matrices. The fact of the matter is that quaternions are connected with positive definite quadratic forms, while spin matrices and Lorentz transforms are characterized by the Lorentz signature (+, –, –, –).

4. CONNECTION OF TWO-SPINORS AND VECTORS

In final calculations, it is necessary to transform abstract indices into component form. Moreover, it is often more convenient to formulate the result in vector form. To establish the connection between the spinor and vector bases, the Infeld–van der Waerden symbols are used:

$$\begin{aligned} g_{\mathbf{a}}^{\mathbf{AA'}} &:= g_{\mathbf{a}}^{a}\varepsilon_{A}^{\mathbf{A}}\varepsilon_{A'}^{\mathbf{A'}}, \\ g_{\mathbf{AA'}}^{\mathbf{a}} &:= g_{a}^{\mathbf{a}}\varepsilon_{\mathbf{A}}^{A}\varepsilon_{\mathbf{AA'}}^{A'}, \end{aligned} \tag{10}$$

where convolution is performed over abstract indices only.

For the standard Minkowski tetrad and the spin reference frame, we obtain

$$\begin{aligned} g_0^{\mathrm{AB'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = g_{\mathrm{AB'}}^{0}, \\ g_1^{\mathrm{AB'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = g_{\mathrm{AB'}}^{1} \\ g_2^{\mathrm{AB'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix} = -g_{\mathrm{AB'}}^{2}, \\ g_3^{\mathrm{AB'}} &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = g_{\mathrm{AB'}}^{3}. \end{aligned} \tag{11}$$

5. DIRAC FOUR-SPINORS AND LORENTZ TWO-SPINORS

In physical calculations, Dirac four-spinors[1] are often used. However, operations with these objects are extremely cumbersome. One of the basic disadvantages of this formalism is the explicit application of γ matrices which are in essence objects serving to connect the vector and spinor spaces. It is as though our spinor objects "live" in two spaces, vector and spinor spaces. Correspondingly, the desire to transfer to a more unified formalism by rejecting either spinor or tensor indices is observed.

Let us consider the transition to the purely spinor formalism based on Lorentz two-spinors. First, we will construct a four-spinor object based on two-spinors. Then we will demonstrate the capabilities of this formalism. Two examples will be considered: the derivation of invariant spinor relations and the calculation of matrix elements.

[1] The introduction of four-spinors can probably be motivated by the desire to construct an object for which the operation of spatial reflection can be implemented conveniently.

5.1. Construction of Four-Spinor Formalism

Let us construct the implementation of the four-spinor formalism based on Lorentz two-spinors[2].

We denote by small Greek letters four-spinor indices; by capital Latin letters, two-spinor indices (as usual); and by small Latin letters, tensor indices.

Let us write the Dirac four-spinor as

$$\psi^{\alpha} = \begin{pmatrix} \varphi^{A} \\ \pi^{A'} \end{pmatrix}, \tag{12}$$

where φ^A and $\pi^{A'}$ are the Lorentz two-spinors.

The adjoint spinor has the form

$$\overline{\psi^{\alpha}} = \bar{\psi}_{\alpha} = (\bar{\pi}_{A}, \bar{\varphi}_{A'}). \tag{13}$$

Let us determine the reflection operator,

$$\hat{P} = \begin{pmatrix} \varphi^{A} \\ \pi^{A'} \end{pmatrix} \longrightarrow \begin{pmatrix} \pi^{A'} \\ \varphi^{A} \end{pmatrix}. \tag{14}$$

We use γ matrices in the chiral representation. Then the explicit form of the γ matrix is

$$\begin{aligned} \gamma_{a\rho}^{\sigma} &= \sqrt{2}\begin{pmatrix} 0 & \varepsilon_{A'R'}\varepsilon_{A}^{S} \\ \varepsilon_{AR}\varepsilon_{A'}^{S'} & 0 \end{pmatrix}, \\ \eta_{\rho}^{\sigma} &= \begin{pmatrix} -\mathrm{i}\varepsilon_{R}^{S} & 0 \\ 0 & \mathrm{i}\varepsilon_{R'}^{S'} \end{pmatrix}, \end{aligned} \tag{15}$$

and

$$\gamma_{ab\rho}^{\sigma} = \begin{pmatrix} \varepsilon_{A'B'}\varepsilon_{R(A}\varepsilon_{B)}^{S} & 0 \\ 0 & \varepsilon_{AB}\varepsilon_{R'(A'}\varepsilon_{B')}^{S'} \end{pmatrix}. \tag{16}$$

We introduce the following notation: $\gamma_5 := \mathrm{i}\eta$.

Using the structure of Dirac four-spinors determined by us, it is possible to construct invariant relations. We will operate with the pair spinor–adjoint four-spinor (ψ and ψ'), and correspondingly, four two-spinors (φ^A, $\varphi_{A'}$, $\pi^{A'}$, and $\bar{\pi}_{A'}$).

5.2. Scalars

The convolutions $\bar{\pi}_A\varphi^A$ and $\bar{\varphi}_{A'}\pi_{A'}$ have the meaning of scalars. Their sum behaves as a scalar, and the

[2] The Dirac four-spinor is often implemented using two three-dimensional spinors (three Pauli spinors). Formally, this is quite possible, since the structure of the spaces $\mathbb{S}_R$ and $\mathbb{S}_{R'}$ of the semispinors of the spinor with a dimension of $n + 1$ coincides with the structure of the space S_ρ of spinors with a dimension of n (n is odd).

difference, as a pseudoscalar:

$$s = \bar{\pi}_A\varphi^A + \bar{\varphi}_{A'}\pi^{A'} = \bar{\psi}_\alpha\psi^\alpha, \tag{17}$$

$$p = \mathrm{i}(\bar{\pi}_A\varphi^A - \bar{\varphi}_{A'}\pi^{A'}) = \mathrm{i}\bar{\psi}_\alpha\gamma^\alpha_{5\beta}\psi^\beta. \tag{18}$$

5.3. Vectors

The combinations $\bar{\pi}^A\pi^{A'}$ and $\varphi^A\bar{\varphi}^{A'}$ have the meaning of vectors. Their sum behaves as a vector, and the difference, as a pseudovector,

$$j^a = \sqrt{2}(\bar{\pi}^A\pi^{A'} + \varphi^A\bar{\varphi}^{A'}) = \bar{\psi}_\alpha\gamma^{a\alpha}_\beta\psi^\beta, \tag{19}$$

$$\tilde{j}^a = \sqrt{2}(\bar{\pi}^A\pi^{A'} - \varphi^A\bar{\varphi}^{A'}) = \bar{\psi}_\alpha\gamma^{a\alpha}_\beta\gamma^\beta_{5\delta}\psi^\delta. \tag{20}$$

5.4. Tensors

A real skew-symmetric tensor can be constructed as follows:

$$a^{ab} = \mathrm{i}(\varphi^{(A}\bar{\pi}^{B)}\varepsilon^{A'B'} - \bar{\varphi}^{(A'}\pi^{B')}\varepsilon^{AB}) = \bar{\psi}_\alpha\sigma^{ab\alpha}_\beta\psi^\beta. \tag{21}$$

These relations seem simpler than the initial ones.

6. CALCULATION OF MATRIX ELEMENTS

Usually, the "Feynman trick" is used for calculating matrix elements in quantum theory; this trick consists in the transformation of the product of spinors into the spur; as a result, the squared matrix element is obtained. Correspondingly, the complexity of calculations increases and the number of calculated elements is proportional to n^2. Moreover, if the complete matrix element is calculated as the sum of many diagrams, or the information on the phase is important, this method is inapplicable.

The proposed alternative is to calculate the matrix element. Let us look at two ways: the application of two-spinors (rejection of tensor indices) and the application of the vector formalism (rejection of spinor indices).

To eliminate the basic obstacle, complex relations for γ matrices, we propose using the two-spinor formalism.

Let us introduce the auxiliary notation based on the sign in the projector $1 \pm \gamma_5$ [3],

$$\psi = \begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}. \tag{22}$$

Correspondingly, the γ matrices are written in the form (see (15))

$$\gamma_a = \begin{pmatrix} 0 & \gamma_{a+}\\ \gamma_{a-} & 0\end{pmatrix}, \quad \hat{p} = \begin{pmatrix} 0 & \hat{p}_+\\ \hat{p}_- & 0\end{pmatrix}, \tag{23}$$

where $\hat{p} := p^a\gamma_a$.

The application of two-spinors is especially justified in the case of the presence of projectors $(1 \pm \gamma_5)$. Therefore, let us consider the simplification of calculations in the case of terms of the form

$$\bar{\psi}_f\gamma^{a_1}\hat{p}_{(a)}\gamma^{a_2}\hat{p}_{(b)}\cdots\gamma^{a_n}\left[\frac{1}{2}(1\pm\gamma_5)\right]\psi_i. \tag{24}$$

Two cases can be singled out: the even and odd numbers of γ matrices,

$$\begin{cases}\bar{\psi}_{f\pm}\gamma^{a_1}_{\mp}\hat{p}_{(a)\pm}\gamma^{a_2}_{\mp}\hat{p}_{(b)}\cdots\gamma^{a_n}_{\pm}\psi_{i\pm},\\ \text{odd number of } \gamma \text{ matrix},\\ \bar{\psi}_{f\mp}\gamma^{a_1}_{\pm}\hat{p}_{(a)\mp}\gamma^{a_2}_{\pm}\hat{p}_{(b)}\cdots\gamma^{a_n}_{\mp}\psi_{i\pm},\\ \text{even number of } \gamma \text{ matrix},\end{cases} \tag{25}$$

Using the two-spinor representation of γ matrices, we obtain the following relations:

$$\gamma^\beta_{a\alpha\pm}\gamma^{a\delta}_{\gamma-} = 2\delta^\delta_\alpha\delta^\beta_\gamma, \tag{26a}$$

$$\gamma^\beta_{a\alpha\pm}\gamma^{a\delta}_{\gamma\pm} = 2(\delta^\beta_\alpha\delta^\delta_\gamma - \delta^\delta_\alpha\delta^\beta_\gamma). \tag{26b}$$

For example, let us prove (26a):

$$\varepsilon_{C'A'}\varepsilon^B_C\varepsilon^C_G\varepsilon^{C'D'} = 2\varepsilon^{D'}_{A'}\varepsilon^B_G, \tag{27}$$

Taking into account the following correspondences,

$$\alpha \longleftrightarrow A', \quad \beta \longleftrightarrow B', \quad \gamma \longleftrightarrow G', \quad \delta \longleftrightarrow D'.$$

we obtain the initial expression.

Thus, sequentially using (25) and (26), we eliminate γ matrices and perform calculations using two-spinors.

After calculations, terms of the following type are obtained:

$$u^\dagger_{f\pm}\hat{p}_{(a)\mp}\,\hat{p}_{(b)\pm}\ldots\hat{e}_{+\text{ or }-}\ldots u_{i\pm}, \tag{28}$$

where e is the polarization.

To obtain a particular result, it is necessary to choose the spinor representation. For example, the standard solution obtained by the decomposition into plane waves can be used. Then in the case of longitudinal polarization, we obtain

$$u_\pm = (\sqrt{E+\varepsilon m} \pm \varepsilon s\sqrt{E-\varepsilon m}) \times \begin{pmatrix}\mathrm{e}^{-\mathrm{i}\varphi/2}\sqrt{1+s\cos\theta}\\ \mathrm{e}^{\mathrm{i}\varphi/2}\sqrt{1-s\cos\theta}\end{pmatrix}, \tag{29}$$

where s is the helicity and ε is the energy sign.

6.1. Example of Calculation of a Matrix Element

Let us calculate the cross section of the reaction

$$\nu + n \longrightarrow p + e^-, \tag{30}$$

The matrix element of this reaction in the standard (V–A) theory has the form

$$M = \frac{G_F}{\sqrt{2}}(\overline{\psi}_e\gamma_a(1+\gamma_5)\psi_\nu)(\overline{\psi}_p\gamma^a(g_V + g_A\gamma_5)\psi_n). \quad (31)$$

Using (25), (26), we reduce (31) to the following form:

$$\begin{aligned} M &= \frac{2G_F}{\sqrt{2}}(u^\dagger_{e\alpha+}\gamma_{a\beta-}{}^{\alpha}u^{\beta}_{\nu+})[(g_A - g_V) \\ &\times(u^\dagger_{p\gamma+}\gamma^{a}_{\delta-}{}^{\gamma}u^{\delta}_{n+} + u^\dagger_{p\gamma-}\gamma^{a}_{\delta+}{}^{\gamma}u^{\delta}_{n-}) + 2g_A u^\dagger_{p\gamma+}\gamma^{a}_{\delta-}{}^{\gamma}u^{\delta}_{n+}] \\ &= \frac{2G_F}{\sqrt{2}}[(g_V - g_A)u^\dagger_{e\alpha+}\gamma_{\alpha\beta-}{}^{\alpha}u^{\beta}_{\nu+}u^\dagger_{p\gamma-}\gamma^{a}_{\delta+}{}^{\gamma}u^{\delta}_{n-} \\ &+ (g_V + g_A)u^\dagger_{e\alpha+}\gamma_{\alpha\beta-}{}^{\alpha}u^{\beta}_{\beta}u^\dagger_{p\gamma+}\gamma^{a}_{\delta-}{}^{\gamma}u^{\delta}_{n+}] \\ &= \frac{4G_F}{\sqrt{2}}[(g_V - g_A)u^\dagger_{e\alpha+}u^{\alpha}_{n-}u^\dagger_{p\beta-}u^{\beta}_{\nu+} + (g_V + g_A) \\ &\times(u^\dagger_{e\alpha+}u^{\alpha}_{\nu+}u^\dagger_{p\beta+}u^{\beta}_{n+} - u^\dagger_{e\alpha+}u^{\alpha}_{n+}u^\dagger_{p\beta+}u^{\beta}_{\nu+})]. \end{aligned} \quad (32)$$

Let us choose the direction angles as follows: $\varphi_\nu = \varphi_n = \varphi_p = \varphi_e = 0$, $\theta_\nu = \theta_n = \pi/2$, θ_e and θ_p are arbitrary.

We introduce the spinors

$$|s_0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \quad |s_1\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \\ |s_2\rangle = \begin{pmatrix}\cos\theta_p/2\\ \sin\theta_p/2\end{pmatrix}, \quad |s_3\rangle = \begin{pmatrix}-\sin\theta_e/2\\ \cos\theta_e/2\end{pmatrix}. \quad (33)$$

Then we can write (see (29))

$$u_{\nu\pm} = \frac{1}{\sqrt{2}}(\sqrt{E_\nu + m_\nu} \pm s_\nu\sqrt{E_\nu - m_\nu})|s_0\rangle, \quad (34)$$

$$u_{n\pm} = \frac{1}{\sqrt{2}}(\sqrt{E_n + m_n} \pm s_n\sqrt{E_n - m_n})|s_1\rangle, \quad (35)$$

$$u_{p\pm} = \frac{1}{\sqrt{2}}(\sqrt{E_p + m_p} \pm s_p\sqrt{E_p - m_p})|s_2\rangle, \quad (36)$$

$$u_{e\pm} = \frac{1}{\sqrt{2}}(\sqrt{E_e + m_e} \pm s_e\sqrt{E_e - m_e})|s_3\rangle. \quad (37)$$

We obtain from (32)

$$\begin{aligned} M &= \frac{G_F}{\sqrt{2}}(\sqrt{E_e + m_e} + s_e\sqrt{E_e - m_e}) \\ &\times(\sqrt{E_\nu + m_\nu} + s_\nu\sqrt{E_\nu - m_\nu})[(g_V - g_A) \\ &\times(\sqrt{E_n + m_n} - s_n\sqrt{E_n - m_n})(\sqrt{E_p + m_p} - s_p\sqrt{E_p - m_p}) \\ &\times\langle s_3|s_1\rangle\langle s_2|s_0\rangle + (g_V + g_A) \\ &\times(\sqrt{E_n + m_n} + s_n\sqrt{E_n - m_n})(\sqrt{E_p + m_p} + s_p\sqrt{E_p - m_p}) \\ &\times(\langle s_3|s_0\rangle\langle s_2|s_1\rangle - \langle s_3|s_1\rangle\langle s_2|s_0\rangle)] = \frac{G_F}{\sqrt{2}} \\ &\times(\sqrt{E_e + m_e} + s_e\sqrt{E_e - m_e})(\sqrt{E_\nu + m_\nu} + s_\nu\sqrt{E_\nu - m_\nu}) \\ &\times[(g_V - g_A)(\sqrt{E_n + m_n} - s_n\sqrt{E_n - m_n}) \\ &\times(\sqrt{E_p + m_p} - s_p\sqrt{E_p - m_p})c\cos\theta_e/2\cos\theta_p/2 \\ &-(g_V + g_A)(\sqrt{E_n + m_n} + s_n\sqrt{E_n - m_n}) \\ &\times(\sqrt{E_p + m_p} + s_p\sqrt{E_p - m_p}) \\ &\times(\sin\theta_e/2\cos\theta_p/2 + \cos\theta_e/2\cos\theta_p/2)]. \end{aligned} \quad (38)$$

Thus, the number of calculated terms is decreased considerably (the order n instead of n^2); moreover, they have a rather simple form.

7. CONCLUSIONS

(1) Semispinors are simpler objects than spinors.

(2) We propose using Lorentz two-spinors instead of Dirac four-spinors.

(3) In relativistic calculations, two-spinors seem more adequate than quaternions.

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