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Towards a Unified Taxonomy and Architecture-Independent Quantum Circuit Compilation for Quantum Finite Automata

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Abstract

As quantum computing matures, concise models are needed to bridge theoretical foundations and practical implementations. Quantum Finite Automata (QFAs) fulfil this role by extending the standard framework of Classical Finite Automata (CFAs) into the quantum domain, providing a compact setting in which to study finite-memory quantum behaviour.

This thesis first revisits the essential notions on both classical automata and quantum information, establishing a common language for readers from either field. It then presents a comprehensive literature review that consolidates the many QFAs variants introduced over the last three decades and arranges them in a unified, consistently named taxonomy. Building on this organisation, the work proposes a compilation algorithm that translates the widely studied Measure Once One-Way Quantum Finite Automata (MO-1QFAs) and Measure Many One-Way Quantum Finite Automata (MM-1QFAs) models into architecture-independent quantum circuit templates, thereby linking abstract automaton descriptions with executable gate-level designs.

Viewed more broadly, the study contributes a computer-science perspective on the quantum landscape and offers an accessible entry point for further research in quantum software by encouraging circuit-level abstraction and enabling systematic comparison of different designs.

Keywords: Quantum finite automata, automata theory, quantum information, quantum computing, quantum circuits, quantum compilation.

Contents

1. Introduction

The accelerating progress of quantum computing has sparked a parallel evolution in theoretical models that aim to describe and harness quantum behaviour within computational frameworks [41]. Among these models, the study of QFAs offers a minimal yet expressive lens for investigating finite-memory quantum systems [4, 79]. Owing to their bounded-state registers, QFAs constitute a rigorous setting for analysing language recognition capabilities under quantum constraints and for crafting algorithms that operate within the limits of current Noisy Intermediate-Scale Quantum (NISQ) technology [92].

QFAs are particularly attractive due to their finite-memory constraints, making them ideal for investigating foundational questions in quantum-computational theory and exploring efficient recognisers for regular and near-Regular Language (REG)s. Moreover, their simplicity renders them amenable to physical implementation on today's hardware, where full-scale quantum algorithms remain impractical [8]. Despite this promise, the diversity of QFA models has led to a fragmented landscape. Disparate notations, inconsistent terminologies, and varied acceptance criteria have made it difficult to compare models, reason about their capabilities, or implement them as executable artefacts.

This thesis tackles the fragmentation challenge through two tightly coupled contributions. First, it provides a coherent and systematic taxonomy of QFAs. Drawing on over three decades of research, the thesis consolidates the principal families of QFAs into a unified nomenclature, identifies key relationships between models, and supplies the conceptual scaffolding required for rigorous analysis and comparison of expressive power, closure properties, and language recognition capabilities.

The second contribution closes the gap between abstract definitions and executable artefacts by introducing a compilation framework that converts high-level descriptions of MO-1QFAs and MM-1QFAs into quantum circuits. The compiler leverages the taxonomy to normalise automaton specifications and then synthesises architecture-independent gate templates whose parameters instantiate the original transition operators. In this way, the compilation algorithm operationalises the taxonomic unification: once models are described within a common schema, they can be mapped uniformly to circuits, enabling empirical evaluation, formal verification, and direct deployment on hardware

The remainder of the thesis is structured as follows. Chapter 2 reviews the necessary background in classical automata theory and quantum information science. Chapter 3 presents the unified taxonomy of QFAs, establishing precise definitions and cataloguing formal properties. Chapter 4 details the circuit compilation framework, with examples illustrating how abstract automata are translated into gate-level designs. Finally, Chapter 5 summarises the findings and outlines prospects for extending the framework to more powerful automata and for integrating QFAs into broader quantum-software stacks.

2. Background

This chapter provides the theoretical and mathematical foundations that connect classical models of computation with quantum information theory, structured around three central domains: classical automata theory, finite-dimensional quantum mechanics, and the gate-based framework of quantum circuits.

We begin by reviewing the algebra of formal languages and the computational limitations and expressiveness of CFAs. The section formalises Deterministic Finite Automata (DFAs), Nondeterministic Finite Automata (NFAs), Probabilistic Finite Automata (PFAs), and Two-Way Finite Automata (2FAs), examining their closure properties, characterisation of REGs, and algebraic structure, which together form a foundational framework for the study of automaton-based computation [60].

Next, we present the finite-dimensional postulates of non-relativistic quantum mechanics. This section covers the fundamental ideas of superposition, entanglement, the probabilistic nature of quantum measurement, and the encoding of information in qubits and quantum states in the context of quantum dynamics, which includes unitary evolution governed by the Schrödinger equation and the handling of decoherence and open systems via density matrices. The exposition follows the axiomatic approach developed by Dirac and von Neumann [44, 118], with particular emphasis on features pertinent to quantum automata, including projective measurements, the no-cloning theorem, and the compositional structure of subsystems [120].

The gate-based circuit model of quantum computation is covered in the last section. It reviews common quantum gates, families of circuits (including parameterised and measurement-based circuits), the implementation of quantum algorithms, and the decomposition of arbitrary unitary operations into standard gate sets [119, 49]. The circuit-level realisation of QFAs covered in later chapters is based on these ideas.

Together, these three pillars form the theoretical foundation upon which the thesis develops its unified treatment of QFAs, their taxonomy, and their compilation into executable circuits.

2.1 Classical Finite Automata and Formal Languages

Finite Automata (FAs) stand at the very origin of algorithmic language processing and they form the canonical recognisers of the REGs, a class originally isolated by Kleene through rational expressions [64]. Their mathematical roots trace back to Chomsky's hierarchy of grammars [34] and to the seminal definition of machine based language recognition given independently by Rabin and Scott [101]. The structural essence of regularity was soon clarified by the Myhill and Nerode equivalence relation, which provides a necessary and sufficient criterion for finite recognisability without reference to a particular device [81, 83]. These results, later unified in modern textbooks [60, 3, 116], still underpin today's compiler front-ends, hardware verification workflows and network protocol design.

Historically four computational patterns crystallised out of the study of CFAs. The first is the DFA, a fully deterministic device that admits a single computation path and thus offers linear time membership testing together with minimal state equivalence [60]. The second is the NFA, which explores many futures in parallel and enjoys an exponential succinctness advantage while remaining equivalent in expressive power to the deterministic model [101]. Probabilistic branching leads to the PFA, an automaton that attaches transition weights and decides membership relative to a rational cut point; with an isolated threshold it still recognises only REGs [89] whereas an arbitrary threshold allows stochastic languages that are not regular [117]. Finally lifting the read head restriction yields two-way variants that may revisit earlier symbols and thereby capture algorithms such as lexical scanners with look ahead, yet they do not surpass regular expressiveness [52].

This section surveys the algebraic landscape in which these machines operate. It begins with precise notions of words, grammars and closure properties, establishing the equivalence between regular grammars, rational expressions and CFAs. It then presents the four concrete models just outlined, emphasising historical context, formal properties such as closure and minimisation, and illustrative examples that will serve as running motifs throughout the thesis. Together these elements motivate later chapters, where classical finite-memory computation is extended first to probabilistic and finally to quantum domains.

2.1.1 Languages, Grammars and Regularity

The study of CFAs rests on an algebraic view of words. Starting from finite alphabets, we assemble strings through concatenation, obtain the free monoid Σ^*, and finally analyse those subsets of Σ^* that arise in computation. This section revisits the classical path from atomic symbols to the full power of regular grammars. Along the way the narrative highlights why each construction matters later for deterministic, nondeterministic and probabilistic machines.

Definition 2.1.1 (Alphabet)**.** A non empty finite set Σ of symbols is called an alphabet [60].

Finite alphabets formalise the intuitive notion of a computer input device that produces a bounded repertoire of symbols. Restricting to finite Σ is essential for decidability results that follow in the automata hierarchy.

Definition 2.1.2 (String and Concatenation)**.** A string, or word, over Σ is a finite sequence $w = a_1a_2 \dots a_n$ with $a_i \in \Sigma$. The empty word is denoted ε. Concatenation $u \cdot v$ appends the sequence of v to u [60].

Concatenation is associative, admits ε as identity and thus endows the set of words with a monoid structure. This algebraic property underpins many closure proofs for language classes.

Notation 2.1.1. The set of all words over Σ forms the free monoid Σ^* under concatenation with identity ε [60].

Definition 2.1.3 (Formal Language)**.** Any subset $L \subseteq \Sigma^*$ is a language [60].

Automata theory evaluates the descriptive cost of specifying L. The smaller the machine needed to decide membership, the simpler the language is deemed.

Concept 2.1.1 (Regular Grammar)**.** A type 3 grammar in Chomsky's hierarchy 2.1 generates the REGs [34].

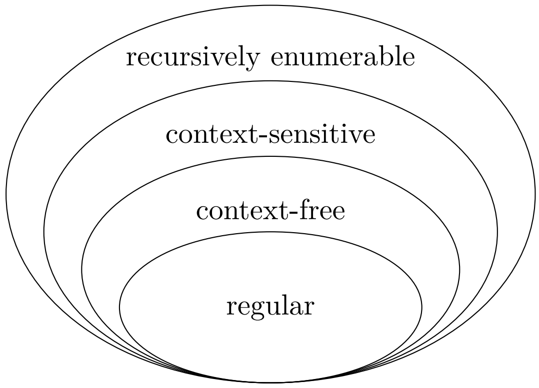


Figure 2.1: Chomsky hierarchy of formal languages

Type 3 productions constrain derivations to one non terminal symbol at most, ensuring finite memory during parsing. The resulting languages coincide with those accepted by finite automata, an insight traced back to Kleene.

Theorem 2.1.1 (Kleene Correspondence)**.** *The following families coincide:*

1. *languages generated by type 3 grammars,*
2. *languages denoted by regular expressions,*
3. *languages recognised by CFAs.*

[64]

Kleene's theorem supplies three interchangeable viewpoints: syntactic derivation, algebraic expression and operational recognition. Switching among them allows proofs to borrow the strengths of each representation, for instance when optimising a DFA derived from a regular expression.

Proposition 2.1.1 (Closure of REGs)**.** *Let $\mathcal{R}$ denote the class of* REG*s. If $L_1, L_2 \in \mathcal{R}$ then so are $L_1 \cup L_2$, $L_1 \cap L_2$, $\overline{L_1}$, $L_1 L_2$ and L_1^* [60].*

Closure under Boolean operations follows directly from set algebra on DFA state sets, while closure under concatenation and star is established with product and power set constructions. These properties justify the pervasive role of REGs in software tools: complex lexical patterns can be decomposed, manipulated and recombined without leaving $\mathcal{R}$.

Theorem 2.1.2 (Myhill-Nerode)**.** *A language $L \subseteq \Sigma^*$ is regular iff the relation $x \equiv_L y \iff \forall z \in \Sigma^* : xz \in L \Leftrightarrow yz \in L$ admits finitely many equivalence classes [60].*

The theorem supplies a minimality criterion: the number of equivalence classes equals the state count of the smallest DFA for L. In compiler construction this bound translates into memory requirements for lexical analysers.

Example 2.1.1 (Simple REG)**.** For $\Sigma = \{0, 1\}$ the set $L = \{w \in \Sigma^* \mid w \text{ ends in } 1\}$ is regular because it is described by the regular expression Σ^*1 [60].

The language of Example 2.1.1 illustrates how a single positional constraint is captured by a two-state DFA, matching the Myhill-Nerode bound.

Observation 2.1.1. REGs admit linear time membership tests and deterministic finite-state representations; therefore they are widely used in lexical tokenisers, model checking and hardware controllers [3].

In summary, regularity offers a balance between expressive adequacy for many practical patterns and tractable analysis, providing the theoretical bedrock on which later subsections build deterministic, nondeterministic and quantum extensions of the finite automaton paradigm.

2.1.2 Deterministic Finite Automata

Among abstract machines the DFA offers the most transparent model of computation. From any configuration the current state and the symbol under the input head select exactly one successor state [60]. This functional behaviour yields predictable memory usage, permits direct hardware realisation and supports efficient software simulation.

Definition 2.1.4 (Deterministic Finite Automaton)**.** A DFA is a quintuple $M = (Q, \Sigma, \delta, q_0, F)$ where Q is a finite set of states, Σ is an input alphabet, $\delta\colon Q \times \Sigma \to Q$ is the transition map, $q_0 \in Q$ is the start state and $F \subseteq Q$ is the set of accepting states [60].

For analysis it is convenient to lift δ to words. The extended map $\hat{\delta}\colon Q \times \Sigma^* \to Q$ satisfies $\hat{\delta}(q, \varepsilon) = q$ and $\hat{\delta}(q, aw) = \hat{\delta}(\delta(q, a), w)$. Structural induction on the length of w shows that $\hat{\delta}$ is total and unique [60].

Proposition 2.1.2 (Unique computation path)**.** *For every input word $w \in \Sigma^*$ the sequence $q_0, \hat{\delta}(q_0, a_1), \hat{\delta}(q_0, a_1a_2), \ldots, \hat{\delta}(q_0, w)$ is the only path in M labelled by w [60].*

Definition 2.1.5 (Accepted language)**.** The language recognised by M is $L(M) = \{\, w \in \Sigma^* \mid \hat{\delta}(q_0, w) \in F \,\}$ [60].

Finite automata gain analytic power from the right invariant congruence

$$x \equiv_L y \iff \forall z \in \Sigma^* \colon xz \in L \Leftrightarrow yz \in L.$$

Theorem 2.1.3 (Myhill Nerode)**.** *A language $L \subseteq \Sigma^*$ is regular iff the relation $\equiv_L$ has finitely many equivalence classes [83].*

The number of classes equals the size of the smallest DFA for L, so state minimisation amounts to quotienting by $\equiv_L$.

Theorem 2.1.4 (DFA minimisation)**.** *Every DFA M possesses a unique minimal equivalent DFA $M_{\min}$ up to isomorphism [80]. The Hopcroft partition refinement algorithm computes $M_{\min}$ in $O(|Q| \log |Q|)$ time [59].*

Because a DFA is finite, many questions are decidable.

Proposition 2.1.3 (Algorithmic questions)**.** *Given DFAs M_1, M_2 over Σ one can decide in polynomial time: emptiness of $L(M_1)$, finiteness of $L(M_1)$, equivalence $L(M_1) = L(M_2)$ and inclusion $L(M_1) \subseteq L(M_2)$ [60].*

Example 2.1.2 (Even number of a symbols)**.** Figure 2.2 depicts the DFA recognising $L = \{\, w \in \{a, b\}^* \mid \text{the number of } a \text{ symbols is even} \,\}$ [60]. The relation $\equiv_L$ has exactly two classes, matching the two states shown.

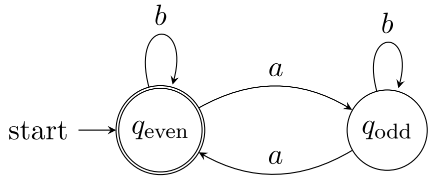


Figure 2.2: DFA for words with an even number of a symbols [60].

Example 2.1.3 (Binary multiples of three)**.** Over $\Sigma = \{0, 1\}$ the set $L = \{\, w \mid w \text{ interpreted as binary represents a multiple of } 3 \,\}$ is regular. A machine with three states tracks the remainder modulo three after each bit is read [116].

DFAs support linear time scanning with constant memory. Compilers use them for lexical analysis, hardware circuits translate transitions into flip flops, and many regular expression engines adopt deterministic evaluation to guarantee predictable performance [3]. Deterministic automata thus form the baseline against which later sections measure the expressive gains and computational costs of nondeterministic, weighted and quantum variants.

2.1.3 Nondeterministic Finite Automata

Where a DFA follows a single computation path, a NFA may branch whenever several successor states are permitted. The branching semantics can reduce the number of states that must be stored explicitly, but it transfers complexity to the acceptance relation [101].

Definition 2.1.6 (Nondeterministic Finite Automaton)**.** A NFA is a quintuple $M = (Q, \Sigma, \delta, q_0, F)$ with finite-state set Q, input alphabet Σ, transition map $\delta \colon Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$, start state $q_0 \in Q$ and set of final states $F \subseteq Q$. Here $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$ adjoins the empty word to allow silent moves [101].

Extending δ to words first requires the ε-closure operator $\mathrm{E}(S) = \big\{\, q \in Q \mid \exists p \in S \colon p \xrightarrow{\varepsilon^*} q \,\big\}$

For $w = a_1 a_2 \dots a_n$ define recursively $S_0 = \mathrm{E}\big(\{q_0\}\big)$ and $S_{i+1} = \mathrm{E}\big(\bigcup_{q \in S_i} \delta(q, a_{i+1})\big)$. The set S_n collects every state reachable after reading w.

Lemma 2.1.1 (Acceptance criterion)**.** *The automaton M accepts w exactly when $S_{|w|} \cap F \neq \varnothing$ [60].*

Nondeterminism does not enlarge expressive power.

Theorem 2.1.5 (Subset construction)**.** *For an NFA with n states the powerset construction produces an equivalent DFA whose state set has size at most 2^n [101].*

Silent moves can be eliminated first; the combined transformation preserves language and produces a DFA whose transition map operates on subsets of Q.

Proposition 2.1.4 (Elimination of ε transitions)**.** *An NFA with ε moves has an equivalent ε-free NFA with the same number of states [60].*

Although determinisation may yield an exponential blow-up, the reverse direction is more benign.

Observation 2.1.2 (Succinctness gap). There exist languages $L_n \subseteq \Sigma^*$ whose minimal NFA requires $\Theta(n)$ states while every equivalent DFA needs $\Theta(2^n)$ states [3].

The gap justifies the preference for NFAs as intermediate representations in regular expression engines that ultimately generate a DFA only for performance critical fragments.

Example 2.1.4 (Words of length one)**.** Figure 2.3 shows a four state NFA that accepts Σ^1. Two silent moves branch from q_0; each branch consumes one symbol then moves to the final state q_f [101].

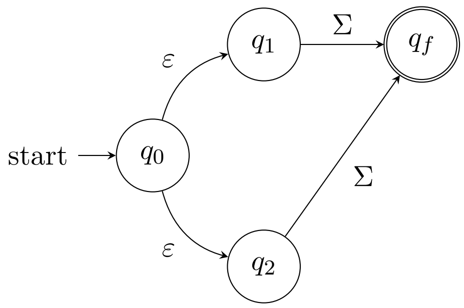


Figure 2.3: NFA for all words whose length equals one [101].

Decision problems for NFAs inherit tractability from their deterministic counterparts because determinisation is constructive; emptiness, finiteness and equivalence remain decidable in polynomial time with respect to the size of the generated DFA [60]. Subsequent sections compare these classical results with the complexity landscape of probabilistic and quantum extensions.

2.1.4 Probabilistic Finite Automata

CFAs represent discrete control without any source of uncertainty. By attaching probabilities to transitions we obtain a model that evolves like a Markov chain driven by the scanned word. These PFAs form the finite-memory analogue of probabilistic Turing machines and supply the algebraic core of hidden Markov models [100].

Definition 2.1.7 (Probabilistic Finite Automaton)**.** A PFA is a quintuple

$$M = (Q, \Sigma, \{\delta(a)\}_{a\in\Sigma}, \mathbf{i}, F)$$

where

- Q is a finite-state set,
- Σ is an input alphabet,
- for every $a \in \Sigma$ the matrix $\delta(a) \in [0,1]^{Q\times Q}$ is row-stochastic, that is $\sum_{q'\in Q} \delta(a)_{q,q'} = 1$ for each $q \in Q$,

- $\mathbf{i} \in [0,1]^Q$ is a row vector that lists the initial state distribution with $\sum_{q\in Q} \mathbf{i}_q = 1$,
- $F \subseteq Q$ is the set of accepting states [100].

Reading a word $w = a_1a_2 \dots a_n$ multiplies the matrices that correspond to its symbols, producing the distribution

$$\mathbf{p}(w) = \mathbf{i}\,\delta(a_1)\delta(a_2)\dots\delta(a_n).$$

The acceptance probability is then $\Pr_M(w) = \sum_{q\in F} \mathbf{p}(w)_q$.

Concept 2.1.2 (Cut point language)**.** For any threshold $\lambda \in [0,1]$ define

$$L(M,\lambda) = \{\, w \in \Sigma^* \mid \Pr_M(w) > \lambda \,\}$$

[100].

Where deterministic and nondeterministic machines yield yes / no answers, a PFA induces a family of cut point languages by varying λ. This observation motivates the notion of stochastic recognition.

Definition 2.1.8 (Stochastic language)**.** A language $L \subseteq \Sigma^*$ is called *stochastic* if there exist a PFA M and a cut point λ such that $L = L(M,\lambda)$ [100]. The family of all stochastic languages is written SL.

Unlike nondeterminism, randomisation enlarges expressive power.

Theorem 2.1.6 (Non regular stochastic languages)**.** *There exists* $L \in \mathsf{SL}\backslash\mathsf{REG}$*; consequently* SL *strictly contains the* REG*s [100].*

The price of this extra power is undecidability.

Proposition 2.1.5 (Undecidable properties)**.** *For unbounded-error PFAs the problems of emptiness, universality and equivalence are undecidable [89].*

Bounding the error restores regularity.

Theorem 2.1.7 (Bounded-error regularity)**.** *If there exists* $\eta > 0$ *such that* $|\Pr_M(w) - \lambda| \geqslant \eta$ *for every* $w \in \Sigma^*$*, then* $L(M,\lambda)$ *is regular [89].*

Even under bounded error, randomised machines can be exponentially more succinct than deterministic ones.

Observation 2.1.3 (State complexity)*.* Some REGs admit PFAs of $\Theta(n)$ states, whereas any equivalent DFA needs $\Theta(2^n)$ states [89].

Example 2.1.5 (Unary majority)**.** Let $\Sigma = \{a,b\}$ and

$$L = \{\, w \in \Sigma^* \mid \#_a(w) > \#_b(w) \,\}.$$

A two-state PFA with cut point $\lambda = \frac{1}{2}$ recognises L by interpreting each a as a biased step toward the accepting state and each b as a step in the opposite direction [100]. The pumping lemma shows that L is not regular, illustrating Theorem 2.1.6.

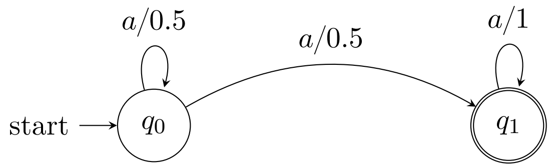


Figure 2.4: PFA that recognises $\{a^k \mid k \geqslant 2\}$ with cut point $\lambda = 0.5$ [100].

Early stochastic automata not only foreshadowed subsequent models of probabilistic computation but also laid the groundwork for practical hidden Markov models in speech and bioinformatics [100, 89]. Taken together, these results show that probability enriches finite-state computation while sharply separating efficient description from decidable analysis, a theme that recurs in quantum and weighted automata studied in the following sections.

2.1.5 Two-Way Variants

Permitting the reading head to move left as well as right gives FAs a bidirectional scanning ability. This operational freedom simplifies many pattern recognisers yet, perhaps surprisingly, does not extend expressive power beyond the REGs [113].

Definition 2.1.9 (Bidirectional Head)**.** Let $\Sigma_\square = \Sigma \cup \{\square\}$, where $\square$ marks the left end of the input. A head movement is specified by $d \in \{-1, 0, 1\}$, meaning left, stay, or right [113].

With the motion primitive in place we distinguish three probabilistic levels of control.

Definition 2.1.10 (2DFA, 2NFA, 2PFA)**.** A bidirectional automaton is a quintuple $M = (Q, \Sigma_\square, \delta, q_0, F)$ that differs only in the type of its transition map δ.

- Two-Way Deterministic Finite Automaton (2DFA) : $\delta \colon Q \times \Sigma_\square \to Q \times \{-1, 0, 1\}$ is a function.
- Two-Way Nondeterministic Finite Automaton (2NFA) : δ maps to finite sets of state-move pairs, introducing nondeterminism [101].
- Two-Way Probabilistic Finite Automaton (2PFA) : for every symbol a the distribution $\delta(\cdot \mid a)$ is row-stochastic over $Q \times \{-1, 0, 1\}$, yielding a Markov decision process [52].

Bidirectional motion changes how a language is processed rather than which languages can be processed.

Theorem 2.1.8 (Expressive Power)**.** *Deterministic, nondeterministic and bounded-error probabilistic bidirectional automata recognise exactly the* REG*s [113, 52].*

Although the recognised class is unchanged, determinising a bidirectional machine can be costly.

Proposition 2.1.6 (Simulation Cost)**.**

- *An n state 2DFA or 2NFA may require $2^{\mathcal{O}(n \log n)}$ states when simulated by a One-Way Deterministic Finite Automaton (1DFA) [113, 101].*

- *A bounded-error 2PFA expands to $\mathcal{O}(n^2)$ states under conversion to a One-Way Probabilistic Finite Automaton (1PFA) [46].*

Bidirectional automata therefore occupy an interesting middle-ground: they match one-way machines in expressive power while often providing exponentially more succinct descriptions.

Observation 2.1.4 (Practical Motivation). Repeated passes over data streams are common in text editors, network-protocol monitors and reversible parsing. Bidirectional automata give a concise formal model of this behaviour [113].

In summary, allowing the head to retrace its steps enriches algorithmic strategy and can yield dramatic savings in state count, yet all such advantages remain confined within the REG frontier that anchors the theory developed in the previous subsections.

2.2 Quantum Mechanics Foundations

Quantum theory delivers a linear-algebraic description of microscopic systems that has withstood every experimental test to date [44, 85]. The finite-dimensional fragment of this theory already suffices for reversible computing, cryptography and the automata models investigated later in the thesis. Accordingly the presentation below restricts attention to finite Hilbert spaces and the unitary maps that act upon them.

The narrative proceeds from kinematic notions to dynamical ones. First come pure quantum states, qubits and Dirac's bra-ket notation, followed by the tensor product that builds composite systems and the Bloch-sphere picture that aids visualisation of single-qubit superposition. Entanglement is then introduced as the key non-classical resource, together with Schmidt rank as a quantitative measure. Measurement postulates appear next, encompassing both projective measurements and the more general Positive Operator-Valued Measures (POVMs) that model noisy detectors. Density operators, partial trace and quantum channels provide the language for open-system dynamics, while Kraus decompositions formalise decoherence and error. Finally the Schrödinger equation is specialised to finite-dimensional unitary evolution, establishing the toolkit used in subsequent chapters on quantum finite automata.

Relativistic extensions handled by quantum field theory, such as those described by the Quantum Fourier Transform (QFT), remain outside the present scope [119]; every example and proof will rely only on the finite-dimensional machinery summarised here.

2.2.1 Qubits and Quantum States

Quantum information is encoded in the state of a two-level system, the Quantum Bit (Qubit). Although physical realisations differ, ranging from photonic polarisations and superconducting circuits to nuclear spins, the mathematical model is uniform: a unit vector in a two-dimensional complex Hilbert space [44, 85]. Dirac's bra-ket notation offers a compact syntax for such vectors and fixes $\{|0\rangle, |1\rangle\}$ as the computational basis used throughout the thesis.

Definition 2.2.1 (Pure qubit state)**.** A pure state of a qubit is a unit vector

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle, \qquad \alpha, \beta \in \mathbb{C},\ |\alpha|^2 + |\beta|^2 = 1.$$

Equivalently one may parameterise the state as

$$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{\mathrm{i}\phi}\sin\frac{\theta}{2}\,|1\rangle, \qquad \theta \in [0, \pi],\ \phi \in [0, 2\pi),$$

where the pair (θ, ϕ) locates $|\psi\rangle$ on the Bloch sphere [85].

The Bloch sphere embeds all pure qubit states on the surface of a unit sphere in $\mathbb{R}^3$; its poles represent the basis vectors while global phases $e^{i\gamma}|\psi\rangle$ collapse to the same point because overall phase has no observable consequence [85]. Figure 2.5 displays the geometry and the two angular coordinates that parameterise a state.

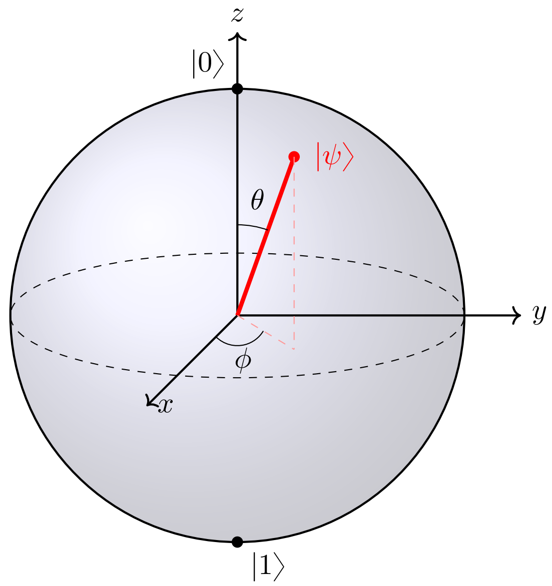


Figure 2.5: The Bloch sphere representation of a qubit.

Composite systems are modelled by tensor products. Two qubits A and B inhabit $\mathcal{H}_{AB} = \mathcal{H}_{2,A} \otimes \mathcal{H}_{2,B}$, enabling entangled states such as the Bell pair $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Mixed states, described by density operators ρ with $\rho \geqslant 0$ and $\operatorname{tr} \rho = 1$, appear later when noise and measurement are introduced; for the purely kinematic discussion here, pure vectors suffice.

The qubit therefore serves as the atomic carrier of quantum information, combining a continuous state space with a discrete computational basis. Subsequent subsections build on this representation, first to formalise measurement and decoherence, then to construct QFAs whose internal registers are composed of a finite number of qubits.

2.2.2 Superposition and Entanglement

Quantum information is encoded in vectors of a finite-dimensional Hilbert space $\mathcal{H}$. Linearity of $\mathcal{H}$ implies that a linear combination of admissible vectors is again an admissible vector.

Definition 2.2.2 (Principle of superposition [85])**.** Let $\{|\psi_i\rangle\}_{i\in I} \subset \mathcal{H}$ be a family of normalised state vectors and let $\{\alpha_i\}_{i\in I} \subset \mathbb{C}$ satisfy $\sum_{i\in I} |\alpha_i|^2 = 1$. The vector $|\Psi\rangle = \sum_{i\in I} \alpha_i |\psi_i\rangle$ is a valid quantum state and is called a superposition of the $|\psi_i\rangle$.

Superposition yields interference phenomena that have no classical counterpart; amplitudes, not probabilities, add. In protocols such as the Deutsch-Jozsa algorithm and amplitude amplification the relative phases between components of $|\Psi\rangle$ are adjusted in order to concentrate probability mass on measurement outcomes that reveal global properties of the input [85].

Remark. For a single qubit any pure state can be written as $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$ with $\theta \in [0,\pi]$ and $\phi \in [0,2\pi)$. This parametrisation maps one-to-one to the Bloch sphere and offers geometric insight into unitary control operations.

Composing systems is formalised by the tensor product. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces for subsystems A and B. A joint pure state lives in $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Some joint states are separable; others are not.

Definition 2.2.3 (Entangled state [61])**.** A pure state $|\Psi\rangle_{AB} \in \mathcal{H}_{AB}$ is entangled if there exist no states $|\psi\rangle_A \in \mathcal{H}_A$ and $|\phi\rangle_B \in \mathcal{H}_B$ such that $|\Psi\rangle_{AB} = |\psi\rangle_A \otimes |\phi\rangle_B$.

Example 2.2.1 (Bell pair [11])**.** The state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle_A |1\rangle_B - |1\rangle_A |0\rangle_B\big)$$

is entangled. Tracing out either subsystem produces the maximally mixed single-qubit density operator $\frac{1}{2}I$, confirming that no local description captures the correlations present in $|\Psi^-\rangle$.

Proposition 2.2.1 (Schmidt criterion [85])**.** *A bipartite pure state is separable if and only if its Schmidt rank equals* 1*.*

Sketch. Write $|\Psi\rangle_{AB} = \sum_i \lambda_i |u_i\rangle_A \otimes |v_i\rangle_B$ with orthonormal families $\{|u_i\rangle\}$, $\{|v_i\rangle\}$ and $\lambda_i > 0$. If only one Schmidt coefficient is non-zero then $|\Psi\rangle_{AB}$ factorises; conversely, factorisation implies a single non-zero coefficient. □

Entanglement is a resource underpinning quintessential quantum protocols. It enables perfect teleportation of an unknown qubit state and doubles the classical capacity of a single qubit through superdense coding [14]. Experimentally, violations of Bell inequalities exclude local hidden-variable theories and confirm the non-classical character of entangled correlations [9].

Remark. Entanglement extends beyond two parties. Greenberger-Horne-Zeilinger states and graph states furnish multipartite resources for measurement-based quantum computation. Resource theories quantify entanglement via monotones such as the von Neumann entropy of the reduced state [61].

Entanglement appears implicitly in later chapters: composite state registers of QFAs may evolve into non-separable configurations even when each register individually undergoes unitary dynamics. Understanding superposition and entanglement is therefore prerequisite for analysing interference patterns and acceptance probabilities in QFAs models.

2.2.3 Measurement and Probabilistic Outcomes

Measurement translates the abstract formalism of quantum mechanics into experimentally accessible numbers. The bridge is supplied by the measurement postulates, stated here for finite-dimensional systems [118, 85].

Definition 2.2.4 (Observable [118])**.** An observable is a Hermitian operator $M \in \mathcal{L}(\mathcal{H})$ that admits the spectral decomposition

$$M = \sum_m m\,\Pi_m,$$

with orthogonal projectors $\{\Pi_m\}$ satisfying $\Pi_m \Pi_{m'} = \delta_{m,m'}\Pi_m$ and $\sum_m \Pi_m = \mathbb{I}$.

Definition 2.2.5 (Projective measurement [23])**.** Let M be an observable with eigenstates $\{|m\rangle\}$ and projectors $\Pi_m = |m\rangle\langle m|$. For an input state $|\psi\rangle$ the probability of obtaining outcome m is

$$P(m) = \langle\psi|\Pi_m|\psi\rangle = |\langle m\rangle\,\psi|^2,$$

and the post-measurement state is

$$|\psi\rangle \longmapsto \frac{\Pi_m\,|\psi\rangle}{\sqrt{P(m)}} = |m\rangle\,.$$

Projective measurements are often too restrictive: realistic detectors have finite resolution, and many protocols require non-orthogonal outcomes. The most general measurement allowed by quantum theory is the POVM, introduced next.

Definition 2.2.6 (POVM [66, 85])**.** A set of positive semi-definite operators $\{E_k\} \subset \mathcal{L}(\mathcal{H})$ is a POVM if $\sum_k E_k = \mathbb{I}$. For an input state $|\psi\rangle$ the probability of outcome k is

$$P(k) = \langle\psi|\,E_k\,|\psi\rangle\,.$$

There exist operators $\{M_k\}$, called Kraus operators, such that $E_k = M_k^\dagger M_k$ and $\sum_k M_k^\dagger M_k = \mathbb{I}$.

Proposition 2.2.2 (State-update rule [66])**.** *When outcome k of a POVM is observed, the (unnormalised) post-measurement state is $M_k\,|\psi\rangle$, and the normalised state equals $M_k\,|\psi\rangle/\sqrt{P(k)}$.*

Example 2.2.2 (Computational-basis measurement)**.** For a single qubit define $E_0 = |0\rangle\langle 0|$ and $E_1 = |1\rangle\langle 1|$. With $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, one finds

$$P(0) = |\alpha|^2, \qquad P(1) = |\beta|^2,$$

in agreement with the Born rule; the post-measurement state collapses to the eigenstate corresponding to the observed outcome.

Linearity of quantum dynamics has a profound implication for information processing.

Theorem 2.2.1 (No-cloning [120, 43])**.** *There exists no unitary operator U and no fixed blank state $|b\rangle$ such that*

$$U\big(|\psi\rangle \otimes |b\rangle\big) = |\psi\rangle \otimes |\psi\rangle$$

holds for every pure state $|\psi\rangle \in \mathcal{H}$.

Sketch. Assume such a unitary U exists and consider two non-orthogonal states $|\psi\rangle$ and $|\phi\rangle$. Because unitaries preserve inner products,

$$\langle\psi|\phi\rangle = \big(\langle\psi|\phi\rangle\big)^2,$$

which implies $\langle\psi|\phi\rangle \in \{0, 1\}$. This contradicts the premise that $|\psi\rangle$ and $|\phi\rangle$ are non-orthogonal; thus no perfect cloner exists. □

The impossibility of perfect cloning secures quantum key distribution, forbids noiseless amplification of unknown signals, and motivates approximate or probabilistic cloning strategies analysed in later chapters [108].

2.2.4 Decoherence and Open Systems

Isolated dynamics represent an idealisation; every realistic device exchanges energy and information with uncontrolled external degrees of freedom [27]. A quantum description that ignores the environment replaces pure vectors by statistical operators.

Definition 2.2.7 (Density operator [85])**.** A density operator on a Hilbert space $\mathcal{H}$ is a positive semidefinite matrix $\rho \in \mathcal{L}(\mathcal{H})$ that satisfies $\operatorname{Tr} \rho = 1$. It encodes an ensemble $\{p_i, |\psi_i\rangle\}$ through $\rho = \sum_i p_i \, |\psi_i\rangle\langle\psi_i|$.

Pure states correspond to projectors $\rho^2 = \rho$; mixed states satisfy $\rho^2 \neq \rho$.

Let $\mathcal{H}_S \otimes \mathcal{H}_E$ denote the Hilbert space of a system S and its environment E. If $|\Psi\rangle_{SE}$ evolves unitarily under U_{SE} the reduced state

$$\rho_S(t) = \operatorname{Tr}_E\big[\, U_{SE}(t)\, \rho_{SE}(0)\, U_{SE}^{\dagger}(t)\big]$$

is generally mixed and its off-diagonal elements in a preferred basis decay, a phenomenon called decoherence [130, 110].

Definition 2.2.8 (Quantum channel)**.** A linear map $\mathcal{E} : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ is a quantum channel if it is completely positive and trace preserving.

Proposition 2.2.3 (Kraus representation [66])**.** *Every completely positive and trace preserving map admits operators $\{L_k\}$ on $\mathcal{H}$ such that*

$$\mathcal{E}(\rho) = \sum_k L_k \rho L_k^{\dagger}, \qquad \sum_k L_k^{\dagger} L_k = \mathbb{I}.$$

Example 2.2.3 (Amplitude-damping channel)**.** For a qubit let $L_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}$ and $L_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$ with $0 \leqslant \gamma \leqslant 1$. The map $\mathcal{E}_\gamma(\rho) = L_0 \rho L_0^{\dagger} + L_1 \rho L_1^{\dagger}$ models spontaneous emission at rate γ. Diagonal elements remain unchanged whereas the coherence term ρ_{01} decays as $\rho_{01} \mapsto \sqrt{1-\gamma}\, \rho_{01}$, illustrating decoherence.

When environmental correlations relax quickly (Markovian limit) the family $\{\rho(t)\}_{t \geqslant 0}$ forms a quantum dynamical semigroup with generator in Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) form [54, 72]:

Definition 2.2.9 (Lindblad master equation)**.** For a system Hamiltonian H and Lindblad operators $\{L_k\}$ the Markovian evolution obeys

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H, \rho] + \sum_k \Big(L_k \rho L_k^{\dagger} - \tfrac{1}{2}\{L_k^{\dagger} L_k, \rho\} \Big).$$

The generator guarantees complete positivity of the channel $e^{t\mathcal{L}}$ for all $t \geqslant 0$ and will serve in later chapters to model noise acting on the internal registers of QFAs.

2.2.5 Unitary Evolution and Quantum Dynamics

In the absence of measurements and uncontrollable perturbations, a quantum system is considered closed. Its state vector evolves according to the postulate of unitary time development [85].

Definition 2.2.10 (Unitary operator)**.** A linear map $U : \mathcal{H} \longrightarrow \mathcal{H}$ is unitary if $U^\dagger U = UU^\dagger = \mathbb{I}$. Unitary operators form a group under composition and preserve inner products: $\langle\psi|\phi\rangle = \langle U\psi|U\phi\rangle$ for all $|\psi\rangle, |\phi\rangle \in \mathcal{H}$.

Dynamics are generated by the system Hamiltonian $H = H^\dagger$ through the time-dependent Schrödinger equation [111]

$$\mathrm{i}\hbar \frac{\mathrm{d}}{\mathrm{d}t} |\psi(t)\rangle = H(t) |\psi(t)\rangle .$$

Proposition 2.2.4 (Propagator)**.** *The formal solution is*

$$|\psi(t)\rangle = U(t, t_0) |\psi(t_0)\rangle , \qquad U(t, t_0) = \mathcal{T} \exp\Big[-\frac{\mathrm{i}}{\hbar} \int_{t_0}^{t} H(t')\, \mathrm{d}t'\Big],$$

where $\mathcal{T}$ denotes the time-ordering operator [85]. The propagator satisfies $U(t, t) = \mathbb{I}$ and the composition law $U(t_2, t_0) = U(t_2, t_1)U(t_1, t_0)$.

Remark (Time independent Hamiltonian)*.* If H is constant one has the closed expression

$$U(t, t_0) = \exp\big[-\mathrm{i}H(t - t_0)/\hbar\big],$$

which implements a one-parameter unitary group generated by H [85].

Example 2.2.4 (Qubit rotation about the z axis)**.** For a single qubit let $H = \frac{\hbar\omega}{2}\sigma_z$. The evolution operator equals

$$U(t, 0) = \exp\big[-\mathrm{i}\omega t\, \sigma_z/2\big] = \begin{pmatrix} e^{-\mathrm{i}\omega t/2} & 0 \\ 0 & e^{\mathrm{i}\omega t/2} \end{pmatrix},$$

which produces a phase rotation of the computational basis and leaves probabilities invariant.

Unitary dynamics conserve superposition amplitudes and entanglement resources. Non unitary modifications originate from coupling to external degrees of freedom or from measurements, modelled in the preceding subsections. A rigorous understanding of unitary time development is essential for analysing quantum algorithms, control protocols and the acceptance dynamics of quantum automata.

2.3 Quantum Gates and Circuits

The gate model formulates quantum computation as a sequence of reversible transformations that act on an ordered register of qubits. Each elementary transformation, or quantum logic gate, is represented by a unitary matrix that preserves the norm of the wavefunction and therefore the probabilistic interpretation of quantum states [84]. By composing gates drawn from a finite universal set such as Hadamard (H), Phase (S), and Controlled-NOT ($CNOT$), any unitary operator on a finite dimensional Hilbert space can be approximated to arbitrary accuracy, providing the foundation on which quantum algorithms like Shor's factoring routine and the QFT are constructed [119, 114].

Beyond abstract universality, physical realisations impose architectural constraints that influence gate granularity, qubit connectivity, and measurement timing. The criteria articulated by DiVincenzo identify coherent state preparation, high fidelity single and two qubit gates, and reliable read-out as indispensable requirements for scalable devices [45]. Present NISQ processors satisfy these conditions only approximately, which motivates depth-aware decompositions, noise-adaptive compilation, and the explicit accounting of ancillary resources [92].

In the context of this thesis, quantum circuits serve as the executable target for the compilation of QFAs. Each transition operator of an automaton is mapped to a concrete gate sequence, and every accepting projector is realised by a register measurement that interfaces classical control with coherent evolution. The remainder of this section surveys the primitive gate library, canonical circuit families, and decomposition techniques that together supply the hardware-agnostic substrate on which the automata-to-circuit translation of Chapter 4 is built.

2.3.1 Common Quantum Gates

Quantum logic gates are unitary operators acting on one or more qubits. They constitute the elementary instruction set of the gate model of quantum computation [85]. A single-qubit gate realises a rotation of the Bloch vector, whereas a multi-qubit gate can generate entanglement, an intrinsically non-classical resource [119]. This subsection recalls the canonical gate library, presents the corresponding matrices, and illustrates their circuit symbols .

Single-qubit gates: Pauli X, Y, Z

The Pauli gates X, Y, and Z implement rotations by π about the x, y, and z axes of the Bloch sphere, respectively [85]. In the computational basis $\{|0\rangle, |1\rangle\}$ they take the forms

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The gate X exchanges the computational basis states and therefore plays the role of a quantum *NOT* [84]. The gate Z leaves $|0\rangle$ invariant while mapping $|1\rangle$ to $-|1\rangle$; it is consequently referred to as a phase flip [84]. The gate Y combines a bit flip with an intrinsic phase: $Y|0\rangle = i|1\rangle$ and $Y|1\rangle = -i|0\rangle$ [84]. All three operators square to the identity (up to a global phase) and mutually anticommute, properties that underpin stabiliser-based error-correction protocols [55].

$$|q\rangle \; — \boxed{X} —$$

Figure 2.6: Circuit symbol for the Pauli X gate. Replacing the label by Y or Z yields the symbols for the remaining Pauli rotations.

Hadamard (H)

The H gate realises a rotation by $\frac{\pi}{2}$ about the axis $(x + z)/\sqrt{2}$ on the Bloch sphere [41]. Its unitary matrix in the computational basis is

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Acting on the basis states it produces the equal-weight superpositions $H|0\rangle = |+\rangle$ and $H|1\rangle = |-\rangle$, where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ define the Hadamard basis [85]. A second application reverses the transformation since $H^2 = I$ up to global phase. Geometrically, the operation transfers quantum states between the Z and X axes of the Bloch sphere, sending $|0\rangle$ and $|1\rangle$ to the equator and vice versa. Because it satisfies $HZH = X$ and $HXH = Z$, the gate exchanges the roles of bit-flip and phase-flip operators and is therefore indispensable for basis changes and interference patterns in algorithms such as Deutsch-Jozsa and Grover search [42, 56, 84].

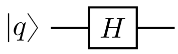


Figure 2.7: Circuit symbol for the H gate .

Phase rotations: Phase (S), T and continuous R_z

Phase rotations apply a relative phase to the state $|1\rangle$ while leaving $|0\rangle$ unchanged [55]. The S gate introduces a phase of $\pi/2$ and the T gate a phase of $\pi/4$; their unitaries are

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix},$$

so that $S|1\rangle = i|1\rangle$ and $\mathrm{T}|1\rangle = e^{i\pi/4}|1\rangle$ [84]. Because $S^2 = Z$ and $\mathrm{T}^4 = Z$, the operations may be viewed as fractional powers of the Pauli Z rotation [26].

Neither gate is involutory; instead their adjoints are

$$S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}, \qquad T^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\pi/4} \end{pmatrix}.$$

The discrete set $\{H, S, \mathrm{CNOT}\}$ generates the Clifford group, which is not universal for quantum computation, but adjoining the non-Clifford T yields a universal gate library capable of approximating any single-qubit unitary to arbitrary precision [26, 40]. In fault-tolerant architectures the cost of magic-state distillation makes the T gate the dominant resource, so circuit depth is often quantified by its T-count [47].

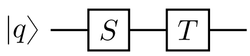


Figure 2.8: Sequential application of an S gate followed by a T gate.

More generally, a rotation about a Pauli axis $\sigma_\alpha \in \{X, Y, Z\}$ through an angle θ is

$$R_\alpha(\theta) = \exp\left(-i\tfrac{\theta}{2}\sigma_\alpha\right),$$

with explicit examples

$$R_x(\theta) = \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) & -i\sin\left(\frac{\theta}{2}\right) \\ -i\sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix}, \qquad R_z(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}.$$

The parameterised family $\{R_y(\theta), R_z(\phi)\}$ forms a convenient basis for variational circuits, and any single-qubit unitary admits the Euler decomposition $U = R_z(\lambda)\, R_y(\theta)\, R_z(\phi)$ up to global phase [85, 62]. Specialising $\theta = 0, \phi = \pi/2$ or $\pi/4$ recovers the S and T operations, respectively.

Multi-qubit gates: entangling operations

Operations acting on two or more qubits can create quantum correlations that admit no classical description [11]. Together with arbitrary single-qubit rotations they supply universality for quantum computation [119].

The $CNOT$ gate flips the target qubit conditioned on the control being in the state $|1\rangle$ [85]. In the ordered basis $\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ its unitary is

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

implementing the map $|a, b\rangle \mapsto |a, a \oplus b\rangle$. Applied to a superposition it generates entanglement, for example

$$\left(\tfrac{|0\rangle+|1\rangle}{\sqrt{2}}\right) \otimes |0\rangle \xrightarrow{CNOT} \tfrac{|00\rangle+|11\rangle}{\sqrt{2}},$$

a Bell state [11]. $CNOT$ is self-inverse and belongs to the Clifford group [55].

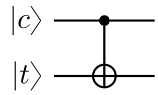


Figure 2.9: Circuit symbol for the $CNOT$ gate with control $|c\rangle$ and target $|t\rangle$.

The Controlled-Z (CZ) gate applies a phase flip to the joint state $|11\rangle$; its matrix is

$$CZ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

[85]. It is related to $CNOT$ by a Hadamard conjugation on the target, $CZ_{(c,t)} = H_t\, CNOT_{(c,t)}\, H_t$ [119], and arises natively in several hardware platforms where controlled phase interactions dominate [8].

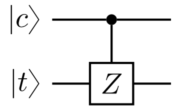


Figure 2.10: Symbol for the CZ gate .

The Swap ($SWAP$) gate exchanges the quantum states of two qubits, $|a, b\rangle \mapsto |b, a\rangle$, and is represented by the unitary matrix

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

[85]. Although $SWAP$ is a genuine two-qubit entangling gate, its action can be realised using exactly three $CNOT$s. A standard decomposition is

$$SWAP_{(1,2)} = CNOT_{1,2}\, CNOT_{2,1}\, CNOT_{1,2}$$

[119].

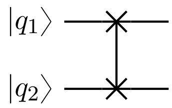


Figure 2.11: Symbol for the $SWAP$ gate .

The three-qubit Toffoli (Controlled-Controlled-X) ($CCNOT$) gate flips a target conditioned on two controls being in $|1\rangle$. It is universal for reversible classical computation [13] and admits a fault-tolerant decomposition using six $CNOT$s, seven T and single-qubit H gates [7].

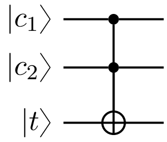


Figure 2.12: Symbol for the $CCNOT$ gate .

The Fredkin (Controlled-SWAP) ($CSWAP$) gate swaps two targets conditioned on a single control and can be synthesised from $CCNOT$s [50].

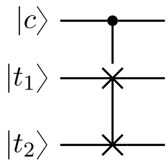


Figure 2.13: Symbol for the $CSWAP$ gate [50].

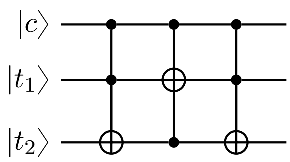


Figure 2.14: Decomposition of the $CSWAP$ gate into three $CCNOT$s gates [50].

Any multi-qubit unitary can be approximated to arbitrary precision using single-qubit rotations together with $CNOT$ or CZ gates [119]. This decomposition forms the foundation of the quantum circuit model, where arbitrary computations are expressed as sequences of elementary operations. As a result, sets such as $\{H, T, CNOT\}$ are universal for quantum computing [85]. High-level gates are routinely compiled into the R_z/R_x+$CNOT$ basis, which aligns with hardware primitives such as IBM's U_3+$CNOT$ set [39, 49].

2.3.2 Types of Quantum Circuits

Similar to their classical counterparts, quantum circuits can be naturally categorised according to their computational function and structural features. The circuit model of quantum computation supports a richer taxonomy reflecting the distinctive characteristics of quantum information, such as superposition, entanglement, and measurement-induced collapse, in contrast to the traditional classification of classical circuits into categories like combinational and sequential [85].

The representative classes of quantum circuits are surveyed in this subsection, with distinctions made based on the kind of gates employed, the presence of parameterisation, the function of measurement, and the interaction with classical control. To demonstrate the circuit structure and operational semantics of each category within the larger context of quantum algorithm design, a canonical example is given.

Single-Qubit Circuits

A circuit that operates on a single qubit evolves the state vector on the Bloch sphere through one-qubit gates only. Such circuits are the elementary building blocks of the gate-based quantum-computing model and re-appear inside larger algorithms during state preparation, basis transformations and parameterised rotations used in variational ansätze [62].

Every element of the special unitary group SU(2), up to a global phase, admits the Z-Y-Z Euler decomposition

$$U \;=\; R_z(\alpha)\, R_y(\beta)\, R_z(\gamma),$$

with real angles α, β, γ. Hence any finite gate library that realises R_z and R_y is universal for single-qubit synthesis. A commonly employed minimal choice is the set $\{H, S, T\}$, where H exchanges the computational and Hadamard eigen-bases, and the phase gates S and T supply the irrational phase that yields density on SU(2) [119, 40]. On fault-tolerant hardware the cost of exact synthesis is dominated by the number of T gates, while approximation to precision ε can be achieved with $\mathcal{O}\big(\log^c(1/\varepsilon)\big)$ gates through the Solovay-Kitaev algorithm [40].

Although a lone qubit cannot generate entanglement, single-qubit circuits are invaluable for hardware calibration, tomography and demonstrations of quantum behaviour such as phase accumulation and axis-dependent rotations [10]. The example below prepares a non-trivial state from $|0\rangle$, measures in the computational basis and routes the classical result to subsequent control logic.

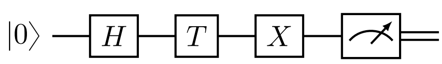


Figure 2.15: Single-qubit circuit that prepares the state $X\,T\,H\,|0\rangle$ and measures it in the computational basis; the outcome probabilities equal the squared moduli of the final amplitudes [85].

The circuit realises the transformation $|\psi\rangle \;=\; X\,T\,H\,|0\rangle$ and then performs a projective measurement in the $\{|0\rangle, |1\rangle\}$ basis; the outcome probabilities are the squared moduli of the amplitudes of $|\psi\rangle$ [85].

Multi-Qubit Circuits

The circuit model attains its full expressive power only when the logical register hosts at least two qubits. As soon as an operation couples distinct lines the global evolution can no longer be factorised into single-qubit operators, and quantum correlations enter as a computational resource. A single entangling gate such as a $CNOT$, CZ, or $SWAP$ already maps separable basis states onto superpositions whose amplitudes refuse to decompose across subsystems [119]. In the axiomatic framework of quantum theory entanglement supplies the nonclassical advantage that powers quantum algorithms, error-correcting codes, and digital simulation [61].

The canonical two-qubit routine prepares a Bell pair. Beginning from the computational basis state $|00\rangle$, a Hadamard rotation on the first line followed by a $CNOT$ controlled by that line yields

$$\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle\big),$$

one of the four maximally entangled Bell states [11]. A single controlled operation therefore suffices to convert local superposition into nonlocal correlation.

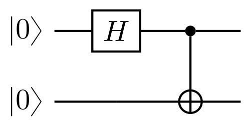


Figure 2.16: Preparation of a Bell state. A Hadamard gate acts on the first qubit and a controlled NOT entangles the register, producing the state $(|00\rangle + |11\rangle)/\sqrt{2}$.

Realistic algorithms rely on considerably deeper entangling fabrics. Quantum adders, stabiliser encoders, oracle constructions, and variational ansätze all deploy cascades of $CNOT$s (or hardware-native equivalents) distributed across many lines [55, 114, 8]. On NISQ processors the cost of those layers dominates both runtime and accumulated error because two-qubit gates are typically an order of magnitude noisier than single-qubit rotations. Consequently layout-aware compilation, topological mapping, and gate-count optimisation focus primarily on reducing the depth and connectivity overhead of the entangling subgraph.

Entanglement is therefore at once the enabler and the principal bottleneck of scalable quantum computation. The synthesis of efficient multi-qubit circuits demands a balance between algebraic decompositions that minimise entangling depth and architectural constraints that restrict admissible couplings. Ongoing research in optimal synthesis, error mitigation, and noise-adaptive layout seeks to narrow this gap and deliver practical advantages for near-term workloads [85].

Parameterised (Variational) Circuits

A Parameterised Quantum Circuit (PQC) is a quantum circuit whose elementary gates depend continuously on tunable parameters, typically rotation angles, that can be updated during classical post-processing [90]. These circuits underpin the family of hybrid quantum-classical methods known as Variational Quantum Algorithms (VQAs), as well as many quantum machine-learning architectures [33]. By treating the gate angles as free variables, a classical optimiser iteratively adjusts them so as to minimise a cost functional estimated from repeated quantum measurements [90]. The same constructs are also referred to as variational or ansatz circuits in the literature [62].

Hardware-efficient parametrisations usually consist of alternating layers of single-qubit rotations and entangling gates, chosen to match the native connectivity of a given processor [62]. This layered design strikes a compromise between expressibility and implementability, since it allows the circuit depth to scale linearly with the number of layers while maintaining a regular pattern that is friendly to compilation. An elementary two-qubit layer of such an ansatz is displayed in Figure 2.17. Each qubit undergoes a rotation about the Y axis by an angle θ_i, is entangled through a $CNOT$, and is finally rotated about the Z axis by an angle ϕ_i; the parameter set $\{\theta_i, \phi_i\}$ is updated during optimisation [90]. Stacking L identical layers enlarges the reachable manifold of states and therefore the representational power of the ansatz, albeit at the expense of an increased number of variational parameters and a higher exposure to noise [115].

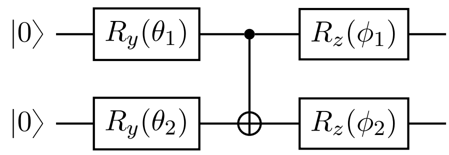


Figure 2.17: A single two-qubit layer of a hardware-efficient PQC. Local Y-rotations with angles θ_1 and θ_2 create superposition, a controlled-NOT entangles the register, and Z-rotations with angles ϕ_1 and ϕ_2 supply additional degrees of freedom.

PQCs constitute the workhorse for ground-state estimation via the Variational Quantum Eigensolver (VQE), for combinatorial optimisation in the Quantum Approximate Optimisation Algorithm, and for discriminative or generative models in quantum machine learning [48]. They exemplify the hybrid paradigm in which a quantum processor provides a differentiable, high-dimensional function while a classical computer searches the associated parameter space [33]. Research directions now focus on enhancing expressibility without incurring barren-plateau gradients, mitigating noise through tailored ansätze, and devising derivative estimation schemes that reduce measurement overhead [115].

Measurement-Based Circuits

In the measurement-based quantum computation model, also known as the one-way quantum computer, computation is driven entirely by adaptive single-qubit measurements performed on a highly entangled resource state [103]. A universal resource is the two-dimensional cluster state, although lower-dimensional graphs suffice for restricted tasks and proof-of-principle demonstrations [28]. The protocol separates state preparation from logical processing: first a cluster is created via nearest-neighbour controlled-Z gates; subsequently, a sequence of measurements, whose bases are classically adjusted in real time according to earlier outcomes, realises the desired unitary evolution up to Pauli by-product operators [104].

To illustrate the mechanism, consider a linear three-qubit cluster. Each qubit is initialised in the $|+\rangle$ eigenstate of X, entangled with its neighbour through a controlled-Z, and then the first two qubits are measured while the third remains unmeasured. Conditional feed-forward ensures that the randomness inherent in projective measurement translates into deterministic logical action on the output qubit.

Because any gate-based circuit admits translation to a measurement pattern on a suitable cluster, measurement-based quantum computation is computationally equivalent to the standard circuit model [28]. The separation between offline entanglement generation and online adaptive measurements has motivated fault-tolerant schemes compatible with photonic and modular architectures, where long-range two-qubit gates are challenging but high-fidelity single-qubit operations and measurements are readily available [104].

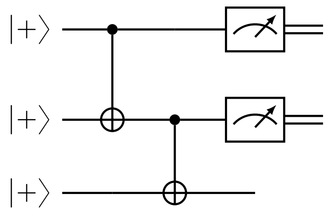


Figure 2.18: Three-qubit linear cluster used to demonstrate measurement-based quantum computation. After entangling the register with controlled-Z gates, qubits 1 and 2 are measured in suitably chosen bases. Feed-forward Pauli corrections on qubit 3 map the stochastic measurement results onto a deterministic logical operation equivalent to a single-qubit gate.

Hybrid Quantum-Classical Circuits

In the NISQ era, noise levels and qubit counts limit the depth of purely quantum workloads, motivating Hybrid Quantum-Classicals (HQCs) in which coherent evolutions alternate with classical post-processing [92]. A prototypical instance is the VQE: the quantum processor prepares a parameterised state and measures observables, while a classical optimiser updates the parameters so as to minimise the energy expectation value [33]. Real-time feedback is likewise essential in active quantum error correction, where measurement outcomes (syndrome bits) are interpreted by classical logic that decides the corrective operations to apply before further decoherence accrues [63].

HQCs exploit complementary strengths. Quantum circuitry explores Hilbert spaces that grow exponentially with qubit number, whereas classical computation affords flexible control flow, arbitrary bandwidth memory, and mature optimisation heuristics [92]. A single optimisation step consists of executing the quantum circuit, collecting measurement data to evaluate the cost function, and using a classical routine to compute new parameters for the subsequent quantum run [33]. The global algorithm is therefore a sequence of partial quantum circuits interleaved with classical kernels rather than a monolithic unitary [92]. Nonetheless, circuit diagrams can depict one iteration explicitly by inserting mid-circuit measurement symbols and classically controlled gates, making the data flow transparent [63].

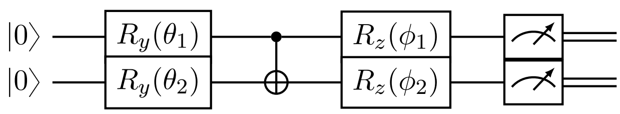


Figure 2.19: Single iteration of a VQE rendered as an HQC circuit. Parameterised rotations and entangling gates prepare the quantum state; projective measurements extract expectation values that are forwarded to a classical optimiser, which in turn updates the angles $\{\theta_i, \phi_i\}$ before the next invocation of the circuit.

Hybrid strategies generalise beyond energy minimisation. The quantum approximate optimisation algorithm tunes discrete decision variables, adaptive phase-estimation tightens eigenvalue estimates, and Bayesian feedback protocols steer quantum sensors towards the Heisenberg limit, all through iterative quantum-classical loops [33]. Current research optimises the communication latency between the two domains, integrates machine-learning controllers that predict favourable parameter updates, and develops analytic gradient estimators that reduce the number of quantum evaluations required per step. Mastering such co-designed workflows is essential for extracting useful performance from near-term processors and for paving a smooth transition towards fully fault-tolerant architectures.

2.3.3 Example Quantum Algorithms as Circuits

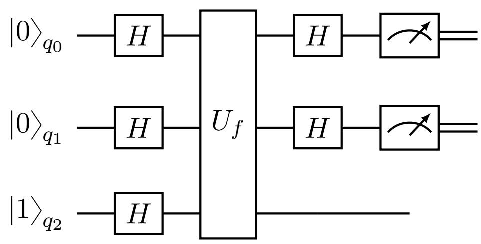
Quantum circuits provide a concrete framework in which the abstract principles of superposition, interference and entanglement manifest as algorithmic resources. By arranging single-qubit rotations and entangling two-qubit gates into ordered layers, designers can synthesise unitary transformations that would be impractical to describe or simulate classically. To illustrate this correspondence we revisit three paradigmatic algorithms routinely cited as milestones in quantum computing: the Deutsch-Josza (DJ) algorithm, Grover amplitude amplification and the Quantum Fourier Transform (QFT) [85]. The DJ algorithm exploits phase kickback to distinguish balanced from constant Boolean oracles in a single query, establishing an exponential gap in query complexity. Grover's procedure rephrases the search task as a coherent rotation toward the marked state, achieving quadratic speed-up through iterative applications of an oracle-controlled phase inversion and a diffusion operator. The QFT realises a basis change that maps computational-basis vectors to phase-encoded superpositions, serving as the central subroutine in period-finding and phase-estimation protocols. Each example exposes a distinct mechanism by which a modest set of one- and two-qubit gates outperforms the best known classical procedures [92]. The presentation below emphasises the circuit topologies, the precise ordering of gates, the accumulation and cancellation of relative phases, the interference patterns induced by measurement and the explicit scaling of gate counts with register size; these details are indispensable when later mapping higher-level computational models such as QFAs onto executable hardware circuits that must respect connectivity, coherence times and error budgets.

Deutsch-Jozsa Algorithm

The DJ algorithm furnishes one of the earliest demonstrations of an exponential separation between quantum and deterministic classical query complexity [42]. The promise problem is to decide whether a Boolean function $f : \{0,1\}^n \to \{0,1\}$ is constant or balanced, given oracle access to $U_f : |x\rangle\,|y\rangle \mapsto |x\rangle\,|y \oplus f(x)\rangle$ [37]. A deterministic classical algorithm requires $2^{\,n-1} + 1$ queries in the worst case, whereas the quantum procedure returns a definitive answer with a single evaluation [42].

The register comprises n input qubits initialised in $|0^{\otimes n}\rangle$ and one work qubit prepared in $|1\rangle$ [85]. A layer of Hadamard gates places the input into an equal superposition and the work qubit into $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ [42]. After the oracle acts, a second Hadamard layer on the input register implements an n-qubit Walsh-Hadamard transform that converts relative phases into computational-basis amplitudes [37]. Measuring the input yields $|0^{\otimes n}\rangle$ if f is constant and any other string if f is balanced, achieving certainty in a single query [42].

Because U_f leaves the work qubit in $|-\rangle$ its effect can be viewed as a relative phase $(-1)^{f(x)}$ applied to $|x\rangle$ [42]. The second Hadamard layer therefore interferes amplitudes constructively on $|0^{\otimes n}\rangle$ when all phases coincide (constant case) and destructively otherwise (balanced case), exemplifying global property extraction by quantum interference [85].



In the diagram U_f acts on all qubits; the work line q_2 returns to $|-\rangle$ independently of f and is discarded prior to measurement [85]. The algorithm thus highlights how carefully engineered phase relations allow a quantum circuit to evaluate a global promise with a single oracle invocation.

Grover's Search Algorithm

Grover's algorithm addresses the unstructured search problem by locating a single marked element among $N = 2^n$ possibilities with a quadratic query reduction relative to classical exhaustive search [56]. The procedure begins by preparing the uniform superposition $|s\rangle = H^{\otimes n}|0^{\otimes n}\rangle$, on which two reflections are alternately applied: an oracle reflection $U_f = I - 2|\omega\rangle\langle\omega|$ that flips the phase of the marked state $|\omega\rangle$ and a diffusion reflection $D = 2|s\rangle\langle s| - I$ that inverts amplitudes about the global mean [25]. The combined Grover operator $G = D\,U_f$ effects a rotation by angle 2θ in the two-dimensional subspace spanned by $|\omega\rangle$ and $|s_\perp\rangle$, where $\sin\theta = 1/\sqrt{N}$; after $r \approx \frac{\pi}{4}\sqrt{N}$ iterations the probability of observing $|\omega\rangle$ peaks at virtually unity, establishing an optimal $O(\sqrt{N})$ query complexity bound proved tight by the hybrid argument [24]. At the circuit level each iteration consists of n Hadamards, an n-qubit controlled-phase oracle and a sequence $H^{\otimes n}X^{\otimes n}Z_{0^n}X^{\otimes n}H^{\otimes n}$ realising the diffusion step with only one- and two-qubit gates using an $(n-1)$-control Z on the all-zero state [85]. Ancilla-free decompositions of the multi-control Z are feasible at $O(n)$ Toffoli-depth using standard gate libraries, while fault-tolerant implementations can trade depth for T-count by leveraging relative-phase Toffolis and measurement-based uncomputation. Resource estimation therefore shows that a full Grover search over $n = 40$ qubits, sufficient for key-search problems, would require roughly 10^6 T gates when executed on a surface-code protected architecture, orders of magnitude below the cost of competing classical brute-force methods. The canonical circuit and a worked example for $n = 2$ are given below, with $|11\rangle$ designated as the marked item.

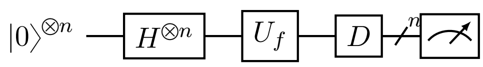


Figure 2.20: Grover iteration circuit for an n-qubit register showing the oracle reflection U_f followed by the diffusion operator D.

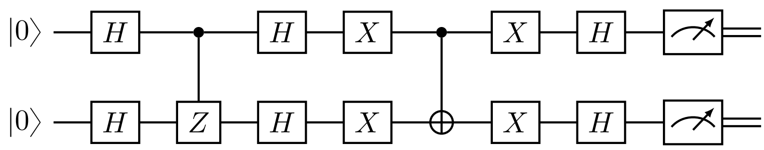


Figure 2.21: Explicit two-qubit Grover circuit with $|11\rangle$ marked, illustrating the oracle phase flip via a controlled-Z and the subsequent diffusion step.

The first Hadamards create the equal-amplitude superposition, the controlled-Z implements the oracle phase flip on $|11\rangle$, and the remaining gates perform diffusion. Repeating the entire block the prescribed number of times concentrates amplitude on the marked state, demonstrating how coherent amplitude amplification converts modest gate counts into a quadratic performance advantage over classical search.

Quantum Fourier Transform (QFT)

The (QFT) is the unitary analogue of the classical discrete Fourier transform, acting directly on the amplitudes of an n-qubit register to encode computational-basis indices into relative phases. It serves as the core subroutine in algorithms such as Shor integer factoring and quantum phase estimation, where interference in the Fourier basis extracts periodicity and eigen-phase information from a black-box unitary. Formally the transform maps

$$|j\rangle \longmapsto \frac{1}{2^{n/2}} \sum_{k=0}^{2^n-1} e^{2\pi i jk/2^n} \, |k\rangle,$$

thereby re-expressing the computational basis in a phase-weighted superposition that can be measured to reveal frequency-domain structure. An exact gate-level realisation decomposes into n layers, each beginning with a Hadamard on qubit q_ℓ followed by controlled-phase rotations $R_k = \mathrm{diag}(1, e^{2\pi i/2^k})$ from q_ℓ onto successively less-significant qubits; this yields the canonical $n(n+1)/2$ gate count that is asymptotically optimal for a fully coherent Fourier transform without ancillary qubits. In practice final qubit-order reversal is implemented by a sequence of SWAPs or absorbed into classical post-processing, while approximate variants truncate small-angle rotations to reduce depth on noisy hardware at the cost of bounded fidelity loss.

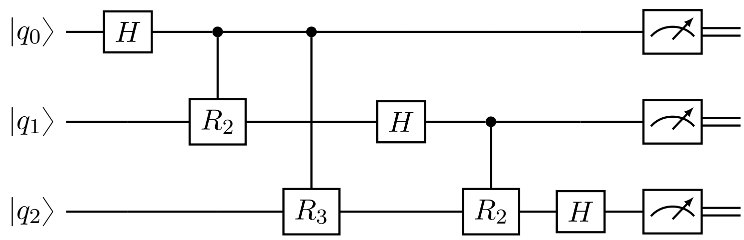


Figure 2.22: Three-qubit QFT circuit without final SWAP gates.

The exact circuit thus achieves an exponential data-size compression relative to a naïve classical implementation that would require 2^n complex multiplications, and its highly regular structure makes it an ideal benchmark for compilation strategies and fault-tolerant resource estimation in larger algorithmic pipelines that incorporate QFAs and other automata-based primitives.

2.3.4 Decomposition of Arbitrary Unitaries into Quantum Circuits

Quantum circuit synthesis asks how an arbitrary n-qubit unitary $U \in \mathrm{U}(2^n)$ can be realised exactly or approximately by a finite sequence of gates drawn from a universal library, typically the Clifford $+\,T$ set. The task is fundamental for compilation, resource estimation and verification, because every high-level quantum algorithm must eventually be expressed at this gate level. Exact synthesis is information-theoretically constrained: a generic unitary carries $4^n - 1$ real parameters, so any gate sequence that exhausts the reachable space requires $\Omega(4^n)$ elementary rotations [112].

A constructive strategy meeting this lower bound factorises U into a product of two-level unitaries that each alter amplitudes in a two-dimensional subspace spanned by computational basis states $|i\rangle$ and $|j\rangle$ [105]. Fedoriaka's elimination algorithm arranges these two-level factors so that $|i\rangle$ and $|j\rangle$ differ in exactly one qubit [49]. Under this condition every factor becomes a single-qubit rotation on the differing qubit, controlled on the remaining $n-1$ lines. The rotation is therefore a fully controlled gate $C^{n-1}\big(R_y(\theta)\big)$ or $C^{n-1}\big(R_z(\phi)\big)$ that can be further decomposed into $\mathcal{O}(n)$ $CNOT$s and single-qubit phases, or implemented directly on platforms with native multi-control support [119].

Choosing the ordering of basis states according to a binary Gray code guarantees that successive pairs differ in exactly one bit, allowing the two-level factors to march systematically across the matrix while never increasing the Hamming distance beyond one [30]. Conceptually the matrix is first permuted by a Gray-order permutation P, the product PUP^{T} is decomposed, and the permutation is later undone by relabeling or by a fixed swap network whose depth scales linearly with n. The algorithm applies $\frac{1}{2}\,2^n\,(2^n-1)$ two-level rotations, giving a gate count $\Theta(4^n)$ that saturates the optimal asymptotic bound up to constant factors [112].

Several practical optimisations reduce the constant overhead. Consecutive X controls on the same line often cancel, global phases can be deferred to a single final R_1 rotation and sparsity in U eliminates whole classes of two-level operations [49]. When U is block diagonal or tensor factorable the synthesis cost drops to polynomial in n; for example, if $U = \bigotimes_{k=1}^{n} U_k$ only n single-qubit gates are required [119].

Alternative exact techniques achieve the same asymptotic footprint while sometimes yielding circuits with more regular structure. The Cosine-Sine Decomposition (CSD) recursively partitions U into uniformly controlled rotations, leading to the quantum Shannon family of circuits that are well suited to optimisation by templates and peephole rules [74]. Householder-reflection methods similarly match the $\Theta(4^n)$ bound but distribute entangling gates differently, which can be advantageous under restricted qubit connectivity [112].

In applications where approximation is acceptable the Solovay-Kitaev theorem guarantees that any exact decomposition can be truncated and then refined to precision ε with length $\mathcal{O}\big(\log^c(1/\varepsilon)\big)$ over a fixed universal set, with $c \approx 3.97$ independent of n [40]. State-of-the-art compilers layer numerical optimisation on top of analytical backbones such as CSD. They invoke iterative KAK factorizations, constraint solvers or the ZX-calculus to compress depths further on noisy intermediate-scale quantum hardware while monitoring accumulated approximation error [58].

Exact Gray-code elimination therefore provides a conceptually transparent synthesis that is provably optimal in the worst case, yet the exponential scaling highlights why practical quantum advantage relies on structured operators such as the QFT, oracle reflections or variational ansätze whose decompositions grow only polynomially with problem size [92].

Example 1: two-qubit random unitary. Consider the numerically specified 4×4 matrix

$$U_{\text{ex}} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{pmatrix},$$

labelled in Gray order $\{|00\rangle, |01\rangle, |11\rangle, |10\rangle\}$. Fedoriaka's procedure eliminates the off-diagonal entries in three steps:

1. Apply $C^1\big(R_y(\theta_1)\big)$ on q_0 controlled by q_1 to zero the $(1, 0)$ element.
2. Apply $C^1\big(R_z(\phi_1)\big)$ on q_0 controlled by q_1 to adjust the relative phase.
3. Apply a single-qubit phase $R_z(\gamma)$ on q_1 to fix the global determinant.

A minimal gate-level implementation is illustrated in Figure 2.23. After local cancellations the circuit contains two $CNOT$s and five single-qubit rotations, underscoring that small instances can be substantially cheaper than the worst-case bound.

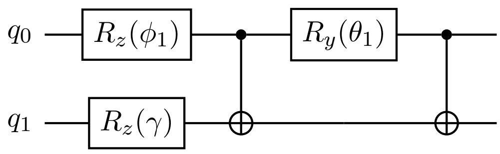


Figure 2.23: Gray-code two-qubit decomposition of U_{ex}. The fully controlled rotations emerge as $CNOT$ sandwiches with single-qubit phases.

Example 2: block-diagonal three-qubit unitary. Let

$$U_{\text{blk}} = \begin{pmatrix} U_0 & 0 \\ 0 & U_1 \end{pmatrix}, \qquad U_0 = R_y(\alpha),\ U_1 = R_z(\beta),$$

where the most significant qubit indexes the blocks. The matrix is already sparse; only two fully controlled single-qubit rotations are needed:

$$C^2\big(R_y(\alpha)\big) \quad \text{and} \quad C^2\big(R_z(\beta)\big).$$

Each decomposes into six $CNOT$s and four elementary rotations in the Clifford $+\,T$ set. The total depth therefore scales linearly with n instead of exponentially, demonstrating how exploitable structure in U suppresses the gate count far below the Gray-code worst case.

Example 3: tensor-product unitary. If $U = U_1 \otimes U_2 \otimes \cdots \otimes U_n$ the circuit is simply the parallel application of the n single-qubit matrices U_k. No entangling gates are required, so the synthesis cost is exactly n elementary rotations. This instance saturates the best possible scaling for a non-trivial n-qubit operator, confirming that exponential overhead is not intrinsic but rather tied to entangling structure.

These examples translate the abstract complexity analysis into concrete gate sequences, supplying reference circuits and resource counts that can be validated experimentally or within automated compilers.

3. Quantum Finite Automata

The study of QFAs arose from the question of how far the finite-memory paradigm of CFAs extends when the transition map is replaced by symbol-controlled unitary operators followed by quantum measurement. Since the seminal formulations of measure-once and measure-many one-way machines [65, 79], this minimalist model has revealed capabilities that distinguish it sharply from its deterministic counterpart; examples include exponential state savings in promise-problem recognition [4] and interactive proof power that eludes classical finite verifiers of identical size [86].

This chapter offers a concise taxonomy that tracks the historical broadening of the model while keeping a uniform notation. We begin with the foundational one-way variants, then investigate how two-way head motion, hybrid quantum-classical control or bounded counters enlarge the language classes recognised, occasionally reaching beyond the regular languages [29, 5, 128]. The survey closes with less common frameworks, such as postselection, promise settings and ω-word machines, that continue to challenge classical intuitions [1, 109, 18].

Throughout the discussion each device is specified as a fixed-size tuple whose components highlight the role of the quantum state space, the observable structure and the head movement policy. Key results on expressive power, closure properties and decidability are presented, culminating in a comparative table that juxtaposes state complexity bounds and closure behaviour across all surveyed models. This structured overview equips the reader with the precise vocabulary and conceptual tools required for the later translation of QFAs into executable quantum circuits developed in Chapter 4.

3.1 Detailed Models of Quantum Finite Automata

Among the numerous variants of QFAs, the measure-once and measure-many models stand out as two of the most foundational and widely studied frameworks. Respectively known as MO-1QFA and MM-1QFA, these automata provide a minimalistic yet powerful theoretical playground for investigating the principles of quantum computation with finite memory. Their significance lies not only in their historical development as some of the earliest quantum models for language recognition [79, 65], but also in their role as archetypal examples in the study of quantum-classical computational boundaries.

A MO-1QFA performs unitary operations throughout the input reading process and conducts a single measurement only at the end of the computation. This model, introduced by Moore and Crutchfield [79], can recognize only a restricted class of REGs, such as the so-called group languages [29]. In contrast, the MM-1QFA, introduced by Kondacs and Watrous [65], allows measurements after each symbol is processed, enabling it to recognize a strictly larger class of REGs, though still not the full class.

The primary importance of MO-1QFA and MM-1QFA is not just theoretical. These models are particularly suitable for demonstrating techniques for the compilation of quantum automata into quantum circuits, as we will explore in Chapter 4. Due to their simpler architecture—one-way movement, discrete time steps, and finite-dimensional Hilbert spaces—these automata provide an ideal framework for illustrating how abstract automaton transitions can be implemented using quantum gates and projective measurements. In this thesis, they will serve as canonical models to illustrate the compilation strategy, setting a baseline for comparison with more advanced or generalized QFA variants.

3.1.1 Measure Once One-Way Quantum Finite Automaton

Introduction

MO-1QFAs represent one of the simplest models of quantum computation in the realm of automata theory. Introduced by Moore and Crutchfield in 2000 [79], MO-1QFAs evolve solely through unitary transformations corresponding to the input symbols and perform a single measurement at the end of the computation. This model has been further characterized by Brodsky and Pippenger [29], and it is known for its conceptual simplicity as well as for its limitations. Notably, when restricted to bounded error, MO-1QFAs recognize exactly the class of group languages—a proper subset of the REGs. In contrast, the measure-many variant [65] employs intermediate measurements and exhibits different acceptance capabilities.

Formal Definition

An MO-1QFA is formally defined as a 5-tuple

$$M = (Q, \Sigma, \delta, q_0, F),$$

where:

- Q is a finite set of states,
- Σ is a finite input alphabet, typically augmented with a designated end-marker (e.g., \$),
- $\delta : Q \times \Sigma \times Q \to \mathbb{C}$ is the transition function such that, for every symbol $\sigma \in \Sigma$, the matrix
$$U_\sigma, \quad \text{with} \quad (U_\sigma)_{q,q'} = \delta(q, \sigma, q'),$$
is unitary [79],
- $q_0 \in Q$ is the initial state, and
- $F \subseteq Q$ is the set of accepting states.

For an input string $x = x_1 x_2 \cdots x_n$, the computation proceeds by applying the corresponding unitary matrices sequentially:

$$|\Psi_x\rangle = U_{x_n} U_{x_{n-1}} \cdots U_{x_1} |q_0\rangle .$$

After reading the entire input, a measurement is performed using the projection operator

$$P = \sum_{q \in F} |q\rangle\langle q| ,$$

so that the acceptance probability is defined as

$$p_M(x) = \|P\,|\Psi_x\rangle\|^2.$$

Alternative characterizations, including formulations using the Heisenberg picture, have been discussed in [95, 91].

Strings Acceptance

A string x is accepted by an MO-1QFA if the acceptance probability $p_M(x)$ exceeds a predetermined cut-point λ. In a bounded error setting, there exists a margin $\epsilon > 0$ such that:

$$\begin{aligned} \forall x \in L: &\quad p_M(x) \geqslant \lambda + \epsilon, \\ \forall x \notin L: &\quad p_M(x) \leqslant \lambda - \epsilon. \end{aligned}$$

Under the unbounded error regime, MO-1QFAs can accept some nonREGs (for instance, solving the word problem over the free group) [29]. The precise acceptance behavior thus depends on whether a cut-point or a bounded error framework is adopted.

Set of Languages Accepted

When MO-1QFAs are restricted to bounded error acceptance, they recognize exactly the class of *group languages*—languages whose syntactic semigroups form groups [29]. This class forms a strict subset of the REGs, emphasizing the inherent limitation of the MO-1QFA model. In contrast, by relaxing the error bounds, one may design MO-1QFAs that accept a broader range of languages, albeit often at the cost of increased computational complexity.

Closure Properties

The class of languages accepted by MO-1QFAs under bounded error exhibits robust closure properties. Specifically, this class is closed under:

- Inverse Homomorphisms [29],
- Word Quotients [29],
- Boolean Operations (union, intersection, and complement) [51, 15].

Additional algebraic properties, including aspects related to the pumping lemma and the structure of the accepting probabilities, have been further elaborated in works such as Ambainis et al. [6] and Xi et al. [121].

Summary of Advantages and Limitations

MO-1QFAs are praised for their simplicity. Since the quantum state evolves unitarily until a single measurement is made, the model avoids the complications associated with intermediate state collapses. This simplicity has practical implications; for example, recent experimental work has shown that custom control pulses can significantly reduce error rates in IBM-Q implementations [73], and photonic implementations have further demonstrated the feasibility of MO-1QFAs in optical setups [31]. On the downside, the acceptance power of MO-1QFAs is limited—when operating under bounded error, they recognize only the group languages, which form a strict subset of the REGs. In contrast, MM-1QFAs offer greater acceptance power but at the cost of increased model complexity [65, 17].

Example

Consider a simple MO-1QFA defined over the unary alphabet $\Sigma = \{a\}$ with state set $Q = \{q_0, q_1\}$, initial state q_0, and accepting state set $F = \{q_1\}$. Let the unitary operator corresponding to the symbol a be defined by the rotation matrix:

$$U_a = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

for a fixed angle θ. For an input string a^n, the state evolves as:

$$|\Psi_{a^n}\rangle = U_a^n |q_0\rangle.$$

The acceptance probability is computed as:

$$p_M(a^n) = \|P\,|\Psi_{a^n}\rangle\|^2,$$

where the projection operator P is given by $P = |q_1\rangle\langle q_1|$. By appropriately choosing θ, the automaton can be tuned so that $p_M(a^n)$ exceeds the cut-point λ (for example, $\lambda = \frac{1}{2}$) if and only if a^n belongs to the target language. This example illustrates the essential mechanism of MO-1QFAs, as described in [79, 29].

Additional Topics

Learning and Optimization: Recent work by Chu et al. [35] has introduced methods that combine active learning with non-linear optimization to approximately learn the parameters of MO-1QFAs. These techniques provide insights into how one can recover the unitary transformations and state structure from observed data.

Complexity and Minimization: The problem of minimizing the number of states in an MO-1QFA was originally posed by Moore and Crutchfield [79]. Subsequent work by Mateus, Qiu, and Li [75] has established an EXPSPACE upper bound for the minimization problem, framing it as a challenge in solving systems of algebraic polynomial (in)equations.

Experimental Implementations: Experimental realizations of MO-1QFAs have also been explored. Lussi et al. [73] demonstrated an implementation on IBM-Q devices using custom control pulses, while photonic approaches have been reported in [31]. These works highlight both the practical challenges and the potential advantages of implementing MO-1QFAs on current quantum hardware.

Future Directions: Future research may focus on further enhancing experimental implementations, developing more robust learning algorithms for MO-1QFAs, and exploring new minimization techniques that could lead to more efficient automata. Extensions that combine features of MO-1QFAs and MM-1QFAs may also provide richer language recognition capabilities and deepen our understanding of quantum computational models.

3.1.2 Measure Many One-Way Quantum Finite Automaton

Introduction

MM-1QFAs are a variant of quantum finite automata in which a measurement is performed after reading each input symbol. Introduced by Kondacs and Watrous in 1997 [65], MM-1QFAs allow the automaton to collapse its quantum state at intermediate steps, thereby potentially influencing the computation dynamically. Although this mechanism can enhance the detection of accepting or rejecting conditions during the run, under the bounded error regime MM-1QFAs are known to recognize only a proper subset of the REGs [29]. Recent work, such as by Lin [71], has provided elegant methods for addressing the equivalence problem of MM-1QFAs, further enriching our understanding of their computational properties.

Formal Definition

A Measure Many One-Way Quantum Finite Automaton (MM-1QFA) is defined as a 6-tuple

$$M = (Q, \Sigma, \delta, q_0, Q_{acc}, Q_{rej}),$$

where:

- Q is a finite set of states,
- Σ is a finite input alphabet, typically augmented with an end-marker (e.g., \$),
- $\delta : Q \times \Sigma \times Q \to \mathbb{C}$ is the transition function, where for each symbol $\sigma \in \Sigma$ the corresponding matrix

  $$U_\sigma, \quad \text{with} \quad (U_\sigma)_{q,q'} = \delta(q, \sigma, q'),$$

  is unitary [65],
- $q_0 \in Q$ is the initial state,
- $Q_{acc} \subseteq Q$ is the set of accepting (halting) states, and
- $Q_{rej} \subseteq Q$ is the set of rejecting (halting) states.

After each symbol is read, the automaton's current state is measured with respect to the decomposition

$$E_{acc} = \text{span}\{|q\rangle : q \in Q_{acc}\}, \quad E_{rej} = \text{span}\{|q\rangle : q \in Q_{rej}\}, \quad E_{non} = \text{span}\{|q\rangle : q \in Q \backslash (Q_{acc} \cup Q_{rej})\}.$$

If the measurement outcome lies in E_{acc} or E_{rej}, the computation halts immediately with acceptance or rejection, respectively. This definition, adapted from Kondacs and Watrous [65] and refined in Lin [71], forms the basis of the MM-1QFA model.

Strings Acceptance

For an input string $x = x_1 x_2 \cdots x_n$, the MM-1QFA processes each symbol sequentially. At each step i, the unitary operator U_{x_i} is applied, followed by a measurement:

- If the measurement result falls in E_{acc}, the automaton immediately accepts x.

- If it falls in E_{rej}, the automaton rejects x.
- If the result lies in E_{non}, the computation continues with the next symbol.

The overall acceptance probability of x is the cumulative probability of all computation paths that eventually lead to an accepting state. In a bounded error framework, there exists a margin $\epsilon > 0$ such that for every $x \in L$, the acceptance probability satisfies

$$p_M(x) \geqslant \lambda + \epsilon,$$

and for every $x \notin L$,

$$p_M(x) \leqslant \lambda - \epsilon,$$

where λ is a predetermined cut-point (commonly set to $\frac{1}{2}$) [65, 29].

Set of Languages Accepted

Under the bounded error constraint, MM-1QFAs recognize a proper subset of the REGs. In particular, the languages accepted by MM-1QFAs must satisfy specific algebraic properties that restrict their expressive power. Although MM-1QFAs can, in some cases, recognize nonREGs when allowed unbounded error, the bounded error condition confines them to a class that is comparable to that of group languages [29, 65]. This limitation underscores the trade-off between the increased measurement frequency and the resultant reduction in language recognition capability.

Closure Properties

The language class recognized by MM-1QFAs with bounded error is known to enjoy several closure properties:

- It is closed under complement and inverse homomorphisms [29].
- It is closed under word quotients [29].
- However, the class is not closed under arbitrary homomorphisms [65, 15].

Recent work by Lin [71] further refines our understanding of these closure properties by addressing the equivalence problem for MM-1QFAs, thereby linking the structural properties of the recognized languages to the underlying automata.

Summary of Advantages and Limitations

MM-1QFAs offer notable advantages:

- The use of intermediate measurements can enable earlier detection of acceptance or rejection, potentially reducing the average computation time.
- The dynamic collapse of the quantum state provides a different balance between quantum coherence and classical decision-making.

Nevertheless, there are significant limitations:

- The frequent measurements interrupt the quantum evolution, which can limit the automaton's ability to harness quantum interference effectively.

- As a result, under bounded error conditions, MM-1QFAs recognize only a restricted subset of the REGs.
- The complexity of analyzing and minimizing MM-1QFAs remains high, with state minimization posing an EXPSPACE challenge [75] and lower bound results highlighting the inherent state complexity [2].

Moreover, when compared to MO-1QFAs, MM-1QFAs may offer greater recognition power in some unbounded error scenarios but at the cost of increased computational and implementation complexity [65, 17].

Example

Consider an MM-1QFA defined over the alphabet $\Sigma = \{a\}$ with the state set $Q = \{q_0, q_1, q_2\}$, where q_0 is the initial state, $Q_{acc} = \{q_2\}$, and $Q_{rej} = \{q_1\}$. Let the unitary operator for the symbol a be given by:

$$U_a = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The MM-1QFA processes an input string such as aa as follows:

1. Starting in state $|q\rangle$, the operator U_a is applied and a measurement is performed. The measurement may collapse the state into:
   - E_{acc} (state q_2) - leading to immediate acceptance,
   - E_{rej} (state q_1) - leading to immediate rejection, or
   - E_{non} (state q_0, in this example) - allowing the computation to continupe.
2. If the first measurement yields a non-halting result, the second symbol is processed in a similar manner. The overall acceptance probability is the sum of the probabilities of all computation paths that eventually result in an outcome within E_{acc}.

This example demonstrates the stepwise measurement process that characterizes MM-1QFAs [65, 71].

Additional Topics

Equivalence and Decision Problems: Lin [71] presents a simplified approach for deciding the equivalence of two MM-1QFAs by reducing the problem to comparing initial vectors, thereby streamlining the decision process.

State Complexity and Lower Bounds: Lower bound results for One-Way Quantum Finite Automaton (1QFA), such as those by Ablayev and Gainutdinova [2], provide insights into the inherent state complexity challenges that also impact MM-1QFAs.

Experimental Considerations: While experimental implementations have predominantly focused on MO-1QFAs due to their relative simplicity, future work may explore the adaptation of techniques (e.g., custom pulse shaping as demonstrated in [73]) to the more complex MM-1QFA framework.

3.2 Main Models of Quantum Finite Automata

3.2.1 One-Way Quantum Finite Automaton

1QFA are the quantum analog of classical one-way finite automata where the input tape is read from left to right without revisiting symbols. They provide a model for finite quantum computation that is more restricted than two-way quantum finite automata but often simpler to implement and analyze.

Among the most studied variants of 1QFA are the MO-1QFA and the MM-1QFA, which are addressed in detail in Section 3.1. These automata differ primarily in the timing of their measurements: MO-1QFA perform a single measurement at the end of the computation, while MM-1QFA perform measurements after reading each input symbol.

Beyond these foundational models, several other types of 1QFA have been proposed, each offering unique computational perspectives or enhancements. We present a comprehensive overview of such models, each in its own subsubsection.

Measure-Only One-Way Quantum Finite Automaton (MON-1QFA)

The MON-1QFA is a special model in which only measurement operations are used for computation, with no intermediate unitary evolutions. This model simplifies quantum computation by relying solely on projective measurements.

Formal Definition A MON-1QFA is defined as a tuple $A = (Q, \Sigma, \rho_0, \{P_\sigma\}_{\sigma\in\Sigma}, Q_{acc})$ where:

- Q is a finite set of states,
- Σ is a finite input alphabet,
- ρ_0 is the initial quantum state (density matrix),
- P_σ is the measurement operator for each input symbol σ,
- $Q_{acc} \subseteq Q$ is the set of accepting states.

Strings Acceptance Acceptance is determined by applying the appropriate projective measurement after each symbol and measuring the final state. Acceptance can be defined with a bounded-error threshold.

Sets of Languages Accepted The class of languages accepted by MON-1QFA is strictly less powerful than the class of REGs. It corresponds to a particular class of REGs known as literally idempotent piecewise testable languages [16].

Closure Properties The languages recognized by MON-1QFA are not closed under union or complementation [16].

Advantages and Limitations They offer a hardware-friendly model due to the absence of unitaries, but are strictly less powerful than MO-1QFA and MM-1QFA.

Comparison Compared to MO-1QFA and MM-1QFA, MON-1QFA are less powerful due to the absence of unitary evolution.

Example An example is the language of all strings over {a, b} with an even number of a's. It can be recognized by a suitable choice of projective measurements.

Additional Topics Measure-only models relate to trace monoids with idempotent generators and have been used in language algebraic characterizations [38, 16].

One-Way Quantum Finite Automata with Two Observables (1QFA(2))

Introduction The One-Way Quantum Finite Automaton with Two Observables (1QFA(2)) model introduces a second observable to the MO-1QFA framework, enhancing the capacity to distinguish input strings. It can be seen as an intermediate enhancement over classical MO-1QFA.

Formal Definition A 1QFA(2) is defined similarly to a standard MO-1QFA, but it includes two projective measurements applied in alternation during computation:

$$A = (Q, \Sigma, \rho_0, \{U_\sigma\}, \{P_1, P_2\}, Q_{acc})$$

where the two measurements P_1 and P_2 alternate throughout the computation.

Strings Acceptance A string is accepted based on the final outcome after alternating between the two measurements. Acceptance is typically defined with bounded error.

Sets of Languages Accepted The class of languages recognized is still a proper subset of REGs, although strictly more than MO-1QFA.

Closure Properties These automata do not have known closure under union or intersection.

Advantages and Limitations The addition of a second observable enables more refined discrimination of inputs, albeit still with limited computational power.

Comparison The model is more expressive than MO-1QFA but remains less powerful than general MM-1QFA.

Additional Topics This line of work aligns with the broader research aim of incrementally extending the expressive power of 1QFA models, as seen in the approach of [36].

Two-Tape One-Way Quantum Finite Automata with Two Heads (2tQFA(2))

Introduction The Two-Tape One-Way Quantum Finite Automaton with Two Heads (2T1QFA(2)) model incorporates two tapes and two heads, enabling cross-comparison between symbols of the input and reference tape, enhancing the recognition of complex patterns.

Formal Definition Formally, a 2T1QFA(2) is a 7-tuple $A = (Q, \Sigma, \delta, q_0, Q_{acc}, Q_{rej}, \mathbb{T})$, where $\mathbb{T}$ denotes the tape set. The quantum evolution occurs in tandem over both tapes with corresponding heads.

Strings Acceptance Acceptance is determined through measurement at halting, typically with bounded error.

Sets of Languages Accepted 2T1QFA(2) can recognize some nonREGs, exceeding the capabilities of classical one-way automata and standard 1QFA models [53].

Closure Properties Closure properties remain largely unexplored, with no known results under union or complement.

Advantages and Limitations The addition of a second tape expands the computational power substantially, though it also introduces practical complexity.

Comparison Outperforms classical and most one-way quantum models in expressive power.

Additional Topics This model exemplifies a practical direction in extending 1QFA expressivity by architectural enhancement.

Nondeterministic Quantum Finite Automata (NQFA)

Introduction Non-Deterministic Quantum Finite Automaton (NQFA) introduce nondeterminism in the quantum setting. Unlike probabilistic nondeterminism, here the nondeterminism arises from quantum measurement outcomes and amplitude branches.

Formal Definition An NQFA is a 5-tuple $A = (Q, \Sigma, \psi_0, \{U_\sigma\}, Q_{acc})$, and uses a cutpoint acceptance mode, where a string x is accepted if $\mathbb{P}(x) > \lambda$ for some threshold λ.

Strings Acceptance NQFA recognize strings based on acceptance with nonzero amplitude, allowing acceptance of nonREGs with bounded error [123].

Sets of Languages Accepted This model strictly recognizes more than REGs, offering power comparable to or exceeding classical nondeterministic finite automata.

Closure Properties NQFA are not closed under complement or union.

Advantages and Limitations The model is powerful yet lacks constructive methods for deterministic acceptance, making some verification tasks harder.

Comparison More expressive than MO-1QFA, MM-1QFA, and even probabilistic finite automata in some cases.

Reversible Quantum Finite Automata (RevQFA)

Introduction Reversible One-Way Quantum Finite Automaton (RevQFA) enforce reversibility in state transitions, in line with quantum mechanics principles, where computation steps are invertible.

Formal Definition Defined similarly to MO-1QFA, but all transitions are reversible, and state evolution is enforced to be unitary across all paths. Measurement occurs only at the end.

Strings Acceptance A string is accepted based on the final state post unitary evolution.

Sets of Languages Accepted Recognizes all REGs [124], differing from many other 1QFA models.

Closure Properties Closed under Boolean operations due to equivalence with classical deterministic automata.

Advantages and Limitations They enjoy a strong correspondence to classical reversible automata with quantum efficiency benefits, though do not surpass REGs.

Comparison Unlike most 1QFA, RevQFA can simulate any DFA, thus closing the gap between quantum and classical finite automata in expressiveness.

Additional Topics The interest in RevQFA stems from the desire to harness quantum reversibility for computation, a core direction explored by [36].

3.2.2 Two-way Quantum Finite Automata

QFAs can be classified based on the measurement policy and head movement direction. In this subsection, we focus exclusively on Two-Way Quantum Finite Automata (2QFAs) models that operate under pure quantum evolution: namely, the Measure Once Two-Way Quantum Finite Automaton (MO-2QFA) and Measure Many Two-Way Quantum Finite Automaton (MM-2QFA). These automata extend the capabilities of their one-way counterparts by allowing bidirectional movement of the tape head, while differing in how often measurements are performed during computation.

Measure-Once Two-Way Quantum Finite Automata (MO-2QFA)

The MO-2QFA was formally introduced by Xi et al. [121] as the natural two-way extension of the (MO-1QFA). It performs unitary transformations while scanning the input in both directions but conducts a projective measurement only once—at the end of the input.

Formal Definition. An MO-2QFA is defined as a 5-tuple $M = (Q, \Sigma, \delta, q_0, Q_a)$ where:

- Q is a finite set of quantum states,
- Σ is a finite input alphabet,
- $\delta : Q \times \Gamma \times Q \times D \to \mathbb{C}$ is the transition function with $\Gamma = \Sigma \cup \{\#, \$\}$ and $D = \{-1, 0, +1\}$ indicating tape head movement,
- $q_0 \in Q$ is the initial state,
- $Q_a \subseteq Q$ is the set of accepting states.

The unitary evolution is enforced through conditions of orthogonality and separability as outlined in [121].

Strings Acceptance. Acceptance is defined via a single projective measurement after the entire input string, delimited by endmarkers, is processed. A string is accepted with a probability computed from the projection onto the accepting subspace. The model supports acceptance with bounded error, cutpoint, and exact acceptance depending on configuration [121].

Set of Languages Recognised. The class of languages recognized by MO-2QFA strictly contains the languages recognized by MO-1QFA, and it includes some nonREGs. In fact, MO-2QFA can recognize proper supersets of group languages and supports complex operations such as intersection and reversal [121].

Closure Properties. The languages recognized by MO-2QFA are closed under union, intersection, complement, and reversal. Notably, the closure under intersection and union can be achieved by direct sum and tensor product constructions, respectively [121].

Advantages and Limitations. The main advantage of MO-2QFA lies in its improved recognition power over one-way models. However, the measurement at the end implies potentially high computational complexity for simulating classical behavior. Its limitation includes the reliance on exact unitary constraints and nontrivial implementation challenges [121].

Comparison. Compared to MO-1QFA, the MO-2QFA is strictly more powerful due to its bidirectional scanning ability. It is less expressive than MM-2QFA in general, since the latter allows more frequent measurements and thus a richer computational structure.

Example. An example from [121] shows that MO-2QFA can recognize the language $L = \{a^n b^n \mid n \geqslant 1\}$ with bounded error—something not possible with any one-way QFA model.

Additional Topics. The authors suggest future directions include studying the closure under concatenation in more general terms and optimizing state complexity for specific regular operations [121].

Measure-Many Two-Way Quantum Finite Automata (MM-2QFA)

Introduction. The MM-2QFA was introduced by Kondacs and Watrous [65] and represents the most powerful pure QFA model in terms of language recognition. It performs a measurement at each computational step and allows head movement in both directions.

Formal Definition. An MM-2QFA is a 6-tuple $M = (Q, \Sigma, \delta, q_0, Q_a, Q_r)$ where:

- Q is the finite set of states partitioned into Q_n (non-halting), Q_a (accepting), and Q_r (rejecting) states,
- $\delta : Q \times \Gamma \times Q \times D \to \mathbb{C}$ is the transition function,
- The machine performs a projective measurement at each step to determine whether to accept, reject, or continue.

Well-formedness constraints are necessary to ensure unitarity and validity of transition amplitudes [65].

Strings Acceptance. A string is accepted if, during any computational step, the machine transitions into an accepting state. This supports acceptance with bounded error, cutpoint, and exact modes depending on the configuration of the measurement [65].

Set of Languages Recognised. The MM-2QFA can recognize nonREGs such as $L_{eq} = \{a^n b^n \mid n \geqslant 0\}$ with bounded error in linear time. This is a significant enhancement over any classical 2PFA or 1QFA model [65].

Closure Properties. The class of languages recognized by MM-2QFA is not known to be closed under union or intersection. This is a key limitation of the model despite its high expressive power.

Advantages and Limitations. The model can recognize nonREGs efficiently, offering exponential advantages in space over classical automata. However, it is more complex to analyze due to frequent measurements and lacks closure under basic operations [65, 94].

Comparison. MM-2QFA is strictly more powerful than all one-way models (MO-1QFA, MM-1QFA) and more powerful than MO-2QFA in general. However, its structure is harder to simulate and analyze, limiting its practical implementation.

Example. Kondacs and Watrous [65] present a MM-2QFA that recognizes L_{eq} in linear time with bounded error—a feat requiring exponential time for classical two-way probabilistic automata.

Additional Topics. Extensions to hybrid models such as Two-Way Quantum Finite Automata with Classical States (2QCFAs) and Quantum Interactive Proof (QIP) with 2QFA verifiers have been proposed to harness the strengths of MM-2QFA while addressing its limitations [88, 94].

3.2.3 Hybrid Quantum Finite Automata

Hybrid quantum finite automata (HQFA) are finite-state machines that combine a quantum state component with a classical state component. In essence, an HQFA consists of a quantum system and a classical finite automaton that operate together, communicating information between them during the computation [68].

In all these models, the goal is to leverage a small quantum memory together with classical states to recognize languages, potentially with far fewer states than a purely classical automaton would require [128].

Several important hybrid QFA models have been introduced. Ambainis and Watrous [5] first proposed the two-way quantum finite automaton with classical states (2QCFA), which augments a two-way deterministic finite automaton with a constant-size quantum register.

More advanced hybrids include multi-tape extensions like the Two-Tape Quantum Finite Automaton with Classical States (2TQCFA) and k-Tape Quantum Finite Automaton with Classical States (kTQCFA), which increase computational power by leveraging multiple input tapes [126]. All one-way hybrid QFA models (One-Way Quantum Finite Automaton with Classical States (1QCFA), 1QFAC, One-Way Quantum Finite Automaton with Control Language (CL-1QFA)) recognize exactly the REGs [68, 128]. However, they can be significantly more *succinct* in terms of state complexity than classical models [122].

One-Way Quantum Finite Automaton with Classical States (1QCFA)

The 1QCFA model, introduced by Zheng, Qiu, Li, and Gruska [128], augments a classical one-way finite automaton with a quantum component. It can be seen as a one-way restriction of the 2QCFA model. The 1QCFA features two-way communication between classical and quantum parts: the classical state influences the quantum operation applied, and the quantum measurement outcome affects the next classical state.

Formal Definition: A 1QCFA is a 9-tuple:

$$A = (Q, S, \Sigma, C, q_1, s_1, \{\Theta_{s,\sigma}\}, \delta, S_a)$$

where:

- Q: finite set of quantum basis states,
- S: finite set of classical states,
- Σ: input alphabet,
- C: measurement outcomes,
- $q_1 \in Q$, $s_1 \in S$: initial quantum and classical states,
- $\Theta_{s,\sigma}$: quantum operation with outcomes in C,
- $\delta : S \times \Sigma \times C \to S$: classical transition function,
- $S_a \subseteq S$: accepting states.

Computation begins in $(s_1, |q_n\rangle)$, proceeds symbol-by-symbol. On input $x = x_1x_2\cdots x_n$, the machine updates classical and quantum states based on outcomes $c_i \in C$ generated by each Θ_{s_i,x_i}, applying $\delta(s_i, x_i, c_i)$ at every step. Acceptance is determined by whether the final classical state is in S_a [68].

Strings Acceptance: A 1QCFA accepts string x with probability based on outcome paths $c_1c_2\cdots c_n$. If the classical state ends in S_a, it accepts. Languages are recognized with bounded error $\varepsilon < 1/2$ if acceptance probability is $\geqslant 1 - \varepsilon$ for all $x \in L$ and $\leqslant \varepsilon$ for all $x \notin L$ [68, 128].

Sets of Languages Recognized: 1QCFA recognize exactly the class of REGs [128]. Any REG can be recognized by some 1QCFA with certainty. Moreover, they can be exponentially more succinct than DFA for some REGs [122].

Closure Properties: The class of languages recognized by 1QCFA is closed under union, intersection, and complement, as it coincides with the REGs [68].

Advantages and Limitations: While 1QCFA do not exceed DFA in language power, they are often exponentially more state-efficient. For instance, certain periodic languages can be recognized with a single qubit: rotating the quantum state by $2\pi/p$ for each a and measuring at the end to detect if the state returned to its initial position (full rotation) [122, 21].

Comparison Between Models: 1QCFA generalize CL-1QFA and 1QFAC. The main difference lies in bidirectional communication. In comparison to 2QCFA, 1QCFA are weaker, since they cannot move backward on the input or recognize non-REGs [68].

Example: A 1QCFA can recognize the language $L = \{a^n : n \equiv 0 \mod p\}$ using a single qubit: rotating the quantum state by $2\pi/p$ for each a and measuring at the end to detect if the state returned to its initial position (full rotation) [20].

Additional Topics: 1QCFA equivalence is decidable [68]. State trade-offs between classical and quantum resources have been studied extensively [99, 122]. Future work includes refining state complexity bounds and exploring minimal configurations.

One-Way Quantum Finite Automaton with Control Language (CL-1QFA)

Introduction: The CL-1QFA (Quantum Finite Automata with Control Language) model, introduced by Bertoni, Mereghetti, and Palano [77], consists of a quantum component responsible for unitary operations and measurements, and a classical DFA that processes the sequence of measurement outcomes. The role of the control language is to guide acceptance: a word is accepted if and only if the sequence of measurement outcomes belongs to a REG defined by the control automaton.

Formal Definition: A CL-1QFA is a 6-tuple:

$$A = (Q, \Sigma, \{U_\sigma\}_{\sigma\in\Sigma}, q_0, M, L)$$

where:

- Q: finite set of quantum basis states,
- Σ: input alphabet,
- U_σ: unitary transformation applied upon reading symbol σ,
- $q_0 \in Q$: initial quantum state,
- M: projective measurement with outcomes in a finite set Γ,
- $L \subseteq \Gamma^*$: regular control language recognized by a classical DFA.

On input $x = x_1 x_2 \cdots x_n$, the automaton applies U_{x_i} and measures after each step, producing an output string $y \in \Gamma^n$. The word x is accepted if $y \in L$.

Strings Acceptance: Acceptance depends entirely on whether the sequence of quantum measurement results belongs to the control language L. This model generally uses bounded-error acceptance. However, exact acceptance is possible for certain languages, depending on the control DFA and quantum measurements [77].

Sets of Languages Recognized: CL-1QFA recognize exactly the class of REGs [68]. Though they do not surpass the regular class in power, their structure allows for different expressive strategies by decoupling quantum operations from classical verification.

Closure Properties: The class of languages recognized is closed under union, intersection, and complement because the control language is regular and the measurement outcomes are deterministic modulo quantum probabilities [68].

Advantages and Limitations: Advantages include clear separation between quantum processing and classical control, which simplifies modular design and analysis. The main limitation is that the model cannot recognize non-REGs and does not allow dynamic feedback between quantum and classical components [68].

Comparison Between Models: CL-1QFA can be simulated by 1QCFA [68], but not vice versa. Unlike 1QCFA, CL-1QFA do not allow two-way communication: measurement outcomes do not affect the ongoing quantum state. This makes CL-1QFA structurally simpler, but less expressive in practice.

Example: To recognize $L = (ab)^*$, a CL-1QFA can measure each input symbol's quantum effect and produce a binary output: '0' for a, '1' for b. The control DFA can then accept only if the output string alternates properly and has even length [77].

Additional Topics: Variants of CL-1QFA using unary control languages have been recently proposed to explore succinctness and unary acceptance conditions [78]. These help analyze minimal state configurations and potential hardware implementations.

Two-Way Quantum Finite Automaton with Classical States (2QCFA)

Introduction: The 2QCFA model, introduced by Ambainis and Watrous [5], is a hybrid automaton that consists of a classical two-way deterministic finite control and a constant-size quantum register. It was designed to exploit quantum computation while maintaining classical control over input movement. This makes the model both powerful and physically realizable, allowing the classical part to handle head movement and state tracking, while the quantum component processes information probabilistically.

Formal Definition: A 2QCFA is a 9-tuple:

$$A = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_a, S_r)$$

where:

- Q: finite set of quantum basis states,
- S: finite set of classical states,
- Σ: input alphabet,
- Θ: quantum transition function defining unitary operators or measurements,
- δ: classical transition function based on current state, symbol, and measurement outcome,
- $q_0 \in Q$, $s_0 \in S$: initial quantum and classical states,
- S_a, S_r: sets of accepting and rejecting classical states.

The classical control can move the tape head both left and right. The quantum state is manipulated via unitary transformations or measurements, which are determined by the classical state and scanned symbol. Decisions are made based on both the classical and quantum information.

Strings Acceptance: 2QCFA accept strings using bounded error or with one-sided error. For example, languages like $L_{eq} = \{a^n b^n \mid n \geqslant 1\}$ can be accepted with one-sided bounded error in expected polynomial time [5].

Sets of Languages Recognized: 2QCFA can recognize certain non-REGs, including L_{eq} and palindromes over unary alphabets, which makes them strictly more powerful than classical DFA or one-way QFA [5, 68].

Closure Properties: The class of languages recognized by 2QCFA is not closed under union or intersection, due to the constraints of the probabilistic error bounds and two-way head movement. However, they maintain closure under reversal and concatenation in specific cases [68].

Advantages and Limitations: Advantages include greater recognition power than 1QCFA and succinctness for certain problems. For example, 2QCFA can recognize L_{eq} with only a constant-size quantum register and logarithmic classical states [107]. However, they are generally limited to languages where probabilistic techniques suffice, and their runtime is often polynomial in the worst case [106].

Comparison Between Models: 2QCFA generalize 1QCFA by allowing two-way head movement, which significantly increases computational power. In contrast to CL-1QFA, they use dynamic feedback from the quantum measurements to the classical state transitions. 2QCFA are more expressive but harder to analyze due to interaction complexity [129].

Example: The language $L_{eq} = \{a^n b^n \mid n \geqslant 1\}$ can be recognized by a 2QCFA by using the quantum register to randomly check positions and probabilistically verify balance between a's and b's through repeated subroutines [5].

Additional Topics: Future work includes better understanding the time complexity of 2QCFA algorithms and developing minimization techniques. Variants include alternating 2QCFA and state-succinct encodings [129, 106].

Two-Tape Quantum Finite Automaton with Classical States (2TQCFA)

Introduction: The 2TQCFA model (Two-Tape Quantum-Classical Finite Automata) extends the 2QCFA by using two input tapes instead of one. Introduced by Zheng, Li, and Qiu [126], this model enhances computational power by enabling comparisons and synchronized traversal of two input strings. The quantum component remains fixed in size, while the classical controller can move the heads on both tapes and perform transitions based on measurements.

Formal Definition: A 2TQCFA is formally a tuple similar to a 2QCFA but with two input tapes:

$$A = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_a, S_r)$$

with the following distinctions:

- Two input heads, each reading a separate string from Σ^*,
- Classical state $s \in S$ determines movement and operation on each tape head,
- Quantum operations Θ depend on the symbols scanned by both heads and classical state.

As in 2QCFA, the automaton evolves through interactions between classical and quantum transitions, but the two-tape structure allows for cross-input comparisons.

Strings Acceptance: 2TQCFA can accept languages using bounded-error acceptance, typically with one-sided error. A notable example includes the language $L = \{w\#w \mid w \in \{a,b\}^*\}$, which is non-regular and not recognizable by 2QCFA, but accepted by 2TQCFA using synchronous traversal of both input halves [126].

Sets of Languages Recognized: The language recognition power of 2TQCFA includes certain context-free and non-REGs not recognizable by 2QCFA. Thus, 2TQCFA strictly extends the power of 2QCFA under bounded-error acceptance [126].

Closure Properties: Due to the added complexity of two-tape processing, closure properties are less well-defined. However, the model is still limited by finite memory and cannot recognize arbitrary context-free languages [68].

Advantages and Limitations: The primary advantage is an increased ability to perform input comparisons, useful for palindromes or equality checks. The main limitations include increased implementation complexity and difficulties in analyzing language classes and performance bounds.

Comparison Between Models: 2TQCFA extend 2QCFA in power by enabling comparisons across two tapes. Unlike kTQCFA (which generalize even further), 2TQCFA remain practical for checking mirrored or related substrings. Compared to 1QCFA and CL-1QFA, they are significantly more powerful in terms of language recognition.

Example: To recognize $L = \{w\#w\}$, a 2TQCFA reads w on the first tape and stores information in the quantum register. It then compares this with the second half of the input on the second tape. Probabilistic subroutines are used to ensure correctness with bounded error [126].

Additional Topics: Variants of multi-tape quantum automata have been studied to explore even richer classes. Open problems include characterizing all non-REGs recognizable by 2TQCFA with polynomial expected runtime.

k-Tape Quantum Finite Automaton with Classical States (kTQCFA)

Introduction: The kTQCFA model generalizes the two-tape quantum-classical automaton to an arbitrary finite number k of input tapes. This model, proposed in subsequent works building upon the 2TQCFA model [126], enhances the automaton's ability to process complex language patterns by allowing simultaneous access to multiple strings. Each tape is read by an independent head, all coordinated by a classical control unit and a constant-size quantum register.

Formal Definition: A kTQCFA is defined similarly to a 2TQCFA but with k tapes and k input heads. The formal components include:

$$A = (Q, S, \Sigma, \Theta, \delta, q_0, s_0, S_a, S_r)$$

with modifications:

- k input tapes, each with its own head,
- Classical state transitions δ depend on the symbols read from all k heads and outcomes of quantum operations,
- Quantum transitions Θ may vary based on any combination of input symbols and classical state.

The automaton reads the tapes simultaneously and updates its classical and quantum states accordingly, with acceptance determined by reaching a state in S_a.

Strings Acceptance: Acceptance is based on bounded-error criteria. This model can accept languages requiring coordinated comparisons across multiple strings, such as interleaving or mirror structures, which are beyond the capability of 2QCFA or 2TQCFA.

Sets of Languages Recognized: kTQCFA can recognize certain languages that lie outside the class of regular and some context-free languages. It provides a hierarchical extension in power with increasing k, where $k = 1$ corresponds to 1QCFA and $k = 2$ to 2TQCFA [68].

Closure Properties: Due to increasing complexity with larger k, closure properties are less explored. They inherit the limited closure of 2TQCFA but allow more expressive constructions for language families.

Advantages and Limitations: The main advantage of kTQCFA is scalability of pattern comparison and cross-tape logic. However, this comes at a cost: managing multiple heads and quantum-classical interactions becomes increasingly complex, both analytically and in potential physical realization.

Comparison Between Models: kTQCFA generalize all previously discussed models. While more powerful, they are also less practical for current quantum computing technologies. Unlike 1QCFA or CL-1QFA which are implementable with simpler setups, kTQCFA require sophisticated synchronization mechanisms.

Example: A 3TQCFA can accept a language like $L = \{(x, y, z) \mid x = y = z\}$ by comparing the three inputs simultaneously, performing probabilistic checks using quantum subroutines and classical tracking over each input position.

Additional Topics: Future work may include classification of languages based on minimal k required, complexity of simulations by smaller models, and physical feasibility of multi-tape implementations in quantum automata.

3.2.4 Quantum Finite Automata with Counters

Quantum finite automata with counters extend the computational power of QFA by incorporating classical or quantum counters into the system. These models provide hybrid computational capabilities where quantum state transitions are influenced by the counter value and vice versa, enabling the recognition of certain non-REGs with bounded error which classical counterparts fail to recognize.

In the literature, various models of QFA with counters have been proposed. This subsection explores the prominent models including Quantum Finite One-Counter Automaton (QF1CA), Two-Way Quantum Finite One-Counter Automaton (2QF1CA), Quantum Finite k-Counter Automaton (1QFkCA), and Real-Time Quantum One-Counter Automaton (RTQ1CA), detailing their structure, properties, capabilities, and limitations based on foundational work such as [22, 67, 88, 32].

Quantum Finite One-Counter Automata (QF1CA)

A QF1CA is a one-way QFA that uses a classical counter, capable of incrementing or decrementing its value and testing for zero. This model merges quantum transitions with classical counter logic, providing a new pathway for recognizing languages beyond the regular set [67].

Formal Definition Formally, a QF1CA consists of a finite set of states Q, an input alphabet Σ, a classical counter with values in $\mathbb{Z}$, and a transition function $\delta : Q \times \Sigma \times \{0,1\} \times Q \times \{-1,0,1\} \to \mathbb{C}$ where $\{0,1\}$ indicates whether the counter is zero or not. The system operates unitarily, with counter updates contingent on the current state and symbol read.

Strings Acceptance QF1CA can accept strings using bounded-error probabilistic acceptance. They can recognize non-REGs such as $L_1 = \{w \in \Sigma^* : \text{equal number of 0's and 1's in } x\}$ when augmented with additional structure in the input [22].

Sets of Languages Accepted The class of languages accepted by QF1CA with bounded error properly includes the class of languages accepted by classical deterministic and probabilistic one-counter automata [22].

Closure Properties Closure properties are limited and not thoroughly investigated; however, QF1CA do not maintain closure under union or intersection due to non-closure in the classical probabilistic case.

Advantages and Limitations A notable advantage is the ability to recognize certain context-free languages with bounded error. However, limitations stem from counter-based non-reversibility and measurement-induced collapses which reduce robustness.

Comparison Compared to 1QFA or MO-1QFA, QF1CA exhibit significantly higher computational power due to the counter's memory augmentation.

Example A QF1CA recognizing the language L_1 as shown above was constructed in [22], showing correct acceptance probabilities distinguishing it from deterministic models.

Additional Topics Current research investigates the influence of quantum control on counter updates and the simulation of classical pushdown automata using counters in hybrid quantum settings.

Two-Way Quantum Finite One-Counter Automata (2QF1CA)

Introduction 2QF1CA enhances the QF1CA model by allowing two-way head movement on the input tape, significantly expanding computational capabilities. This flexibility enables the automaton to reprocess information with context, analogous to two-way classical finite automata but equipped with quantum transitions and a counter.

Formal Definition A 2QF1CA is defined by a tuple $(Q, \Sigma, \delta, q_0, Q_a, Q_r)$, where δ maps configurations including direction: $\delta : Q \times \Sigma \times \{0,1\} \times Q \times \{-1,0,1\} \times \{-1,0,1\} \to \mathbb{C}$. The last component indicates the head movement (-1 for left, 0 for stay, 1 for right), and the counter updates accordingly.

Strings Acceptance 2QF1CA can recognize more complex languages such as L_2 from [22], composed of multiple L_1 segments demarcated by control symbols. These languages are not recognizable by 1QF1CA or classical probabilistic variants.

Sets of Languages Accepted These automata can accept languages outside deterministic and probabilistic one-counter automata capabilities, establishing a broader language class, including some context-sensitive languages under bounded error.

Closure Properties Closure under complement and intersection is not generally guaranteed due to quantum nondeterminism and measurement dependencies. Formal closure results remain limited.

Advantages and Limitations The key strength of 2QF1CA lies in its bidirectional input scanning which provides significant advantages in language parsing. However, unitarity and interference management become more complex.

Comparison Compared to QF1CA, the two-way model shows enhanced language recognition at the cost of more complex design and verification.

Example Recognition of L_2 involving interleaved structures demonstrates the superiority of 2QF1CA over classical and one-way quantum models as outlined in [22].

Additional Topics Further topics include automaton minimization, real-time simulation constraints, and efficient quantum algorithm implementation.

One-Way Quantum Finite k-Counter Automata (1QFkCA)

Introduction The 1QFkCA model generalizes the QF1CA by including k classical counters. Each counter is independently incremented, decremented, or checked against zero, enabling multi-dimensional memory augmentation in the quantum control logic [32].

Formal Definition Formally, a 1QFkCA is given by a transition function $\delta : Q \times \Sigma \times \{0,1\}^k \times Q \times \{-1,0,1\}^k \to \mathbb{C}$ with the counter vector defining current zero/non-zero statuses and updates.

Strings Acceptance Languages involving multiple numeric relationships, such as $L = \{a^n b^n c^n \mid n \geqslant 1\}$, can be recognized in bounded error by appropriately configured 1QFkCA.

Sets of Languages Accepted These automata recognize a subset of context-sensitive languages and are more powerful than all one-counter automata, quantum or classical.

Closure Properties Closure under intersection and union becomes feasible with k counters, particularly when structured synchronization is used in parallel counters.

Advantages and Limitations Their capability to recognize complex dependencies is advantageous, but the exponential state complexity and entangled counter management are practical limitations.

Comparison Compared to QF1CA, this model is exponentially more powerful but with higher operational complexity.

Example In [32], a 1QFkCA was shown to recognize the language $a^n b^n c^n$ via three synchronized counters incremented and decremented according to the current segment of the input.

Additional Topics Potential topics include quantum counter compression, fault tolerance in counters, and counter sharing protocols in hybrid quantum-classical systems.

Realtime Quantum One-Counter Automata (rtQ1CA)

Introduction RTQ1CA represents a restricted subclass of QF1CA in which the input head moves strictly right at each step, processing the input in real-time. This model explores trade-offs between real-time operation and computational power [32].

Formal Definition Defined similarly to QF1CA but with a strict constraint on the transition direction (right only). The transition function thus omits head direction: $\delta : Q \times \Sigma \times \{0, 1\} \times Q \times \{-1, 0, 1\} \rightarrow \mathbb{C}$.

Strings Acceptance Though more limited, RTQ1CA can still recognize several non-REGs with carefully crafted transition amplitudes and counter updates.

Sets of Languages Accepted Their accepted languages lie strictly between those of MO-1QFA and QF1CA due to the real-time restriction.

Closure Properties Due to strict real-time behavior and interference effects, closure properties are even more restricted.

Advantages and Limitations The main advantage is speed and simplicity in implementation, but at a significant cost to recognition power compared to QF1CA or 2QF1CA.

Comparison RTQ1CA are less powerful than general QF1CA, but more powerful than classical real-time automata due to quantum parallelism.

Example An RTQ1CA can probabilistically accept strings with a balanced number of 0's and 1's using only real-time passes and interference.

Additional Topics Real-time simulation fidelity and circuit-based implementations of RTQ1CA models are open areas of study.

3.2.5 Generalised Quantum Finite Automata

Generalised Quantum Finite Automata (1gQFAs) extend the standard QFAs by replacing the usual unitary-based state transitions with the most general physically admissible maps—namely, trace-preserving quantum operations. This modification permits non-unitary evolution, allowing the automata to simulate probabilistic and classical automata while still operating with finite memory. Nevertheless, it has been shown that both the measure-once and measure-many versions of 1gQFA recognize exactly the REGs (with bounded error) [69].

Measure Once Generalised Quantum Finite Automaton

A Measure Once Generalised Quantum Finite Automaton (MO-1gQFA) generalizes the traditional MO-1QFA by allowing each input symbol to trigger a trace-preserving quantum operation (instead of a unitary transformation) on the system. In this model, no measurement is performed during the reading of the input; a single projective measurement is executed only at the end to decide acceptance or rejection [69].

Formal Definition An MO-1gQFA is defined as the quintuple

$$M = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma}, P_{acc}\},$$

where

- $\mathcal{H}$ is a finite-dimensional Hilbert space,
- Σ is a finite input alphabet,
- $\rho_0 \in D(\mathcal{H})$ is the initial density operator,
- For each $\sigma \in \Sigma$, the state transition is given by the trace-preserving quantum operation
$$\mathcal{E}_\sigma(\rho) = \sum_k \mathcal{E}_{\sigma,k}\, \rho\, \mathcal{E}_{\sigma,k}^\dagger, \quad \text{with } \sum_k \mathcal{E}_{\sigma,k}^\dagger \mathcal{E}_{\sigma,k} = I,$$
- P_{acc} is a projector on the accepting subspace of $\mathcal{H}$ (with the complementary projector $P_{rej} = I - P_{acc}$).

On an input string $x = \sigma_1\sigma_2\cdots\sigma_n \in \Sigma^*$ the automaton evolves as

$$\rho_x = \mathcal{E}_{\sigma_n} \circ \mathcal{E}_{\sigma_{n-1}} \circ \cdots \circ \mathcal{E}_{\sigma_1}(\rho_0),$$

and a final measurement in the basis $\{P_{acc}, P_{rej}\}$ is performed. The acceptance probability is defined by

$$f_M(x) = \mathrm{Tr}\big(P_{acc}\, \rho_x\big).$$

Strings Acceptance A string $x \in \Sigma^*$ is accepted by M if the acceptance probability meets the specified criterion. Common acceptance criteria include:

1. **Bounded Error:** There exist a threshold $\lambda \in (0, 1]$ and an error margin $\epsilon > 0$ such that
$$f_M(x) \geqslant \lambda + \epsilon \quad \text{if } x \in L,$$
$$f_M(x) \leqslant \lambda - \epsilon \quad \text{if } x \notin L.$$
2. **Cutpoint Acceptance:** x is accepted if $f_M(x) > \lambda$, where λ is an isolated cutpoint.
3. **Exact Acceptance:** In certain constructions (e.g., when simulating a deterministic finite automaton) one has $f_M(x) = 1$ for accepted strings and $f_M(x) = 0$ for rejected strings.

Set of Languages Accepted It has been proved that under the bounded error criterion, MO-1gQFA recognize precisely the class of REGs. That is, for every REG there exists an MO-1gQFA recognizing it, and every language recognized by an MO-1gQFA is regular [69].

Closure Properties The class of languages recognized by MO-1gQFA is closed under several standard operations:

- **Union and Intersection:** By suitable constructions (e.g., via direct sums and tensor products), if L_1 and L_2 are recognized by MO-1gQFA then so are $L_1 \cup L_2$ and $L_1 \cap L_2$.
- **Complementation:** Replacing P_{acc} with its complement $I - P_{acc}$ yields an automaton for the complement language.
- **Inverse Homomorphism and Concatenation with REGs:** These operations preserve the regularity of the language.

Summary of Advantages and Limitations The MO-1gQFA model is advantageous due to its structural simplicity—requiring only a final measurement—and its ability to simulate classical probabilistic automata exactly via general trace-preserving operations. However, despite the broadened operational framework, its computational power remains confined to recognizing REGs (with bounded error). Furthermore, the state minimization problem for MO-1gQFA is known to be EXPSPACE-hard [75].

Example An example is provided by the simulation of a deterministic finite automaton (DFA) for the language

$$L = a^* b^*.$$

Here, one chooses

$$\mathcal{H} = \text{span}\{|q_n\rangle, |q_n\rangle, \dots, |q_n\rangle\},$$

sets the initial state as $\rho_0 = \sum_i \pi_i \, |q_n\rangle\langle q_i|$ (with $\{\pi_i\}$ given by the DFA's initial distribution), and defines each operation $\mathcal{E}_\sigma$ so that for each basis state $|q_n\rangle$,

$$\mathcal{E}_\sigma\big(|q_n\rangle\langle q_i|\big) = \sum_j A(\sigma)_{ij} \, |q_n\rangle\langle q_j| \,,$$

where $A(\sigma)$ is the stochastic matrix corresponding to the DFA's transition function. The final measurement is performed using

$$P_{acc} = \sum_{q_i \in F} |q_n\rangle\langle q_i| \,,$$

where F is the set of accepting states. This construction ensures that the acceptance probability $f_M(x)$ replicates the behavior of the DFA [69].

Additional Topics Further research on MO-1gQFA includes the equivalence problem, where necessary and sufficient conditions are derived based on the linear span of the reachable density operators. In addition, advanced state minimization techniques have been developed, reducing the minimization problem to solving systems of polynomial inequalities with an EXPSPACE upper bound [76].

Measure Many Generalised Quantum Finite Automaton

Introduction Measure Many Generalised Quantum Finite Automaton (MM-1gQFA) extend the MO-1gQFA model by performing a measurement after processing each input symbol. In this model, after each trace-preserving quantum operation corresponding to a symbol, a projective measurement is executed that partitions the state space into three mutually orthogonal subspaces—namely, the accepting subspace, the rejecting subspace, and the non-halting subspace. If the outcome lies in the accepting or rejecting subspace, the computation halts immediately; otherwise, it continues with the next symbol [69].

Formal Definition An MM-1gQFA is defined as the 6-tuple

$$M = \{\mathcal{H}, \Sigma, \rho_0, \{\mathcal{E}_\sigma\}_{\sigma\in\Sigma\cup\{¢,\$\}}, \mathcal{H}_{acc}, \mathcal{H}_{rej}\},$$

where

- $\mathcal{H}$ is a finite-dimensional Hilbert space that decomposes as

$$\mathcal{H} = \mathcal{H}_{acc} \oplus \mathcal{H}_{rej} \oplus \mathcal{H}_{non},$$

  with $\mathcal{H}_{acc}$ and $\mathcal{H}_{rej}$ denoting the accepting and rejecting subspaces, and $\mathcal{H}_{non}$ the non-halting subspace;

- Σ is a finite input alphabet, and the symbols ¢ and \$ serve as the left and right end-markers, respectively;

- $\rho_0 \in D(\mathcal{H})$ is the initial state with $\mathrm{supp}(\rho_0) \subseteq \mathcal{H}_{non}$;

- For each $\sigma \in \Sigma \cup \{¢, \$\}$, the state transition is given by the trace-preserving quantum operation

$$\mathcal{E}_\sigma(\rho) = \sum_k \mathcal{E}_{\sigma,k}\, \rho\, \mathcal{E}_{\sigma,k}^\dagger, \quad \text{with } \sum_k \mathcal{E}_{\sigma,k}^\dagger \mathcal{E}_{\sigma,k} = I;$$

- After each $\mathcal{E}_\sigma$, a projective measurement is performed with respect to the orthogonal projectors $\{P_{non}, P_{acc}, P_{rej}\}$ onto $\mathcal{H}_{non}$, $\mathcal{H}_{acc}$, and $\mathcal{H}_{rej}$, respectively.

For an input string $x \in \Sigma^*$ (presented as $¢\, x\, \$$), the automaton processes each symbol sequentially. If, at any step, the measurement projects onto $\mathcal{H}_{acc}$ (or $\mathcal{H}_{rej}$), the computation halts with acceptance (or rejection). Otherwise, if the outcome is in $\mathcal{H}_{non}$, the automaton continues processing the next symbol.

Strings Acceptance The acceptance of an input string x is defined by the cumulative probability that the automaton halts in an accepting configuration. The common acceptance criteria include:

1. **Bounded Error:** There exist $\lambda \in (0, 1]$ and $\epsilon > 0$ such that

$$\begin{aligned} &\text{if } x \in L, \quad \text{cumulative acceptance probability} \geqslant \lambda + \epsilon, \\ &\text{if } x \notin L, \quad \text{cumulative acceptance probability} \leqslant \lambda - \epsilon. \end{aligned}$$

2. **Cutpoint/Exact Acceptance:** As in the MO-1gQFA model, acceptance may also be defined via an isolated cutpoint or by requiring exact acceptance.

Set of Languages Accepted It has been established that MM-1gQFA, despite the intermediate measurements after each symbol, recognize exactly the class of REGs (with bounded error). Thus, the frequency of measurements does not extend the language recognition power beyond that of MO-1gQFA [69].

Closure Properties MM-1gQFA are closed under standard operations. In particular, if L_1 and L_2 are recognized by MM-1gQFA then:

- They are closed under *union* and *intersection* (by appropriate constructions using direct sums or tensor products),
- They are closed under *complementation* (by swapping the roles of $\mathcal{H}_{acc}$ and $\mathcal{H}_{rej}$),
- And they are closed under other operations such as inverse homomorphism.

Summary of Advantages and Limitations The MM-1gQFA model offers the flexibility of making intermediate measurements, which may simplify the design of some automata. However, like MO-1gQFA, its computational power remains limited to REGs under the bounded error regime. Furthermore, the state minimization problem for MM-1gQFA is EXPSPACE-hard [75].

Example For example, consider an MM-1gQFA designed to recognize

$$L = \{w \in \{a, b\}^* \mid \text{the last symbol of } w \text{ is } a\}.$$

In this automaton, after each input symbol the machine performs a measurement. If a measurement outcome projects onto $\mathcal{H}_{acc}$ (indicating that the current configuration is accepting) and no previous measurement forced a rejection, the automaton eventually halts with acceptance. This construction guarantees that the cumulative acceptance probability meets the bounded error condition exactly when the input ends with an a [69].

Additional Topics Recent research on 1gQFA has addressed the equivalence problem, providing necessary and sufficient conditions based on the linear span of the reachable density operators. Moreover, advanced state minimization techniques have been developed, reducing the minimization problem to solving systems of polynomial inequalities with an EXPSPACE upper bound [76]. Future directions include exploring further generalizations and their potential applications in modeling noisy quantum systems.

3.2.6 Interactive Automata Based on Quantum Interactive Proof Systems

Interactive automata based on quantum interactive proof systems offer a striking demonstration of how even extremely resource-limited verifiers—modeled by quantum finite automata (qfa's)—can, through interaction with a powerful prover, recognize nontrivial languages. Two principal models have been developed in this area:

- Quantum Interactive Proof (QIP) systems, in which the verifier's internal moves remain hidden (private-coin), and

- Quantum Arthur-Merlin (QAM) systems, where the verifier publicly announces his next move (public-coin).

In these models, the verifier is typically a two-way qfa, though one-way variants have also been considered. The seminal works by Nishimura and Yamakami [86, 87] and Zheng, Qiu, and Gruska [127] have established detailed protocols and complexity separations that reveal the potential of interactive proofs even when the verifier possesses only finite-dimensional quantum memory.

Formal Definition A general QIP system with a qfa verifier is defined as a pair (P, V), where:

Verifier. The verifier V is given by

$$V = \big(Q,\, \Sigma \cup \{¢, \$\},\, \Gamma,\, \delta,\, q_0,\, Q_{acc},\, Q_{rej}\big),$$

with the following components:

- Q is a finite set of inner states partitioned as $Q = Q_{non} \cup Q_{acc} \cup Q_{rej}$;
- Σ is the input alphabet, and ¢ and \$ denote the left and right endmarkers, respectively;
- Γ is the communication alphabet;
- δ is the transition function. For each configuration (q, σ, γ), the verifier changes its state, updates the tape head position (with moves in $\{-1, 0, 1\}$), and writes a new symbol in the communication cell according to complex amplitudes given by $\delta(q, \sigma, \gamma, q', \gamma', d)$;
- $q_0 \in Q$ is the initial state;
- Q_{acc} and Q_{rej} are the sets of halting (accepting and rejecting) states.

The verifier's overall Hilbert space, denoted by $\mathcal{H}_V$, is spanned by basis states of the form

$$|q, k, \gamma\rangle, \quad q \in Q,\ k \in \mathbb{Z},\ \gamma \in \Gamma.$$

Prover. The prover P is specified by a family of unitary operators

$$\{U_{x,P,i}\}_{i \geqslant 1},$$

acting on the prover's private Hilbert space $\mathcal{H}_P$. In some variants (denoted by the restriction ⟨c-prover⟩), the prover's unitaries are required to have only 0-1 entries, effectively making the prover deterministic.

QAM Systems In the QAM variant, the verifier is additionally required to announce his next move via the communication cell, rendering the system a public-coin protocol. Thus, while the basic structure of (P, V) remains the same, a QAM system is denoted as

$$\text{QAM}(\langle\text{restriction}\rangle) \quad \text{or} \quad \text{QIP}(\text{public}),$$

and the transition function δ is designed so that, after each move, the pair (q', γ', d) is revealed to the prover.

Strings Acceptance An interactive proof system (P, V) accepts an input string $x \in \Sigma^*$ if, after a prescribed sequence of interaction rounds, the verifier eventually performs a halting measurement that yields an accepting configuration with high probability. Formally, the system recognizes a language $L \subseteq \Sigma^*$ if the following conditions hold:

- **Completeness:** For every $x \in L$, there exists a prover strategy P such that the verifier accepts x with probability at least $1 - \epsilon$, where $\epsilon < 1/2$.
- **Soundness:** For every $x \notin L$, for every prover strategy P^*, the verifier rejects x with probability at least $1 - \epsilon$.

Variants of acceptance include definitions via an isolated cutpoint or exact acceptance (i.e., acceptance with probability 1), but the bounded-error model is standard.

Set of Languages Accepted The language recognition power of these interactive systems depends on the verifier's model and the nature of the interaction:

- When the verifier is a **1qfa** (one-way qfa), it has been shown that

  $$\mathrm{QIP}(\mathrm{1qfa}) = \mathrm{REG},$$

  meaning that even with interaction the system recognizes only the REGs [86].
- In contrast, when the verifier is a **2qfa** (two-way qfa), the interactive proof system can recognize languages that are not regular. For example, several protocols with 2qfa verifiers operating in expected polynomial time have been shown to outperform classical AM systems with 2pfa verifiers [127, 87].
- In the public-coin (QAM) variant, where the verifier reveals its next move, the additional information sometimes further enhances the system's power, and comparisons with classical Arthur-Merlin systems have been established.

Closure Properties The language classes defined by QIP and QAM systems exhibit robust closure properties:

- They are closed under *union* and *intersection*, typically via parallel composition (using direct sums or tensor products).
- They are closed under *complementation* (by exchanging the roles of Q_{acc} and Q_{rej} in the verifier's design).
- They are also closed under other operations such as inverse homomorphism.

These properties are established through constructions that combine multiple protocols while preserving the bounded-error guarantees.

Summary of Advantages and Limitations Interactive proof systems with qfa verifiers offer several compelling advantages:

- **Finite Quantum Resources:** The verifier operates with a finite-dimensional quantum system, making the model realistic for devices with limited quantum memory.

- **Enhanced Recognition via Interaction:** Even though a standalone qfa (especially a 1qfa) may recognize only REGs, interaction with a powerful prover can significantly boost the verifier's ability, particularly when using two-way qfa verifiers.

- **Flexibility through Protocol Variants:** By varying whether the system is a QIP (private-coin) or QAM (public-coin) system, and by imposing restrictions on the prover (quantum vs. classical), one can fine-tune the computational power and compare with classical interactive proof systems.

However, there are also limitations:

- **Limited Power of One-Way Verifiers:** When restricted to one-way qfa verifiers, the system's power is confined to the REGs.

- **Potentially High Interaction Complexity:** Protocols with two-way qfa verifiers can require a large (sometimes exponential) number of rounds or running time.

- **Technical Complexity:** The design and analysis of these interactive protocols are intricate, involving careful balancing of quantum and classical information.

Example An illustrative example is the QIP protocol for the language

$$\texttt{Pal\#} = \{x\#x^R \mid x \in \{0,1\}^*\},$$

which comprises even-length palindromes separated by a delimiter. In the protocol described in [87], the verifier (modeled as a 2qfa) interacts with a quantum prover in the following way:

1. The verifier scans the input (framed by the endmarkers ¢ and \$) and, based on its transition function δ, generates a superposition reflecting potential midpoints.

2. Through a sequence of rounds, the verifier requests the prover to indicate the position of the center. In the QAM variant, the verifier publicly announces his next move to assist the prover.

3. Finally, the verifier applies a QFT to consolidate the information and performs a measurement. If the input is indeed of the form $x\#x^R$, the verifier accepts with high probability; otherwise, it rejects.

This example clearly demonstrates how interaction compensates for the verifier's limited memory, enabling recognition of a nontrivial language.

Additional Topics Several open problems and future research directions emerge from this line of work:

- **Round Complexity:** How does limiting the number of interaction rounds (e.g., as in QIP#(k)) affect the recognition power and efficiency?

- **Prover Restrictions:** What are the precise differences in computational power when the prover is restricted to classical behavior (⟨c-prover⟩) versus full quantum capability?

- **Public vs. Private Protocols:** Further analysis is needed to understand the trade-offs between QIP (private-coin) and QAM (public-coin) systems.
- **Resource-Bounded Protocols:** Tightening the upper and lower bounds on running time and state complexity for these systems remains a challenging task.

These issues continue to be central to the ongoing exploration of the interplay between interaction and quantum finite automata.

- **Limited-Round Interactive Systems (QIP#(k)):** In some works (e.g., by Nishimura and Yamakami), the number of interaction rounds is explicitly bounded. These models, often denoted by QIP#(k) (with k indicating the maximum number of rounds), allow a more refined complexity classification of interactive protocols.
- **Interactive Proof Systems with Semi-Quantum Verifiers:** Another significant model is the one in which the verifier is not a full-fledged quantum finite automaton but a *semi-quantum* two-way finite automaton (2QCFA). In such systems—as studied, for instance, by Zheng, Qiu, and Gruska—the verifier possesses both classical and quantum states, using limited quantum resources alongside classical processing. These systems (sometimes denoted QAM(2QCFA) in the public-coin setting) have been shown to recognize languages beyond those recognizable by two-way probabilistic finite automata.
- **Variants Based on Prover Restrictions:** Some works also examine the effect of restricting the prover to *classical* behavior (i.e., using only 0-1 unitary operators, sometimes denoted by the restriction $\langle$c-prover$\rangle$). This yields interactive models that can be compared with their fully quantum counterparts.

3.2.7 Multi-letter Models

Multiletter Quantum Finite Automata (Multi-Letter Quantum Finite Automaton (ML-QFA)) are a generalization of traditional quantum finite automata, where transitions depend on multiple letters read from the input, rather than a single letter. This allows them to capture more complex patterns in the input string.

Multi-Letter Quantum Finite Automaton (ML-QFA)

Multiletter QFA were introduced to extend the capability of classical and quantum models by applying unitary operations based on the last k letters read rather than just one. This enables them to recognize languages outside the reach of traditional measure-once or measure-many 1QFA models [12].

Formal Definition A k-letter ML-QFA is a 5-tuple $A = (Q, Q_{acc}, |\psi_0\rangle, \Sigma, \mu')$, where:

- Q is a finite set of states,
- $Q_{acc} \subseteq Q$ is the set of accepting states,
- $|\psi_0\rangle$ is the initial superposition of states with unit norm,
- Σ is the input alphabet,

- $\mu' : (\Sigma \cup \{\Lambda\})^k \to U(\mathbb{C}^n)$ is a function assigning a unitary operator to every k-tuple of symbols.

The transition applies the unitary associated with the last k letters read. For input $\omega = x_1 x_2 \ldots x_n$, the computation evolves through unitary applications as specified in Eq. (1) and acceptance is determined using a projection operator P_{acc} [96].

Strings Acceptance Acceptance is defined by exact probability, cutpoint (strict or non-strict), or bounded-error depending on how $P_A(\omega) = \|P_{acc} U_\omega |\psi_0\rangle \|^2$ compares to a threshold λ.

Sets of Languages Accepted ML-QFA can recognize a proper superset of REGs compared to MO-1QFA and MM-1QFA. Notably, they can recognize the language $(a+b)^*a$ which is not recognizable by MO-1QFA or MM-1QFA [12].

Closure Properties The class of languages recognized by ML-QFA is not closed under union, intersection, or complement, especially under non-strict cutpoint semantics [96].

Advantages and Limitations ML-QFA demonstrate higher computational power with fewer states in certain scenarios. However, equivalence and minimization are complex and computationally hard. For non-strict cutpoints, the emptiness problem is undecidable [97].

Comparison ML-QFA are more expressive than MO-1QFA and MM-1QFA under the same acceptance criteria, but less so than two-way or general quantum automata with additional memory models [98].

Example The language $(a+b)^*a$ is a canonical example recognized by 2-letter ML-QFA, using a transition function dependent on the last two characters read [12].

Additional Topics Research is ongoing in determining exact hierarchies, equivalence testing, and applying ML-QFA in quantum protocol verification [70, 97].

Multiletter Measure-Many Quantum Finite Automata (ML-MMQFA)

Introduction Multi-Letter Measure Many Quantum Finite Automaton (ML-MMQFA) combine the power of measure-many acceptance strategies with multiletter transitions, extending both classical MM-1QFA and ML-QFA. In this model, a measurement is performed after each quantum evolution step, but the evolution itself depends on the last k letters read. This hybrid structure allows ML-MMQFA to accept more complex languages than ML-QFA or MM-1QFA alone [70].

Formal Definition A k-letter ML-MMQFA is a 7-tuple $A = (Q, Q_{acc}, Q_{rej}, |\psi_0\rangle, \Sigma, \mu', \mathcal{O})$, where:

- Q is the finite set of states,

- $Q_{acc} \subset Q$ is the set of accepting states,
- $Q_{rej} \subset Q$ is the set of rejecting states with $Q_{acc} \cap Q_{rej} = \emptyset$,
- $|\psi_0\rangle$ is the initial state,
- Σ is the finite input alphabet,
- $\mu' : (\Sigma \cup \{\Lambda, \pounds, \$\})^k \to U(\mathbb{C}^n)$ assigns a unitary matrix to each k-letter word,
- $\mathcal{O} = \{P_{acc}, P_{rej}, P_{non}\}$ is a projective measurement partitioning the Hilbert space based on Q_{acc}, Q_{rej}, and the non-halting subspace.

Computation starts with the end-marked input $\pounds x_1 x_2 \dots x_n\$$, and proceeds by interleaving unitary evolutions with projective measurements.

Strings Acceptance For a word $\omega = x_1 x_2 \dots x_n$, the acceptance probability is:

$$P_A(\omega) = \sum_{i=0}^{n+1} \left\| P_{acc} U_{x_i} \left(\prod_{j=i-1}^{0} P_{non} U_{x_j} \right) |\psi_0\rangle \right\|^2$$

A string is accepted if this probability exceeds a cutpoint (for probabilistic acceptance), or is exactly 1 (for exact acceptance). Both strict and non-strict cutpoints are considered [70].

Sets of Languages Accepted ML-MMQFA can recognize languages beyond the regular class. They accept some languages that are not accepted by classical QFA or even standard MM-1QFA, especially under non-strict cutpoint semantics [96, 70].

Closure Properties The set of languages recognized by ML-MMQFA is not closed under union, intersection, or complementation for general acceptance modes. Notably, these properties depend heavily on the acceptance criteria used (bounded-error, cutpoint, etc.) [96].

Advantages and Limitations ML-MMQFA are more powerful than both ML-QFA and MM-1QFA. However, they inherit the undecidability of the emptiness and equivalence problems for non-strict and strict cutpoint semantics [97, 70]. Their expressive power comes at the cost of analytical and implementation complexity.

Comparison ML-MMQFA strictly subsume ML-QFA under the same input size and acceptance criteria. They are not comparable in power with general QFA models that allow additional memory or two-way movement. Compared to MM-1QFA, ML-MMQFA can accept non-stochastic languages [96].

Example An ML-MMQFA with 2-letter transitions and intermediate measurements can accept the language $(a + b)^*a$ with higher robustness to probabilistic acceptance thresholds than an ML-QFA [12].

Additional Topics Further work focuses on state complexity, succinctness, and simulation algorithms between different classes. The diagonal sum construction plays a key role in analyzing equivalence and decidability [70].

Multi-Letter Reversible Quantum Finite Automaton (ML-RevQFA)

Introduction Multi-Letter Reversible Quantum Finite Automaton (ML-RevQFA) extend the multiletter framework by imposing reversibility constraints on the quantum evolution. These automata are designed so that each computation step is invertible, maintaining the core principle of reversibility from quantum mechanics and enhancing coherence preservation [12].

Formal Definition A k-letter ML-RevQFA is a special case of k-letter ML-QFA where each unitary transformation $\mu'(\omega)$ satisfies the additional constraint that $\mu'(\omega)^{-1} = \mu'(\omega)^\dagger$ for all $\omega \in (\Sigma \cup \{\Lambda\})^k$, and the set of such transformations forms a group under composition.

The automaton is defined as a 5-tuple $A = (Q, Q_{acc}, |\psi_0\rangle, \Sigma, \mu')$, with μ' restricted to reversible unitary matrices.

Strings Acceptance Acceptance is defined as in ML-QFA using a projector P_{acc}. For an input string ω, the acceptance probability is:

$$P_A(\omega) = \|P_{acc} U_\omega |\psi_0\rangle\|^2$$

with U_ω computed from the sequence of reversible unitaries associated with the k-letter substrings [12].

Sets of Languages Accepted ML-revQFA are strictly more limited than general ML-QFA due to their reversibility constraint. They can recognize a subset of REGs and do not accept all REGs with bounded error, particularly under exact acceptance criteria.

Closure Properties The class of languages recognized by ML-revQFA is not closed under union or complement. Reversibility limits the computational power of these models in comparison with more general multiletter QFA models [12].

Advantages and Limitations Reversible automata are appealing for quantum computing implementations due to better coherence and energy efficiency. However, their expressiveness is constrained. ML-revQFA cannot recognize some simple REGs that non-reversible ML-QFA can handle [12].

Comparison ML-revQFA are less powerful than both ML-QFA and ML-MMQFA. They are closely related to reversible classical automata and group automata. Compared to non-reversible models, they generally require more states or cannot recognize the same languages under equivalent semantics [12].

Example It was shown that ML-revQFA cannot accept the language $(a + b)^*a$ even when using multiletter transitions, while a general ML-QFA can accept it with bounded error [12].

Additional Topics Future research may explore connections with quantum error correction, fault-tolerant reversible computing, and applications in energy-efficient quantum hardware design.

3.3 Other Models of Quantum Finite Automata

Beyond the core models of quantum finite automata discussed in the previous sections, the literature also presents several alternative models that explore different computational paradigms, theoretical extensions, or enhancements. While these models are less prominent or less widely used, they offer valuable insights into the boundaries and variations of quantum automata theory.

In this section, we provide a concise overview of some notable variants. Each model is briefly introduced with its main characteristics and distinguishing features, along with references to the original works in which they were proposed. Readers interested in further details are encouraged to consult the cited articles.

3.3.1 Quantum Turing Machines

The Quantum Turing Machine (QTM) is the quantum analog of a classical Turing machine, featuring an infinite tape and a moving head with quantum states and unitary transitions. It was first proposed by Deutsch in 1985 as a general model of quantum computation [41]. A QTM can implement any quantum algorithm and is computationally equivalent to the quantum circuit model (Yao proved that any QTM can be efficiently simulated by quantum circuits and vice versa [125]). Unlike finite automata models, the QTM is not limited to REGs - it has unbounded memory and can recognize non-REGs - but this generality comes at the cost of a much more complex machine description. In practice, QTMs serve mostly as a theoretical cornerstone since simpler models (like quantum circuits) are used for designing algorithms, yet the QTM remains important for defining quantum complexity classes and formalizing the Church-Turing principle in the quantum realm.

3.3.2 Latvian Quantum Finite Automata

The term Latvian Quantum Finite Automaton (LQFA) refers to the one-way quantum finite automaton model introduced by Ambainis and Freivalds (who are Latvian) in 1998 [4]. This model is essentially the *measure-once* 1QFA: the machine's state evolves unitarily as it reads the input, and only after reaching the end of the input is a single projective measurement performed to decide acceptance. (In contrast, the earlier QFA model by Kondacs and Watrous allowed measurements after each step.) The Latvian 1QFA demonstrated that even with a single end-of-input measurement, a quantum automaton can recognize certain REGs with exponentially fewer states than any equivalent deterministic automaton. However, like other 1QFAs, it cannot recognize all REGs. The LQFA is historically significant as one of the first quantum automata models, and its state-efficiency advantages and limitations were studied in subsequent works.

3.3.3 l-valued Finite Automata

An l-valued Finite Automaton (l-VFA) is an automaton model based on multi-valued logic (in particular, on quantum logic), rather than probabilistic or binary state transitions. This model was explored by Ying (2000) and was later formalized and extended by Qiu in 2007 as a "logical" approach to quantum computation [93]. In an l-VFA, the transition function is not strictly deterministic or probabilistic - instead, each transition from a state p to a state q on an input symbol σ is assigned a truth-value from a complete orthomodular lattice L. Intuitively, $\delta(p, \sigma, q)$ may be 0, 1, or some intermediate truth-value in L. A string is accepted by an l-VFA if the aggregated truth-value of all paths leading to an accepting state evaluates to 1 in the lattice sense. This construction generalizes classical finite automata and provides a way to apply quantum logic to automata theory.

3.3.4 l-valued Pushdown Automata

The l-valued Pushdown Automaton (l-VPDA) extends the idea of an l-VFA by adding a pushdown stack, thus enabling recognition of some non-REGs within the l-valued logic framework. This model was introduced alongside l-VFAs by Qiu in 2007 [93] as part of the effort to build automata theory on quantum logic. An l-VPDA operates similarly to a classical pushdown automaton, but its state transitions and stack operations carry truth-values in a lattice L instead of deterministic outcomes.

3.3.5 Quantum Automata with Advice

Quantum Finite Automaton with Advice are variants of 1QFA that are supplemented with an additional input - an advice string or quantum state - that depends only on the input length n and is provided to the automaton to improve its computation. This idea was studied by Yamakami (2014) [124]. In his model, the machine can utilize a pre-prepared quantum advice state during its computation, allowing for potentially improved computational power while still remaining weaker than full quantum Turing machines.

3.3.6 Enhanced Quantum Finite Automata

Enhanced One-Way Quantum Finite Automaton (E-1QFA) is a variant of the one-way QFA where the machine's state can be measured after each symbol is read, rather than restricting measurement to occur only at the end of the input. This model was introduced by Nayak [82] and studied further by Lin [71]. It allows the computation to dynamically adapt based on partial measurement outcomes, making it slightly more powerful than traditional one-way QFAs in certain contexts.

3.3.7 Postselection Quantum Finite Automata

Postselection Quantum Finite Automaton (PQFA) is a theoretical model that augments a quantum finite automaton with the power of *postselection* - the ability to conditionally proceed based on a desired measurement outcome. This powerful but unphysical feature was used to explore computational limits, and the model was studied in depth by Scegulnaja-Dubrovska et al. [109] and originally proposed in the context of quantum complexity by Aaronson [1].

3.3.8 ω Quantum Finite Automata

ω-Quantum Finite Automaton (ω-QFA) extend quantum finite automata to operate on infinite input strings. Bhatia and Kumar (2019) introduced several formal models with different acceptance conditions like Büchi, Rabin, and Streett [18]. These models are important for exploring quantum computation over streams or continuous inputs and show intriguing differences from their classical counterparts.

3.3.9 Promise Problems and Quantum Finite Automata

Promise problems are a generalization of language recognition where an automaton is required to correctly classify inputs from two disjoint sets: the “yes” instances and the “no” instances. This relaxed setting provides a useful framework for analyzing subtle distinctions in computational power, especially when comparing classical and quantum models.

QFA have demonstrated significant advantages in the context of promise problems. These models are often more state-efficient or capable of solving problems that classical automata cannot handle with bounded error. One notable study by Zheng et al. [129] investigates the 2QCFA model and demonstrates its exponential state succinctness over classical counterparts for families of promise problems. For example, they construct a 2QCFA that solves a problem with constant quantum memory and logarithmic classical memory, whereas equivalent classical automata require exponentially more states.

Other works explore theoretical implications of quantum advantages under promises. Rashid and Yakaryilmaz [102] analyze how quantum automata solving promise problems can relate to foundational concepts like contextuality in quantum theory. Bianchi et al. [19] examine the computational complexity of promise problems across classical and quantum finite automata, identifying specific contexts where quantum models are strictly more efficient. Gruska et al. [57] further study promise problems under exact acceptance and show that QFA can solve certain structured promise problems with significantly fewer states than their classical counterparts.

Overall, the study of promise problems has emerged as a rich area to highlight the computational advantages of quantum models, often revealing separations that are not observable in standard language recognition settings.

4. Automata to Circuits

QFAs furnish a concise, mathematically transparent model of finite-memory computation, yet practical algorithms must ultimately be recast as quantum circuits that manipulate qubits through finite sequences of gates and measurements. The purpose of this chapter is to articulate, in a systematic manner, how a quantum automaton defined at the symbolic level is translated into a concrete circuit description suitable for compilation on NISQs hardware. By mapping each automaton primitive onto circuit counterparts we obtain designs that are executable on present devices, support quantitative resource accounting and admit gate-level formal verification.

Section 4.1 outlines the compilation workflow for the MO-1QFA, illustrating how its fundamental components can be encoded within a quantum circuit model. The translation process preserves the computational semantics of the original automaton while making it compatible with standard circuit synthesis techniques.

Section 4.2 extends the methodology to the MM-1QFA, whose intermediate measurements create early-halt branches and classical control flow. Particular attention is devoted to expressing the three-outcome measurement paradigm with standard two-outcome projective tests, to limiting ancilla overhead when discarding rejected branches, and to maintaining language-recognition semantics in the presence of realistic noise.

A template-first compilation philosophy is retained throughout: for an input word of length L the compiler emits a parameterised skeleton in which each placeholder gate is later instantiated with the concrete operator U_{σ_i} attached to the i-th symbol. This separation between structural aspects fixed by the automaton and numerical parameters dictated by the input encourages component reuse across multiple words and eases the deployment of QFAs as high-speed recognisers within larger quantum applications. Two complementary instantiation strategies are considered in Section 4.3. The first, an offline synthesis approach, compiles every operator U_σ ahead of execution and stores the resulting gate sequences as reusable fragments. The second adopts a parameter-loading paradigm in which a generic template containing analytic Euler-angle rotations is populated at runtime with classically computed angles that depend on the input word, thereby reducing memory overhead and enabling just-in-time adaptation to specific problem instances.

Upon completing the chapter the reader will possess a reproducible method for converting any MO-1QFA or MM-1QFA into an architecture-independent gate-level description, together with practical criteria for choosing state encodings, measurement decompositions and synthesis back-ends. These results pave the way for future research on two-way and hybrid models and represent a decisive step toward a unified tool-chain for automata-driven quantum software engineering.

4.1 Measure-Once One-Way Quantum Finite Automaton to Circuit

This section presents the compilation of the MO-1QFA model into an executable quantum circuit, elucidating the precise correspondence between automaton-level abstractions and gate-level constructs. The internal state set Q is represented using $\lceil \log_2 |Q| \rceil$ qubits, encoding each classical state $q \in Q$ into a computational basis vector of the quantum register. Each symbol $\sigma \in \Sigma$ is associated with a unitary matrix U_σ, which governs the evolution of the state vector upon reading σ; these operators are later decomposed into sequences of elementary gates drawn from a universal set (e.g., Clifford+T or $\{\text{CNOT}, R_z, H\}$). The initial state preparation maps the all-zero register to the automaton's start state via minimal gate operations. Finally, the accepting condition is enforced via a projective measurement onto a subspace defined by the set of accepting states F, implemented as a single multi-controlled rotation followed by a standard measurement. This mapping supports systematic synthesis of circuits from automaton descriptions while preserving the semantics of quantum language recognition.

4.1.1 Mapping Automaton Components to Circuit Elements

A MO-1QFA is defined as a tuple

$$A = (Q, \Sigma, \delta, q_0, F),$$

as introduced in Section 3.1.1. The goal is to construct, for any such automaton, a quantum circuit that faithfully reproduces its evolution and acceptance behaviour. This is achieved by mapping each formal component to a corresponding physical construct within the circuit model, preserving the semantics of quantum language recognition.

State Register and Qubit Allocation

The set of internal states Q is encoded over an n-qubit register, where $n = \lceil \log_2 |Q| \rceil$. Each state $q \in Q$ corresponds to a computational basis vector $|q\rangle \in (\mathbb{C}^2)^{\otimes n}$ under a fixed encoding. This representation ensures compatibility with standard gate decompositions and measurement procedures, and facilitates reversible indexing of automaton transitions.

Symbol-Dependent Unitary Evolution

Each symbol $\sigma \in \Sigma$ induces a unitary transformation U_σ defined by the transition function δ. These matrices are assumed to be unitary by definition of the model, and are compiled into native gate sequences using a fault-tolerant universal basis, such as Clifford+T or $\{\mathsf{CNOT}, R_z, H\}$. This decomposition is performed either ahead of time or via parameterised template instantiation, depending on the compilation strategy adopted.

Initialisation Procedure

The computation starts in the automaton's designated initial state q_0, represented as $|q_0\rangle$ on the n-qubit register. Physical initialisation begins with the zero state $|0\rangle^{\otimes n}$, which is then mapped to $|q_0\rangle$ using a preparation circuit. When q_0 is a basis vector in the encoding, this step reduces to applying a sequence of Pauli-X gates.

Measurement and Acceptance

Upon completion of the input traversal, the quantum state is measured against the accepting subspace defined by the set $F \subseteq Q$. The corresponding projector is

$$P_{\text{acc}} = \sum_{q \in F} |q\rangle\langle q| \,,$$

and the final two-outcome measurement $\{P_{\text{acc}}, I - P_{\text{acc}}\}$ determines acceptance (output 1) or rejection (output 0). In circuit terms, this is realised via a controlled operation targeting an ancilla qubit, followed by a standard measurement. Efficient implementations leverage multi-controlled rotations and ancilla reuse to minimise overhead.

Automaton part	**Circuit realisation**	**Explanation**
Q	n-qubit basis	Encode each $q \in Q$ as $\|q\rangle$, with $n = \lceil \log_2 \|Q\| \rceil$.
Σ	unitary U_σ	Reading σ applies U_σ.
δ	set $\{U_\sigma\}$	Transition matrices later decomposed into elementary gates.
q_0	state $\|q_0\rangle$	Prepare register from $\|0\rangle^{\otimes n}$ to $\|q_0\rangle$.
F	projector P_{acc}	Measure $\{P_{\text{acc}}, I - P_{\text{acc}}\}$ for accept/reject.

Table 4.1: Mapping MO-1QFA components to quantum-circuit constructs.

4.1.2 General Compilation Algorithm

The compilation of a MO-1QFA into an executable quantum circuit is divided into two conceptually distinct steps:

1. **Template Generation:** Construct a symbolic circuit template that encodes the automaton's structure using placeholder gates. This template depends only on the automaton and the input word length.
2. **Instantiation:** For a specific input word, substitute each symbolic placeholder with the actual unitary operator defined by the automaton and decompose it into elementary gates.

Algorithm 1 Template Generation for a MO-1QFA Circuit

Require: Automaton $A = (Q, \Sigma, \delta, q_0, F)$, input length L
Ensure: Parametric circuit template with symbolic placeholders
1: $n \leftarrow \lceil \log_2 |Q| \rceil$
2: Initialise n qubits in state $|q_0\rangle$
3: **for** $i = 1$ **to** L **do**
4: Insert symbolic gate $\boxed{U_{x_i}}$
5: **end for**
6: Append projective measurement $\{P_{\text{acc}}, I - P_{\text{acc}}\}$

Algorithm 1 creates a circuit that is independent of the actual input string. Each gate $\boxed{U_{x_i}}$ is a placeholder symbolically representing the unitary matrix to be applied upon reading the i-th symbol. This structure is fully determined by the automaton and remains fixed across all input words of the same length.

To execute the template on a specific word $x = x_1x_2 \dots x_L \in \Sigma^L$, the placeholders $\boxed{U_{x_i}}$ must be instantiated as concrete gates derived from the automaton's transition matrices.

Algorithm 2 Instantiation and Execution of a Compiled MO-1QFA Circuit

Require: Input $x = x_1x_2 \dots x_L$, circuit template, gate library or decomposition scheme for each U_σ
Ensure: Acceptance probability $p_A(x)$
1: **for** $i = 1$ **to** L **do**
2: Replace $\boxed{U_{x_i}}$ with a gate decomposition implementing U_{x_i}
3: **end for**
4: Apply $U_{x_L} \cdots U_{x_1}$ to $|q_0\rangle$
5: Perform measurement $\{P_{\text{acc}}, I - P_{\text{acc}}\}$
6: **return** $p_A(x) = \|P_{\text{acc}} U_{x_L} \cdots U_{x_1} |q_0\rangle\|^2$

The two-phase compilation strategy promotes modularity: the symbolic template can be reused across many inputs, while the instantiation adapts to specific data. This design aligns with scalable quantum software practices and is compatible with both pre-synthesised libraries and dynamic, runtime parameter loading (see Section 4.3).

4.1.3 Step-by-Step Examples

To concretise the abstract compilation scheme described above, we now present explicit examples illustrating the end-to-end translation of MO-1QFA instances into executable quantum circuits. Each example begins with a formal automaton specification and proceeds through template generation, gate instantiation for a specific input word, and—when appropriate—optimised decomposition into native gates. These case studies demonstrate the generality of the approach and clarify how structural automaton features influence circuit depth, gate choice, and measurement configuration.

Single-Letter Alphabet. Consider the MO-1QFA defined by

$$Q = \{q_0, q_1\}, \quad \Sigma = \{a\}, \quad q_0 \text{ initial}, \quad F = \{q_1\},$$

and let the transition unitary associated with the only input symbol be the Hadamard gate

$$U_a = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We compile this automaton for words of length $L = 1$ using Algorithm 1. The process proceeds as follows:

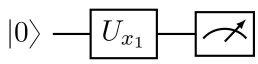
1. **Qubit allocation.** Since $|Q| = 2$, we require $n = \lceil \log_2 2 \rceil = 1$ qubit. The state set is encoded as $|q_0\rangle = |0\rangle$ and $|q_1\rangle = |1\rangle$, so the circuit will consist of a single wire.

2. **Initialisation.** The register is initialised in state $|0\rangle = |q_0\rangle$, which matches the default zero state on most quantum backends.

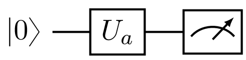
3. **Unitary evolution (template).** The compiler inserts one symbolic placeholder $\boxed{U_{x_1}}$ representing the unitary corresponding to the input symbol at position 1. The result is the circuit shown in subfigure 4.1a.

4. **Instantiation.** For the specific input $x = a$, we substitute $U_{x_1} = H$, yielding the circuit in subfigure 4.1b.

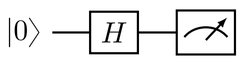
5. **Measurement.** The accepting projector is $P_{\text{acc}} = |1\rangle\langle 1|$. A measurement in the computational basis is applied at the output. The final synthesised circuit—requiring no decomposition since H is native—is shown in subfigure 4.1c. The output state is
$$H|0\rangle = \tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle),$$
so the acceptance probability is
$$p_M(a) = \|P_{\text{acc}} H|0\rangle\|^2 = \left|\tfrac{1}{\sqrt{2}}\right|^2 = \tfrac{1}{2}.$$



(a) Step 1: Template circuit



(b) Step 2: Instantiation



(c) Step 2: Gate-level circuit

Figure 4.1: Compilation stages for Example 4.1.3. The first row shows the symbolic template; the second row shows instantiation and final gate decomposition.

Two-Symbol Word of Length $L = 2$. Consider a MO-1QFA defined by:

$$Q = \{q_0, q_1, q_2\}, \quad \Sigma = \{a, b\}, \quad q_0 \text{ initial}, \quad F = \{q_2\},$$

with transitions specified as follows:

- U_a performs a rotation between $|q_0\rangle$ and $|q_1\rangle$,
- U_b swaps $|q_1\rangle$ and $|q_2\rangle$.

We compile the automaton on the input word $x = ab$, of length $L = 2$.

1. **Qubit allocation.** Since $|Q| = 3$, we require
$$n = \lceil \log_2 3 \rceil = 2 \text{ qubits.}$$
The states q_0, q_1, and q_2 are encoded as $|00\rangle$, $|01\rangle$, and $|10\rangle$ respectively.

2. **Initialisation.** The register is prepared in state $|q_0\rangle = |00\rangle$. This is typically the default zero state and requires no preparation gates.

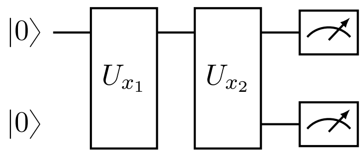
3. **Unitary evolution (template).** For a word of length $L = 2$, the compiler inserts two placeholders $\boxed{U_{x_1}}$ and $\boxed{U_{x_2}}$, representing the unitaries to be applied at each step. The resulting template is shown in subfigure 4.2a.
4. **Instantiation.** For the specific input $x = ab$, we substitute $U_{x_1} = U_a$ and $U_{x_2} = U_b$, yielding the circuit in subfigure 4.2b.
5. **Measurement.** The accepting subspace corresponds to $F = \{q_2\}$, i.e., the basis state $|10\rangle$. The measurement is performed in the computational basis, and the projector is

$$P_{\text{acc}} = |10\rangle\langle 10|\,.$$

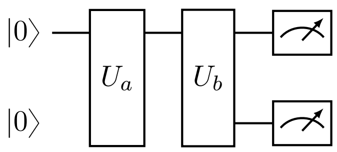
The final synthesised circuit, shown in subfigure 4.2c, assumes a possible decomposition where U_a is implemented as a rotation $\mathcal{R}_X(\theta)$ acting on the subspace $\text{span}\{|00\rangle, |01\rangle\}$, and U_b is realised by a SWAP gate between $|01\rangle$ and $|10\rangle$.

The acceptance probability depends on the choice of rotation angle θ. For instance, if $\theta = \pi/2$, the sequence $U_b U_a |00\rangle$ results in state $|10\rangle$ with probability 1, yielding

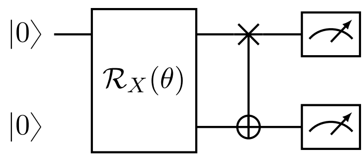
$$p_M(ab) = \|P_{\text{acc}} U_b U_a |00\rangle\|^2 = 1.$$



(a) Step 1: Template circuit



(b) Step 2: After instantiation



(c) Step 2: Gate-level circuit

Figure 4.2: Compilation stages for Example 4.1.3. The first row shows the symbolic template; the second row shows the instantiated and decomposed circuit for input *ab*.

Example 3: Cyclic Automaton, $L = 3$. Consider a MO-1QFA with

$$Q = \{q_0, q_1, q_2\}, \quad \Sigma = \{a\}, \quad q_0 \text{ initial}, \quad F = \{q_0\}.$$

Let the unitary U_a implement a 3-cycle over the basis states:

$$U_a |q_i\rangle = |q_{i+1 \bmod 3}\rangle\,, \quad \text{for } i \in \{0, 1, 2\}.$$

We compile the automaton on the input word $x = aaa$, of length $L = 3$.

1. **Qubit allocation.** The state space has cardinality $|Q| = 3$, so we require

$$n = \lceil \log_2 3 \rceil = 2 \text{ qubits}.$$

The states q_0, q_1, and q_2 can be encoded as basis vectors $|00\rangle$, $|01\rangle$, and $|10\rangle$ respectively.

2. **Initialisation.** The register is initialised in state $|q_0\rangle = |00\rangle$.

3. **Unitary evolution (template).** The compiler inserts three placeholders, one for each input symbol. These gates are labelled $\boxed{U_{x_1}}$, $\boxed{U_{x_2}}$, $\boxed{U_{x_3}}$ and will later be replaced with actual unitaries. The resulting circuit structure is shown in subfigure 4.3a.

4. **Instantiation.** For the input $x = aaa$, all placeholders are replaced by U_a, as every symbol is a. The resulting sequence $U_aU_aU_a$ is shown in subfigure 4.3b.

5. **Cycle recognition and optimisation.** Since U_a implements a perfect 3-cycle, we have

$$U_a^3 = \mathbb{I}.$$

The composition of three applications of U_a returns the system to its original state. Recognising this cyclic structure, the compiler simplifies the circuit by removing all three gates, as shown in subfigure 4.3c. The net result is that the final state equals the initial state, i.e.,

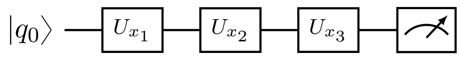
$$|\Psi_{aaa}\rangle = |q_0\rangle ,$$

which lies in the accepting subspace defined by $F = \{q_0\}$. Thus, the input aaa is accepted with probability

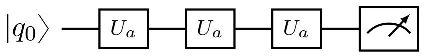
$$p_M(aaa) = \|P_{\text{acc}}\, |q_0\rangle\,\|^2 = 1.$$

In the context of quantum circuit compilation, automata that exhibit cyclic behaviour—such as k-cycles over the state set—allow for a key optimisation: repeated applications of a unitary operator implementing the cycle can often be collapsed. Specifically, if a unitary U satisfies $U^k = \mathbb{I}$, then any sequence of k consecutive applications yields the identity transformation. In such cases, the compiler can detect the cyclic structure statically and eliminate the redundant operations from the circuit. This not only reduces the gate count but also preserves the automaton's transition semantics exactly. Such cycle-aware optimisation plays a crucial role in minimising depth and improving interpretability of automaton-derived circuits.

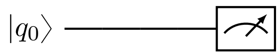
When the number of repetitions is not a multiple of the cycle length, or when intermediate cyclic states influence acceptance, the compiler cannot eliminate the corresponding gates. In such cases, the unitary operator implementing the cycle must be applied explicitly at each relevant input position. These cycle operations are realised as permutations over the encoded state space, typically constructed from SWAP gates or controlled Pauli operations. Although they may introduce repetition, this explicit representation ensures that the circuit precisely mirrors the automaton's behaviour, including partial traversals of cyclic transitions.



(a) Step 1: Template circuit



(b) Step 2: Instantiation



(c) Step 2: Optimised circuit

Figure 4.3: Compilation stages for Example 4.1.3. The first row shows the symbolic template; the second row illustrates the instantiation and the final optimised circuit after cycle detection.

Partial Cycle Without Optimisation. We now consider a variation of Example 4.1.3 in which the cycle is only partially traversed. Let the automaton be defined as

$$Q = \{q_0, q_1, q_2\}, \quad \Sigma = \{a\}, \quad q_0 \text{ initial}, \quad F = \{q_1\},$$

and let U_a be the same 3-cycle unitary:

$$U_a \left|q_i\right\rangle = \left|q_{(i+1) \bmod 3}\right\rangle, \quad \text{for } i \in \{0, 1, 2\}.$$

The input word is now $x = aa$, i.e., only two applications of U_a.

1. **Qubit allocation.** As before, we require

   $$n = \lceil \log_2 3 \rceil = 2 \text{ qubits},$$

   with states encoded as $|00\rangle$, $|01\rangle$, and $|10\rangle$.

2. **Initialisation.** The register is prepared in $|00\rangle$, representing q_0.

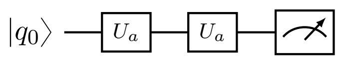
3. **Unitary evolution.** The compiler inserts two unitaries: $U_{x_1} = U_{x_2} = U_a$, as shown in subfigure 4.4a. Since the number of applications is less than the cycle length, the system does not return to the initial state. Therefore, the cycle identity $U_a^3 = \mathbb{I}$ does not apply here, and no optimisation is possible. The complete sequence must be preserved.

4. **Measurement.** After applying U_a twice, the final state is

   $$|\Psi_{aa}\rangle = U_a^2 \left|q_0\right\rangle = \left|q_2\right\rangle.$$

   The acceptance condition is $F = \{q_1\}$, corresponding to $|01\rangle$. Since the final state $|10\rangle$ is orthogonal to the accepting subspace, the acceptance probability is

   $$p_M(aa) = \|P_{\text{acc}} \left|\Psi_{aa}\right\rangle\|^2 = 0.$$



(a) Circuit for $x = aa$ (no optimisation)

Figure 4.4: Example 4.1.3: when the number of cycle steps does not complete the full orbit, the compiler must retain all gates to preserve the automaton's semantics. The input is rejected with probability 1.

4.2 Measure-Many One-Way Quantum Finite Automaton to Circuit

The circuit-level translation of a MM-1QFA follows many of the same principles used for the measure-once case, with a crucial difference: instead of deferring measurement until the end of the computation, a projective measurement is performed after each input symbol is processed. This repeated measurement model introduces non-unitary branches into the computation, enabling the automaton to halt early—either accepting or rejecting—based on intermediate outcomes. As a result, the circuit must incorporate mid-computation measurements, conditional halting logic, and branching structure, reflecting the MM-1QFA's hybrid quantum-classical behaviour.

This section describes how to compile a MM-1QFA into a quantum circuit that correctly reproduces its acceptance semantics, with attention to managing measurement timing, decomposing the three-outcome measurement structure, and ensuring termination guarantees for all input strings.

4.2.1 Mapping Automaton Components to Circuit Elements

The MM-1QFA model is defined by the tuple

$$M = (Q, \Sigma, \delta, q_0, Q_{\mathrm{acc}}, Q_{\mathrm{rej}}),$$

as introduced in Section 3.1.2. Compared to the MO-1QFA case, the key structural difference lies in the measurement strategy: whereas a MO-1QFA performs a single final measurement, a MM-1QFA applies a projective measurement after every symbol.

The first compilation steps closely follow those in Section 4.1, and only diverge when handling measurements:

State Register and Qubit Allocation

The internal state set Q is represented using $n = \lceil \log_2 |Q| \rceil$ qubits, encoding each classical state $q \in Q$ as a computational basis state $|q\rangle$. This is identical to the measure-once case.

Initialisation

The quantum register is initialised in the basis state corresponding to the initial state $q_0 \in Q$. If q_0 is a computational basis vector under the chosen encoding, this requires only a few X gates to flip the default all-zero state to $|q_0\rangle$.

Symbol-Dependent Unitary Evolution

Each input symbol $\sigma \in \Sigma$ is associated with a unitary matrix U_σ acting on the n-qubit register. The operator U_σ is applied before measurement at each step. These matrices are later compiled into native gates as in the MO-1QFA case.

Repeated Measurement

After each unitary U_σ is applied, a projective measurement is performed to decide whether the computation halts. The measurement is defined by three mutually orthogonal subspaces corresponding to:

- the accepting states Q_{acc},
- the rejecting states Q_{rej}, and
- the continuing (non-halting) states $Q_{\text{non}} = Q \backslash (Q_{\text{acc}} \cup Q_{\text{rej}})$.

The corresponding projectors are:

$$P_{\text{acc}} = \sum_{q \in Q_{\text{acc}}} |q\rangle\langle q|, \quad P_{\text{rej}} = \sum_{q \in Q_{\text{rej}}} |q\rangle\langle q|, \quad P_{\text{non}} = I - P_{\text{acc}} - P_{\text{rej}}.$$

The outcome determines whether the computation halts (accepts or rejects) or contines to the next symbol. This makes the circuit fundamentally hybrid: unitary evolution is interrupted by measurements and conditional control logic, which must be tracked classically.

Classical Control Flow

Unlike the MO-1QFA case, which requires only a final measurement, the MM-1QFA circuit must include measurement operations at each time step, as well as classical post-processing to determine whether further computation is needed. In practical implementations, this control flow may be realised via classical conditionals or early termination logic, depending on the hardware platform and circuit model (e.g., mid-circuit measurements with feedback).

Automaton part	**Circuit realisation**	**Explanation**
Q	n-qubit basis states	Encode each $q \in Q$ as $\|q\rangle$ with $n = \lceil \log_2 \|Q\| \rceil$.
Σ	unitaries U_σ	Each input symbol σ applies U_σ to the state register.
δ	set $\{U_\sigma\}$	Transition operators, later decomposed into native gates.
q_0	initial state $\|q_0\rangle$	Prepared from $\|0\rangle^{\otimes n}$ using X gates if necessary.
Q_{acc}, Q_{rej}	repeated three-outcome measurements	After each U_σ, apply a measurement $\{P_{\text{acc}}, P_{\text{rej}}, P_{\text{non}}\}$. Outcome determines halting or continuation.
Control flow	classical feedback	Conditional termination after each step based on measurement results.

Table 4.2: Mapping MM-1QFA components to quantum-circuit constructs.

4.2.2 General Compilation Algorithm

The compilation of a MM-1QFA into a quantum circuit proceeds in two structured steps:

1. **Template Generation:** A symbolic circuit is created to reflect the automaton's structure for input words of fixed length, using placeholders for both unitaries and measurements.
2. **Instantiation and Execution:** For a specific word, the placeholders are replaced with the actual gates implementing the automaton's transitions, and the computation proceeds with intermediate measurements after each symbol.

Let $F = Q_{\mathrm{acc}}$ and $R = Q_{\mathrm{rej}}$ denote, respectively, the sets of accepting and rejecting states. The three-outcome measurement applied after each step is defined by the orthogonal projectors:

$$P_{\mathrm{acc}} = \sum_{q \in F} |q\rangle\langle q|\,, \qquad P_{\mathrm{rej}} = \sum_{q \in R} |q\rangle\langle q|\,, \qquad P_{\mathrm{cont}} = I - P_{\mathrm{acc}} - P_{\mathrm{rej}}.$$

Algorithm 3 Template Generation for a MM-1QFA Circuit

Require: Automaton $A = (Q, \Sigma, \delta, q_0, F, R)$, input length L

Ensure: Parametric circuit template with symbolic placeholders

$n \leftarrow \lceil \log_2 |Q| \rceil$
Initialise n qubits in state $|q_0\rangle$
for $i = 1$ **to** L **do**
    Insert symbolic gate $\boxed{U_{x_i}}$
    Insert symbolic measurement $\{P_{\mathrm{acc}}, P_{\mathrm{rej}}, P_{\mathrm{cont}}\}$
end for

This circuit skeleton remains independent of any particular input string and can be reused for all words of the same length.

To make the template executable, the symbolic gates are instantiated as concrete decompositions according to the input word $x = x_1 x_2 \ldots x_L \in \Sigma^L$. At each step, the automaton either halts or continues depending on the result of the three-outcome measurement.

Algorithm 4 Instantiation and Execution of a Compiled MM-1QFA Circuit

Require: Input $x = x_1 x_2 \ldots x_L$, circuit template, gate library or decomposition scheme for each U_σ

Ensure: Acceptance probability $p_M(x)$

$p \leftarrow 0$ ▷ Cumulative acceptance probability
$w \leftarrow 1$ ▷ Cumulative continuation weight
Initialise quantum state $|\psi_0\rangle = |q_0\rangle$
for $i = 1$ **to** L **do**
    Replace $\boxed{U_{x_i}}$ with gate decomposition for U_{x_i}
    Apply U_{x_i} to current state $|\psi_{i-1}\rangle$ to obtain $|\psi_i\rangle$
    Compute outcome probabilities:

$$p_{\mathrm{acc}}^{(i)} = \| P_{\mathrm{acc}} |\psi_i\rangle \|^2, \quad p_{\mathrm{rej}}^{(i)} = \| P_{\mathrm{rej}} |\psi_i\rangle \|^2, \quad p_{\mathrm{cont}}^{(i)} = \| P_{\mathrm{cont}} |\psi_i\rangle \|^2$$

    $p \leftarrow p + w \cdot p_{\mathrm{acc}}^{(i)}$
    if $p_{\mathrm{cont}}^{(i)} = 0$ **then**
        break ▷ Computation halts in accepting or rejecting subspace
    else
        Collapse $|\psi_i\rangle$ to $\frac{P_{\mathrm{cont}}|\psi_i\rangle}{\|P_{\mathrm{cont}}|\psi_i\rangle\|}$
        $w \leftarrow w \cdot p_{\mathrm{cont}}^{(i)}$
    end if
end for
return $p_M(x) = p$

This two-step strategy preserves the semantics of the original MM-1QFA, interleaving unitary evolution and projective measurements. On platforms supporting mid-circuit measurements and classical branching, this logic can be directly implemented. On static circuit backends, equivalent semantics may be approximated using classically post-processed measurement results or circuit unfolding.

4.2.3 Step-by-Step Examples

To illustrate how a MM-1QFA is compiled into a circuit, we now present explicit examples that walk through each phase of the translation. These examples highlight the distinguishing features of the measure-many model, including intermediate measurements, early halting, and hybrid classical-quantum control. Each circuit begins with the symbolic template generated by the compiler, proceeds through instantiation for a specific input word, and concludes with either a full execution or early termination based on measurement outcomes. The goal is to clarify how the abstract operational semantics of the automaton are faithfully realised in the corresponding gate-level circuit.

Early Acceptance on $x = ab$ Consider a MM-1QFA defined over the state space

$$Q = \{q_0, q_1, q_2\}, \quad \Sigma = \{a, b\}, \quad q_0 \text{ initial}, \quad Q_{\text{acc}} = \{q_2\}, \quad Q_{\text{rej}} = \varnothing.$$

Let the transition unitaries be defined as follows:

- U_a maps $|q_0\rangle \mapsto |q_1\rangle$,
- U_b maps $|q_1\rangle \mapsto |q_2\rangle$.

The goal is to process the input string $x = ab$ of length $L = 2$ and observe early acceptance behaviour.

1. **Qubit allocation.** The automaton has three basis states, so

   $$n = \lceil \log_2 3 \rceil = 2 \text{ qubits}.$$

   We assume an encoding where q_0, q_1, and q_2 correspond to $|00\rangle$, $|01\rangle$, and $|10\rangle$, respectively.

2. **Initialisation.** The register is prepared in $|q_0\rangle = |00\rangle$.

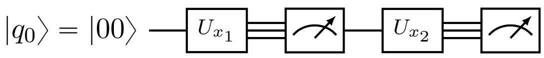
3. **Unitary evolution and template generation.** Since $x = ab$ has length 2, the compiler generates a circuit with two slots for U_{x_1} and U_{x_2}, each followed by a three-outcome measurement. The symbolic circuit template is shown in subfigure 4.5a.

4. **Instantiation.** We substitute U_a and U_b into the template to obtain the concrete circuit for $x = ab$, shown in subfigure 4.5b.

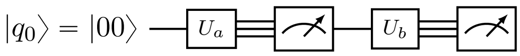
5. **Gate synthesis.** Suppose U_a and U_b are implemented using Pauli-X gates acting on the encoded state transitions (e.g., $|00\rangle \mapsto |01\rangle$ and $|01\rangle \mapsto |10\rangle$). The resulting circuit uses two X gates interleaved with measurements, as shown in subfigure 4.5c.

6. **Measurement semantics.** After applying U_a, the system moves from $|q_0\rangle$ to $|q_1\rangle$. Since $q_1 \notin Q_{\text{acc}} \cup Q_{\text{rej}}$, the first measurement yields CONTINUE. After applying U_b, the system transitions to $|q_2\rangle \in Q_{\text{acc}}$, so the second measurement yields ACCEPT and the computation halts. Thus,

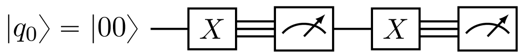
$$p_M(ab) = 1.$$



(a) Step 1: Symbolic circuit template



(b) Step 2: Instantiation for $x = ab$



(c) Step 2: Gate-level synthesis

Figure 4.5: Compilation stages for Example 4.2.3. Input ab causes an early accept: after U_b, the system reaches $|q_2\rangle$, a final state.

Early Rejection on $x = b$ Consider a MM-1QFA defined over the state space

$$Q = \{q_0, q_1\}, \quad \Sigma = \{b\}, \quad q_0 \text{ initial}, \quad Q_{\text{acc}} = \varnothing, \quad Q_{\text{rej}} = \{q_1\}.$$

Let the transition function be defined such that:

- U_b maps $|q_0\rangle \mapsto |q_1\rangle$.

We examine the behaviour of the automaton on the input string $x = b$ of length $L = 1$.

1. **Qubit allocation.** The automaton has two basis states, so the number of required qubits is
$$n = \lceil \log_2 2 \rceil = 1.$$
The classical states are encoded as $|q_0\rangle = |0\rangle$ and $|q_1\rangle = |1\rangle$.

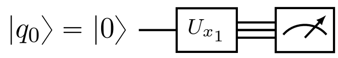
2. **Initialisation.** The register is initialised in the state $|q_0\rangle = |0\rangle$, which requires no gate-level preparation.

3. **Unitary evolution and template generation.** The input $x = b$ consists of a single symbol, so the compiler generates a template with one symbolic unitary gate followed by a measurement. This is shown in subfigure 4.6a.

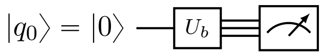
4. **Instantiation.** The placeholder U_{x_1} is replaced by the unitary U_b, as shown in subfigure 4.6b.

5. **Gate synthesis.** Assuming U_b maps $|0\rangle \mapsto |1\rangle$, it can be implemented directly as a Pauli-X gate:
$$U_b = X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

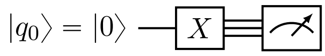
6. **Measurement semantics.** The measurement is performed immediately after U_b. Since $U_b\,|q_0\rangle = |q_1\rangle$ and $q_1 \in Q_{\text{rej}}$, the outcome is REJECT with probability 1. The computation halts after a single step, and the automaton returns output 0:
$$p_M(b) = \|P_{\text{rej}}\,|q_1\rangle\|^2 = 1.$$



(a) Step 1: Symbolic circuit template



(b) Step 2: After instantiation for $x = b$



(c) Step 2: Gate-level synthesis

Figure 4.6: Compilation stages for Example 4.2.3. The input b triggers an immediate transition to $|q_1\rangle$, a rejecting state. The automaton halts after one step and returns REJECT.

4.3 Unitary Operators Instantiation

Once the circuit skeleton is constructed for a given MO-1QFA or MM-1QFA, the remaining compilation step consists in instantiating the placeholder unitaries U_σ with actual gate-level operators. This section presents the main strategies available to achieve such instantiation, highlighting the trade-offs between offline preprocessing and dynamic runtime synthesis.

4.3.1 Offline Synthesis

The *offline synthesis* strategy precomputes a gate decomposition for each unitary matrix U_σ associated with the alphabet Σ. This approach is particularly suitable when the automaton is fixed and used to process multiple inputs of the same language class. The steps are:

- For every $\sigma \in \Sigma$, extract the unitary matrix U_σ defined by the automaton's transition function δ.
- Decompose U_σ into a circuit of elementary gates from a fixed universal set (e.g., Clifford+T or $\{\text{CNOT}, R_z, H\}$).
- Store each gate sequence as a reusable fragment in a gate library.

This method guarantees high performance at runtime, as no decomposition is needed during execution. However, it requires more memory to store all precompiled gate sequences, and it lacks adaptability in contexts where U_σ changes dynamically or is defined procedurally.

4.3.2 Template-Based Parameter Loading

An alternative method is *template-based parameter loading*, where the circuit skeleton includes parametrized gates (e.g., Euler-angle rotations) and the actual rotation angles are injected at runtime based on the specific U_σ required. This is achieved by:

- Designing each U_σ as a composition of generic rotation gates (e.g., $R_z(\theta_1)R_y(\theta_2)R_z(\theta_3)$).
- Computing the angles $\theta_1, \theta_2, \theta_3$ classically using a synthesis algorithm (e.g., ZYZ decomposition) from the matrix representation of U_σ.

- Populating the parametrized gates of the circuit with the computed angles just before execution.

This strategy supports adaptive and memory-efficient compilation, especially useful when the automaton model is generated on-the-fly or when circuits are embedded in larger configurable pipelines. The downside is the runtime overhead incurred by angle computation and dynamic loading.

4.3.3 Hybrid and Optimized Approaches

In practice, a hybrid scheme combining both methods is often adopted. Common unitary matrices with known decompositions can be stored offline, while less frequent or dynamically generated ones are handled through runtime parameter loading. Furthermore, if the automaton contains symmetries (e.g., cyclic state transitions), structural optimizations can reduce the number of distinct unitaries needed, enabling further compression of the circuit template.

4.3.4 Summary

Unitary instantiation closes the automaton-to-circuit translation by assigning concrete quantum operations to each input-driven evolution. Offline synthesis prioritizes speed and repeatability; template-based methods emphasize flexibility and memory economy. The choice depends on the application domain—static recognizers may favor offline strategies, while programmable quantum systems benefit from dynamic parameter loading.

5. Conclusion

This study has revisited the theoretical foundations and the practical compilation of QFAs. Starting from a detailed taxonomy that unifies over thirty years of scattered literature, we framed the principal one-way variants, namely the MO-1QFA and the MM-1QFA, within a single terminology. The resulting classification clarifies relations among models and supplies a consistent language for future analyses.

Building on that framework, we introduced a compilation pipeline that translates high-level QFA specifications into architecture-independent quantum circuits. The pipeline separates template generation from parameter instantiation: templates encode the automaton structure once, whereas numerical parameters are loaded either offline or at run time. This separation permits the execution of QFAs on present NISQ devices without sacrificing portability or reusability.

By producing executable gate-level designs for both MO-1QFAs and MM-1QFAs, the thesis bridges finite-memory language recognisers and quantum software stacks. The approach advances theoretical understanding of automaton-to-circuit translation and delivers practical tools for embedding quantum recognisers into larger workflows, such as verification and pattern-matching tasks.

Future investigations may target two-way and hybrid automata, automated minimisation, and quantitative studies of expressive trade-offs in quantum-classical hybrids. The convergence of automata theory and circuit design outlined here reinforces the position of QFAs as foundational elements of quantum computing and opens avenues for systematic, automata-driven programming on forthcoming hardware generations.

Abbreviations

ω-QFA	ω-Quantum Finite Automaton
kTQCFA	k-Tape Quantum Finite Automaton with Classical States
l-VFA	l-valued Finite Automaton
l-VPDA	l-valued Pushdown Automaton
$CCNOT$	Toffoli (Controlled-Controlled-X)
$CNOT$	Controlled-NOT
$CSWAP$	Fredkin (Controlled-SWAP)
CZ	Controlled-Z
H	Hadamard
$SWAP$	Swap
S	Phase
REG	Regular Language
1DFA	One-Way Deterministic Finite Automaton
1PFA	One-Way Probabilistic Finite Automaton
1QCFA	One-Way Quantum Finite Automaton with Classical States
1QFA	One-Way Quantum Finite Automaton
1QFA(2)	One-Way Quantum Finite Automaton with Two Observables
1QFkCA	Quantum Finite k-Counter Automaton
1gQFA	Generalised Quantum Finite Automaton
2DFA	Two-Way Deterministic Finite Automaton
2FA	Two-Way Finite Automaton
2NFA	Two-Way Nondeterministic Finite Automaton
2PFA	Two-Way Probabilistic Finite Automaton
2QCFA	Two-Way Quantum Finite Automaton with Classical States
2QF1CA	Two-Way Quantum Finite One-Counter Automaton
2QFA	Two-Way Quantum Finite Automaton
2T1QFA(2)	Two-Tape One-Way Quantum Finite Automaton with Two Heads
2TQCFA	Two-Tape Quantum Finite Automaton with Classical States
CFA	Classical Finite Automaton

CL-1QFA	One-Way Quantum Finite Automaton with Control Language
CSD	Cosine-Sine Decomposition
DFA	Deterministic Finite Automaton
DJ	Deutsch-Josza
E-1QFA	Enhanced One-Way Quantum Finite Automaton
FA	Finite Automaton
GKSL	Gorini-Kossakowski-Sudarshan-Lindblad
HQC	Hybrid Quantum-Classical
LQFA	Latvian Quantum Finite Automaton
ML-MMQFA	Multi-Letter Measure Many Quantum Finite Automaton
ML-QFA	Multi-Letter Quantum Finite Automaton
ML-RevQFA	Multi-Letter Reversible Quantum Finite Automaton
MM-1gQFA	Measure Many Generalised Quantum Finite Automaton
MM-1QFA	Measure Many One-Way Quantum Finite Automaton
MM-2QFA	Measure Many Two-Way Quantum Finite Automaton
MO-1gQFA	Measure Once Generalised Quantum Finite Automaton
MO-1QFA	Measure Once One-Way Quantum Finite Automaton
MO-2QFA	Measure Once Two-Way Quantum Finite Automaton
MON-1QFA	Measure-Only One-Way Quantum Finite Automaton
NFA	Nondeterministic Finite Automaton
NISQ	Noisy Intermediate-Scale Quantum
NQFA	Non-Deterministic Quantum Finite Automaton
PFA	Probabilistic Finite Automaton
POVM	Positive Operator-Valued Measure
PQC	Parameterised Quantum Circuit
PQFA	Postselection Quantum Finite Automaton
QAM	Quantum Arthur-Merlin

QF1CA	Quantum Finite One-Counter Automaton
QFA	Quantum Finite Automaton
QFT	Quantum Fourier Transform
QIP	Quantum Interactive Proof
QTM	Quantum Turing Machine
Qubit	Quantum Bit
RevQFA	Reversible One-Way Quantum Finite Automaton
RTQ1CA	Real-Time Quantum One-Counter Automaton
VQA	Variational Quantum Algorithm
VQE	Variational Quantum Eigensolver

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