Assignment - 2

Q2 Laplacian matrix of a graph. Let A be the incidence matrix of a directed graph with n nodes and m edge. The Laplacian matrix associated with the graph is defined as $L = AA^T$ which is the Gram matrix of A^T. It is named after the mathematician Pierre-Simon Laplace

a) Show that $D(v) = v^T L v$ where $D(v)$ is the Dirichlet energy.
A quantitative measure of this is the function of v given by

$$D(v) = \|A^T v\|^2 \quad -①$$

Hence, $D(v) = (A^T v)^T (A^T v)$ $(\because \text{Norm } 2 \text{ for } (A^T v))$

$\therefore D(v) = (v^T A)(A^T v)$

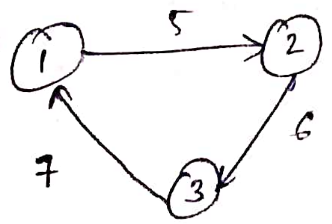
$\therefore D(v) = v^T (AA^T) v$

$\therefore D(v) = v^T L v$ $\quad \because$ Laplacian matrix $L = AA^T$

proved that Dirichlet Energy $= v^T L v$.

b) Describe the entries of L.

Lets consider the following directed graph



Incident Matrix $(A) = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$\therefore L = AA^T = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$

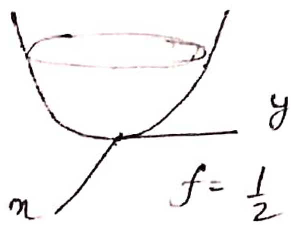
Here,

$$L_{ij} = \begin{cases} \text{degree of node} & ; \text{if } i = j \\ 0 & ; \text{if } i \neq j \text{ \& } i \text{ \& } j \text{ are connected} \\ -1 & ; \text{if } i \neq j \text{ \& } i \text{ and } j \text{ are not connected} \end{cases}$$

Q3

Second derivatives $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$a > 0$ and $ac > b^2$



$f = \frac{1}{2} x^T S x > 0$

The graph of $2f = ax^2 + 2bxy + cy^2$ is a bowl when S is positive definite.

If S has a negative eigen value $\lambda < 0$ the graph goes below 0. There is a maximum if S is negative definite (all $\lambda < 0$, upside down bowl). Or a saddle point when S has both positive and negative eigen values. A saddle point matrix is "Indefinite".

The energy $x^T S x = 2x_1 x_2$ certainly has a saddle point and not a minimum at $(0,0)$. What symmetric matrix S produces this energy? What are its eigen values?

The graph $2f = x^T S x$ is a bowl. There is a minimum if S is positive definite, and a maximum if S is negative definite.

Saddle point : when S has both positive & negative eigen values

Now, $x^T S x = 2x_1 x_2$ has a saddle point and not a min at $(0,0)$.

$x^T S x = a x_1^2 + 2b x_1 x_2 + c x_2^2$ for $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

& $x^T S x = 2x_1 x_2$ is given to us for energy

Thus, $a = c = 0$ and $b = 1$

So, $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now, calculating eigen values for $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$\Rightarrow |S - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - 1 = 0$

$\Rightarrow \boxed{\lambda = \pm 1}$

So, we can see that $\lambda = +1$ and -1 i.e. both positive and negative values which proves that it has saddle point.