Assignment (2)

Q:- (2) Laplacian Matrix of a graph. Let A be the incidence matrix of directed Graph of n nodes and m edges.

Laplacian is defined as $L = AA^T$

Gram Matrix of ~~AA~~ $A = A^TA$

Gram Matrix of $A^T = AA^T$ (Laplacian Matrix)

(Pierre - Simon Laplace)

$A \in R^{n \times m}$

n-nodes m-edges.

(a) Show that $D(V) = V^T L V$ where $D(V)$ is Dirichlet Energy?

<u>Dirichlet's Energy</u>:- $D(V) = ||A^T V||^2$ —— (1)

Hence,

from ($\because ||V||^2 = V^T V$.)

$$D(V) = (A^T V)^T (A^T V)$$

from $\because (AB)^T = (B^T A)$

$$D(V) = V^T (AA^T) V$$

$$\boxed{D(V) = V^T (L) V}$$

Since ($\because L = AA^T$)

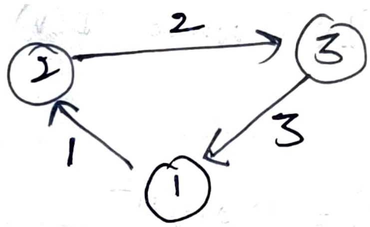
Hence proved

$$\boxed{D(V) = V^T L V}$$

Where L is Laplacian matrix $= AA^T$

2. (b) Describe the Entries of Laplace Matrix.

Let's consider following directed graph (G)



Incident Matrix (A) $= \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

$L = A \cdot A^T$

$A^T = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$

$$L = \overset{A}{\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}} \overset{A^T}{\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}}$$

$$= \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

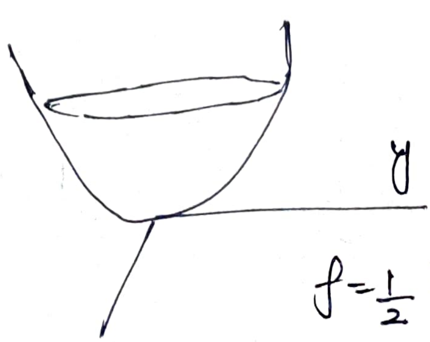
~~So~~ Hence :-

$$\boxed{L_{ij} = \begin{cases} \text{degree of node} & \text{if } i=j \\ -1 & : \text{if } i \neq j \text{ and } i \text{ and } j \text{ are connected} \\ 0 & : \text{if } i \neq j \text{ and } i \text{ and } j \text{ are not connected} \end{cases}}$$

(Q3) Second derivative

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$a > 0 \quad \text{and} \quad ac > b^2$$



$$f = \frac{1}{2} x^T S x > 0$$

The Graph of $2f = ax^2 + 2bxy + cy^2$ is a bowl where S is positive definite and symmetric also.

If $\lambda < 0$, the Graph Goes below zero
Then S is negative definate (upside down bowl)

Matrices having Saddle point (having both positive and negative Eigen values) are "Indefinate"

$x^T S x = 2x_1x_2$ has a Saddle point Matrices and not minimum at $(0,0)$. What Symmetric matrix S produces this energy? What are Eigen value?

The Graph $2f = x^T S x$ is bowl shaped
There Exist minimum if ~~x~~ S is positive definite and maximum if S is negative definite.

$x^T S x = ax_1^2 + 2bx_1x_2 + cx_2^2$ for $S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$x^T S x = 2x_1x_2$

$\therefore$ $a = 0$, $c = 0$, $b = 1$

$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Now calculating Eigen values for $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$|S - \lambda I| = 0$$

$$\begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 - 1 = 0$$

$$(\lambda + 1)(\lambda - 1) = 0$$

$$\boxed{\begin{matrix} \lambda = -1 \\ \lambda = 1 \end{matrix}}$$

Since λ is having both positive and negative Eigen values, it mean and confirms that it has saddle point;

A point where both positive and negative eigen values exists.