Online Learning Without Prior Information

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Outline

Online Linear Optimization

1. Setting:
   - Sample at time t: $(\mathbf{x_t}, y_t)$
   - Hypothesis : $\mathbf{w}$
   - Let $f_t(\mathbf{w}) =< \mathbf{w}, \mathbf{g_t} >$, where $\mathbf{g_t} \in \bigtriangledown f_t(\mathbf{w_t})$
   - Let $R(\mathbf{w}) = \frac{1}{2\eta}\|\mathbf{w}\|_2^2$

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2. Regret:

   $$\text{Regret}(\mathbf{u}) = \sum_{t=1}^{T} f_t(\mathbf{w_t}) - f_t(\mathbf{u})$$

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3. FTRL Algorithm:

   $$\forall t, \mathbf{w}_t = argmin_{\mathbf{w}} \sum_{i=1}^{t-1} f_i(\mathbf{w}) + R(\mathbf{w})$$

   Thus, $\mathbf{w_{t+1}} = \mathbf{w_t} - \eta \mathbf{g_t} = \mathbf{w_t} - \eta \bigtriangledown f_t(\mathbf{w_t})$

Online Linear Optimization

lemma

Let $w_1, w_2, \cdots$ be the sequence of vectors produced by this OLO algorithm, we have

$$
\begin{aligned}
Regret(\mathbf{u}) &= \sum_{t=1}^{T} f_t(\mathbf{w_t}) - f_t(\mathbf{u}) \\
&\leq R(\mathbf{u}) - R(\mathbf{w_1}) + \sum_{t=1}^{T} (f_t(\mathbf{w_t}) - f_t(\mathbf{w_{t+1}})) \\
&\leq \frac{1}{2\eta} \|\mathbf{u}\|_2^2 + \sum_{t=1}^{T} < \mathbf{w_t} - \mathbf{w_{t+1}}, \mathbf{g_t} > \\
&= \frac{1}{2\eta} \|\mathbf{u}\|_2^2 + \sum_{t=1}^{T} \eta \|\mathbf{g_t}\|_2^2
\end{aligned}
$$

Online Linear Optimization

$Regret(\mathbf{u}) \leq \frac{1}{2\eta}\|\mathbf{u}\|_2^2 + \sum_{t=1}^{T} \eta\|\mathbf{g_t}\|_2^2$

Usually, we consider $U = \{\mathbf{u} : \|\mathbf{u}\| \leq B\}$ and let L be such that $\frac{1}{T}\sum_{t=1}^{T} \|\mathbf{g_t}\|_2^2 \leq L^2$, then by setting $\eta = \frac{B}{L\sqrt{2T}}$, we have

$$Regret_T(U) \leq BL\sqrt{2T}$$

Previous works and problem now

$B = max_{\mathbf{u} \in U} \|\mathbf{u}\|$	$L_{max} = max_t \|z_t\|$	Regret bound
V	V	$O(BL_{max}\sqrt{T})$
V	X	$O(BL_{max}\sqrt{T})$
X	V	$O(\|\mathbf{u}\| log(\|\mathbf{u}\|)L_{max}\sqrt{T})$
X	X	??

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Lower bound

For any $\epsilon > 0$,

$$Regret_T(U) \geq O(\|\mathbf{u}\| log(\|\mathbf{u}\|)L_{max}\sqrt{T} + L_{max} exp[(max_t \frac{\|g_t\|}{L(t)})^{1/2-\epsilon}])$$

where $L(t) = max_{t'<t}\|g_{t'}\|$ and L_{max} are unknown in advance.

Lower Bound with unknown L_{max} and unbounded U

Theorem 1

For any $c, k, \epsilon > 0$, there exists a T and an adversarial strategy picking $g_t \in \mathbb{R}$ in response to $w_t \in \mathbb{R}$ such that

$$
\begin{aligned}
R_T(u) &= \sum_{t=1}^{T} g_t w_t - g_t u \\
&\geq (k + c\|\mathbf{u}\| log\|\mathbf{u}\|) L_{max} \sqrt{T} log(L_{max} + 1) + k L_{max} e^{((2T)^{1/2-\epsilon})}) \\
&\geq (k + c\|\mathbf{u}\| log\|\mathbf{u}\|) L_{max} \sqrt{T} log(L_{max} + 1) + k L_{max} e^{[(max_t \frac{\|g_t\|}{L(t)})^{1/2-\epsilon}]})
\end{aligned}
$$

for some $u \in \mathbb{R}$ where $L_{max} = max_{t \leq T}\|g_t\|$ and $L(t) = max_{t' < t}\|g_{t'}\|$

Proof of lower bound

$$R_T(u) = \sum_{t=1}^{T} g_t w_t - g_t u$$

High level concept: for adversary, the goal is to pick a sequence of g_ts such that the algorithm suffer high regret.

- Case 1: when w_t is small ($w_t < \frac{1}{2}\exp\{T^{1/2}/4log(2)c\}$), there is a large u

$$\Rightarrow \text{choose } g_t = -1$$

$$\Rightarrow L_{max} = 1, \quad \max_t \frac{\|g_t\|}{L(t)} = 1$$

- Case 2: when $w_t < \frac{1}{2}\exp\{T^{1/2}/4log(2)c\}$, let $u = 0$

$$\Rightarrow \text{choose } g_t = 2T$$

$$\Rightarrow L_{max} = 2T, \quad \max_t \frac{\|g_t\|}{L(t)} = 2T$$

The algorithm RESCALEDEXP

High level concept: by "guess-and-double" strategy

1. Initialize a guess L for L_{max} to $\|g_1\|$, then we can run a "known-L_{max}" algorithm.
2. The "known-L_{max} algorithm uses the Follow-the-Regularized-Leader(FTRL) framework.
3. If $\|g_t\| > 2L$, then update the guess to $\|g_t\|$.
   The periods during which L is constant is called "epochs".
4. Need to prove that the "known-L_{max}" algorithm does not suffer too much regret when seeing a g_t that violate the assumed bound L.

The algorithm RESCALEDEXP

RESCALEDEXP

Initialize: $k \leftarrow \sqrt{2}$, $M_0 \leftarrow 0$, $w_1 \leftarrow 0$, $t_\star \leftarrow 1$ // $t_\star$ is the start-time of the current epoch.
for $t = 1$ **to** T **do**
  Play w_t, receive subgradient $g_t \in \partial \ell_t(w_t)$.
  if $t = 1$ **then**
    $L_1 \leftarrow \|g_1\|$
    $p \leftarrow 1/L_1$
  end if
  $M_t \leftarrow \max(M_{t-1}, \|g_{t_\star:t}\|/p - \|g\|^2_{t_\star:t})$.
  $\eta_t \leftarrow \frac{1}{k\sqrt{2(M_t + \|g\|^2_{t_\star:t})}}$
  //Set w_{t+1} using FTRL update
  $w_{t+1} \leftarrow -\frac{g_{t_\star:t}}{\|g_{t_\star:t}\|}\left[\exp(\eta_t \|g_{t_\star:t}\|) - 1\right]$ // $= \operatorname{argmin}_w \left[\frac{\psi(w)}{\eta_t} + g_{t_\star:t} w\right]$
  if $\|g_t\| > 2L_t$ **then**
    //Begin a new epoch: update L and restart FTRL
    $L_{t+1} \leftarrow \|g_t\|$
    $p \leftarrow 1/L_{t+1}$
    $t_\star \leftarrow t + 1$
    $M_t \leftarrow 0$
    $w_{t+1} \leftarrow 0$
  else
    $L_{t+1} \leftarrow L_t$

The algorithm RESCALEDEXP

Theorem 2

Let $M_{max} = max_t M_t$. Then if $L_{max} = max_t \|g_t\|$ and $L(t) = max_{t'<t} \|g_t\|$, RESCALEDEXP achieves regret:

$$R_T(\mathbf{u}) \leq O(L_{max} log(\frac{L_{max}}{L_1})[(\|\mathbf{u}\| log(\|\mathbf{u}\|) + 2)\sqrt{T} + exp(8 max_t \frac{\|g_t\|^2}{L(t)^2})])$$

Improved version-lower bound

For any $\gamma \in (1/2, 1], k > 0$, $T_0 > 0$, and any online optimization algorithm picking $w_t \in \mathbb{R}$, there exists a $T > T_0$, a $u \in \mathbb{R}$, and a sequence $g_1, \cdots, g_T \in \mathbb{R}$ with $\|g_t\| \leq max(1, 18\gamma(4k)^{1/\gamma}(t-1)^{1-1/2\gamma})$ on which the regret is:

$$\begin{aligned} R_T(\mathbf{u}) &= \sum_{t=1}^{T} g_t w_t - g_t \mathbf{u} \\ &\geq k\|\mathbf{u}\| L_{max} log^{\gamma}(T\|\mathbf{u}\| + 1)\sqrt{T} \\ &+ max_{t \leq T} L_{max} \frac{L_{t-1}^2}{\|g\|_{1:t-1}^2} exp[\frac{1}{2}(\frac{L_t/L_{t-1}}{288\gamma k^2})^{1/2\gamma - 1}] \end{aligned}$$

Improved version: lower bound

First dimension tradeoff

Fix $\gamma = 1$, for $k > 0$ there is tradeoff between

$$k\|\mathbf{u}\|L_{max}log(T\|\mathbf{u}\|)\sqrt{T}$$

and

$$exp[(\frac{L_t/L_{t-1}}{k^2})]$$

Second dimension tradeoff

Fix k, for $\gamma \in (1/2, 1]$ there is tradeoff between

$$\|\mathbf{u}\|L_{max}log^{\gamma}(T\|\mathbf{u}\|)\sqrt{T}$$

and

$$exp[(L_t/\gamma L_{t-1})^{1/(2\gamma-1)}]$$

Improved version: design algorithm

Still uses the FTRL framework with different regularizer.

$$\mathbf{w_{t+1}} = argmin_{\mathbf{w}\in W}\psi_t(\mathbf{w}) + \sum_{t'=1}^{t} f_{t'}(\mathbf{w})$$

where

$$\psi_t(\mathbf{w}) = \frac{k}{a_t\eta_t}\psi(a_t\mathbf{w})$$

$$\frac{1}{\eta_0^2} = 0, \quad \frac{1}{\eta_t^2} = max(\frac{1}{\eta_{t-1}^2} + 2\|g_t\|_*^2, L_t\|g_{1:t}\|_*)$$

$$a_1 = \frac{1}{(L_1\eta_1)^2}, \quad a_t = max(a_{t-1}, \frac{1}{(L_t\eta_t)^2})$$

a_t and η_t are carefully chosen functions of observed gradients $g_1, \cdots, g_t$ that guarantee the desired asymptotics in the regret bound

Improved version: adaptive regularizers

Definition 1

A convex function $f : W \to \mathbb{R}$ is σ-strongly convex with respect to a norm $\|\cdot\|$ if for all $x, y \in W$ and $g \in \partial f(x)$ we have

$$f(y) \geq f(x) + g \cdot (y - x) + \frac{min(\sigma(x), \sigma(y))}{2} \|x - y\|^2$$

Definition 2

Let W be a closed convex subset of a vector space s.t. $0 \in W$. Any differentiable function $\psi : W \to \mathbb{R}$ that satisfies the following conditions:

1. $\psi(0) = 0$
2. $\psi(x)$ is σ-strongly-convex with respect to some norm $\|\cdot\|$ for some $\sigma : W \to \mathbb{R}$ s.t. $\|x\| \geq \|y\|$ implies $\sigma(x) \leq \sigma(y)$
3. For any C, there exists a B s.t. $\psi(x)\sigma(x) \geq C$ for all $\|x\| \geq B$

is called a $(\sigma, \|\cdot\|)$-adaptive regularizer.

Improved version: adaptive regularizers

Define

$$h(\mathbf{w}) = \psi(\mathbf{w})\sigma(\mathbf{w}), \quad h^{-1}(x) = max_{h(\mathbf{w})\leq x}\|\mathbf{w}\|$$

$$D = max_t \frac{L_{t-1}^2}{(\|g\|_*^2)_{1:t-1}} h^{-1}\left(\frac{5L_t}{k^2 L_{t-1}}\right)$$

Theorem 3

Suppose ψ is a $(\sigma, \|\cdot\|)$-adaptive regularizer and $g_1, \cdots g_T$ is some arbitrary sequence of subgradients, then FTRL with regularizer ψ_t achieves regret

$$R_T(u) \leq kL_{max}\frac{\psi(2uT)}{\sqrt{2T}} + 2L_{max}D + \frac{45L_{max}}{\sigma_{min}}$$

Goal: Choose a $h(x) = \psi(x)\sigma(x)$ s.t. $h^{-1}(x) \approx exp(x)$
and $\psi(2uT)/\sqrt{2T} = O(\|u\|\sqrt{T}log(T\|u\|+1))$

Improved version: γ-optimal

Theorem 4

If ψ is an $(\sigma, \|\cdot\|)$-adaptive regularizer s.t.

$$\psi(x)\sigma(x) \geq \Omega(\gamma log^{2\gamma-1}(\|x\|)) \quad (1)$$

$$\psi(x) \leq O(\|x\| log^{\gamma}(\|x\|+1)) \quad (2)$$

Then for any $k \geq 1$, FTRL with regularizers $\psi_t(\mathbf{w}) = \frac{k}{a_t \eta_t}\psi(a_t \mathbf{w})$ yields regret

$$R_T(u) \leq O[kL_{max}\sqrt{T}\|u\| log^{\gamma}(T\|u\|+1)] + max_t \frac{L_{max}L_{t-1}^2}{\|g\|_{1:t-1}^2} exp[O((\frac{L_t}{k^2\gamma L_{t-1}})^{1/2\gamma-1})]$$

We call regularizers that satisfy these conditions γ-optimal.

Note that this actually match the lower bound for all $\gamma \in (1/2, 1]$.

Improved version: choose adaptive regularizers

Proposition 1

Let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$ be a three-times differentiable function that satisfies

1. $\phi(0) = 0$
2. $\phi'(x) \geq 0$
3. $\phi''(x) \geq 0$
4. $\phi'''(x) \leq 0$
5. $lim_{x \rightarrow \infty} \phi(x)\phi''(x) = \infty$

Then $\psi(\mathbf{w}) = \phi(\|\mathbf{w}\|)$ is a $(\phi''(\|\cdot\|), \|\cdot\|)$-adaptive regularizer.

Improved version: 1-optimal adaptive regularizer

Proposition 2

Let $\phi(x) = (x+1)log(x+1) - x$. Then $\psi(w) = \phi(\|w\|)$ is a 1-optimal, $(\phi''(\|\cdot\|), \|\cdot\|)$-adaptive regularizer
(pf):

1. $\phi(0) = 0$
2. $\phi'(x) = log(x+1)$
3. $\phi''(x) = \frac{1}{x+1}$
4. $\phi'''(x) = -\frac{1}{(x+1)^2}$
5. $\phi(x)\phi''(x) = log(x+1) - \frac{x}{x+1}$

Thus, FTRL with regularizers
$\psi_t(w) = \frac{k}{\eta_t a_t}((\|w\|+1)log(\|w\|+1) - \|w\|)$ achieves the regret:

$$k\|\mathbf{u}\|L_{max}log(T\|\mathbf{u}\|+1)\sqrt{T} + L_{max}max_{t\leq T}\frac{L_{t-1}^2}{\|g\|_{1:t-1}^2}exp[(\frac{5L_t/L_{t-1}}{k^2})]$$

Improved version: γ-optimal adaptive regularizer

Proposition 3

Given $\gamma \in (1/2, 1]$, set $\phi(x) = \int_0^x log^\gamma(z+1)dz$. Then $\psi(w) = \phi(\|w\|)$ is a γ-optimal, $(\phi''(\|\cdot\|), \|\cdot\|)$-adaptive regularizer

Thus, FTRL with regularizers $\psi_t(w) = \int_0^{\|w\|} log^\gamma(z+1)dz$ achieves the regret:

$$R_T(u) \leq O[kL_{max}\sqrt{T}\|u\| log^\gamma(T\|u\|+1)] + max_t \frac{L_{max}L_{t-1}^2}{\|g\|_{1:t-1}^2} exp[O((\frac{L_t}{k^2\gamma L_{t-1}})^{1/2\gamma-1})]$$

and the update rule:

$$w_{t+1} = -\frac{g_{1:t}}{a_t\|g_{1:t}\|}[exp((\eta_t\|g_{1:t}\|/k)^{1/\gamma}) - 1]$$

The algorithm FREEREX

FREEREX, for $\gamma = 1$

Algorithm 1 FREEREX

Input: k.

Initialize: $\frac{1}{\eta_0^2} \leftarrow 0$, $a_0 \leftarrow 0$, $w_1 \leftarrow 0$, $L_0 \leftarrow 0$, $\psi(w) = (\|w\| + 1)\log(\|w\| + 1) - \|w\|$.

for $t = 1$ **to** T **do**

Play w_t, receive subgradient $g_t \in \partial \ell_t(w_t)$.

$L_t \leftarrow \max(L_{t-1}, \|g_t\|)$.

$\frac{1}{\eta_t^2} \leftarrow \max\left(\frac{1}{\eta_{t-1}^2} + 2\|g_t\|^2, L_t\|g_{1:t}\|\right)$.

$a_t \leftarrow \max(a_{t-1}, 1/(L_t\eta_t)^2)$.

//Set w_{t+1} using FTRL update

$w_{t+1} \leftarrow -\frac{g_{1:t}}{a_t\|g_{1:t}\|}\left[\exp\left(\frac{\eta_t\|g_{1:t}\|}{k}\right) - 1\right]$ // $= \operatorname{argmin}_w \left[\frac{k\psi(a_t w)}{a_t\eta_t} + g_{1:t}w\right]$

end for