Linear Classification

MA515: Foundations of Data Science

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Panoramic View of the Talk

1. Classification Problem
2. Linear Regression as Classifier
3. Logistic Regression
4. Linear Discriminant Analysis

Example (Loan approval)

- Given the properties x of a customer like age, income, liability, assets, job
- To predict $f(x)$, the loan approved or not.

Example (Graduate admission)

- Given the properties x of a student like grades, GRE score, university ranking, subject of interest
- To predict $f(x)$, student get admission or not.

Linear Classifier

- A Classifier partitions input space into decision regions
- An input/feature space which is linearly separable can be partitioned by a linear decision boundary.

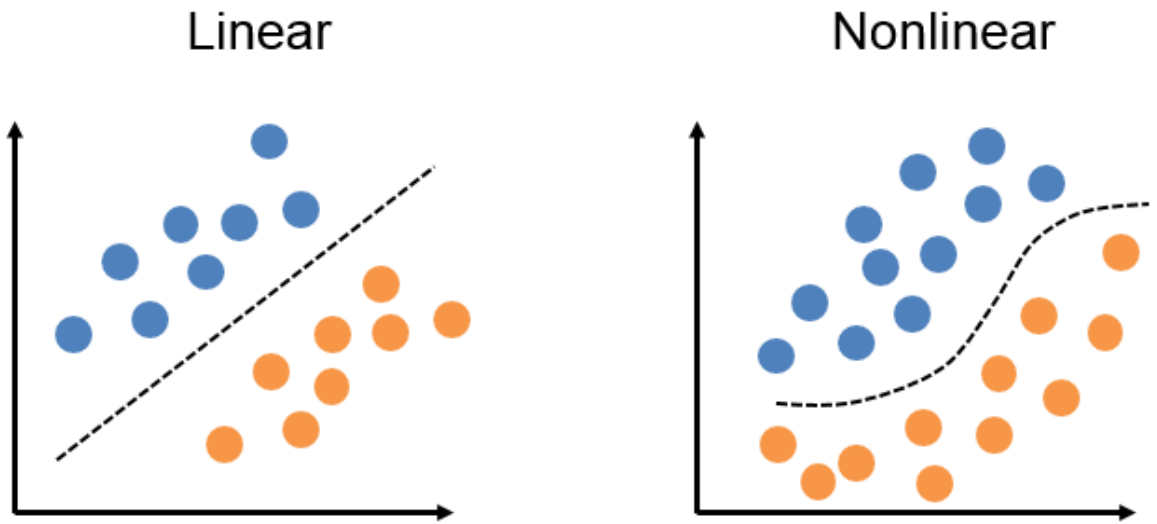


Figure: Linear vs Non-linear Classifier

Linear Regression as Classifier

Linear Regression as Classifier

- Suppose, we have two classes and 1 feature only.
- We substitute the labels with 0 and 1.
- Fit the SLR and if the predicted value is ≤ 0.5, we predict the label as 0 and if it is greater than 0.5, we predict the label as 1.

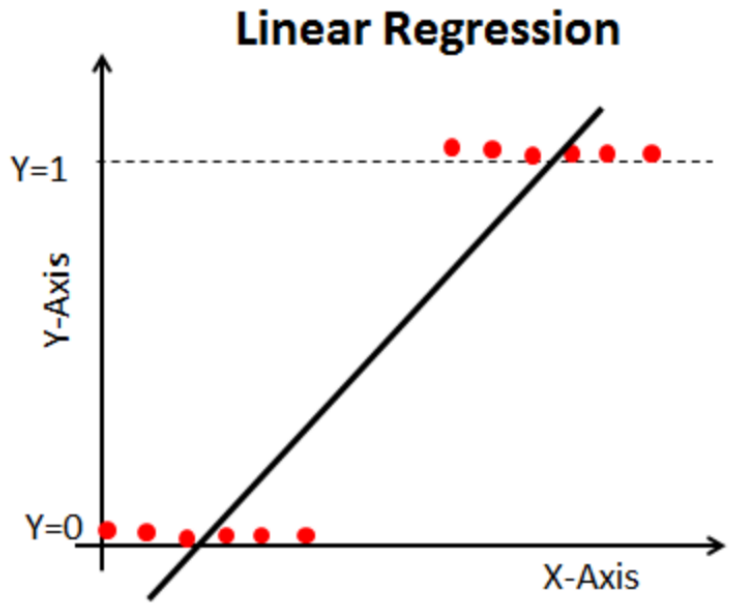


Figure: Linear Regression as Classifier

- Get the model $\hat{f} = X\hat{\beta}$, where

$$\hat{\beta} = (X^T X)^{-1} X^T Y.$$

- For a new data point U, if the predicted value $U\hat{\beta} \leq 0.5$, we predict label 0 and if it is greater than 0.5, we predict label 1.
- Here U is a row vector $U = (u_1, u_2, \ldots, u_p)$.

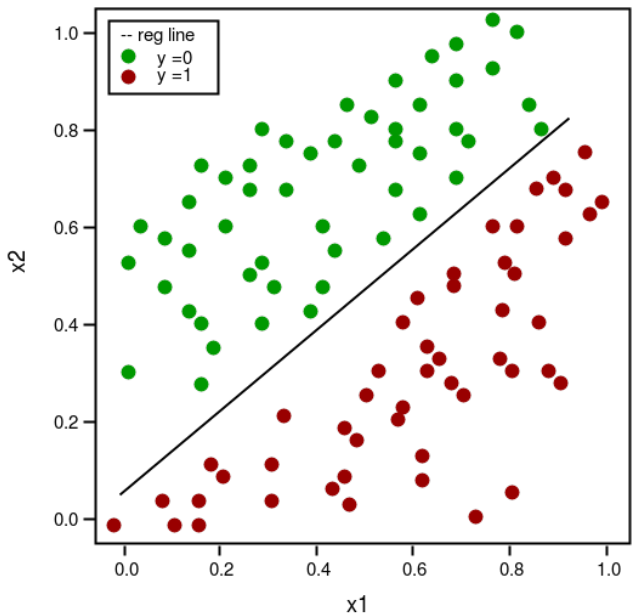


Figure: Linear regression as a classifier with 2 features

Sensitivity of Linear Regression

Linear regression is sensitive to imbalanced data. An Imbalanced Dataset refers to a situation where the number of instances across different classes in a classification problem is not evenly distributed.

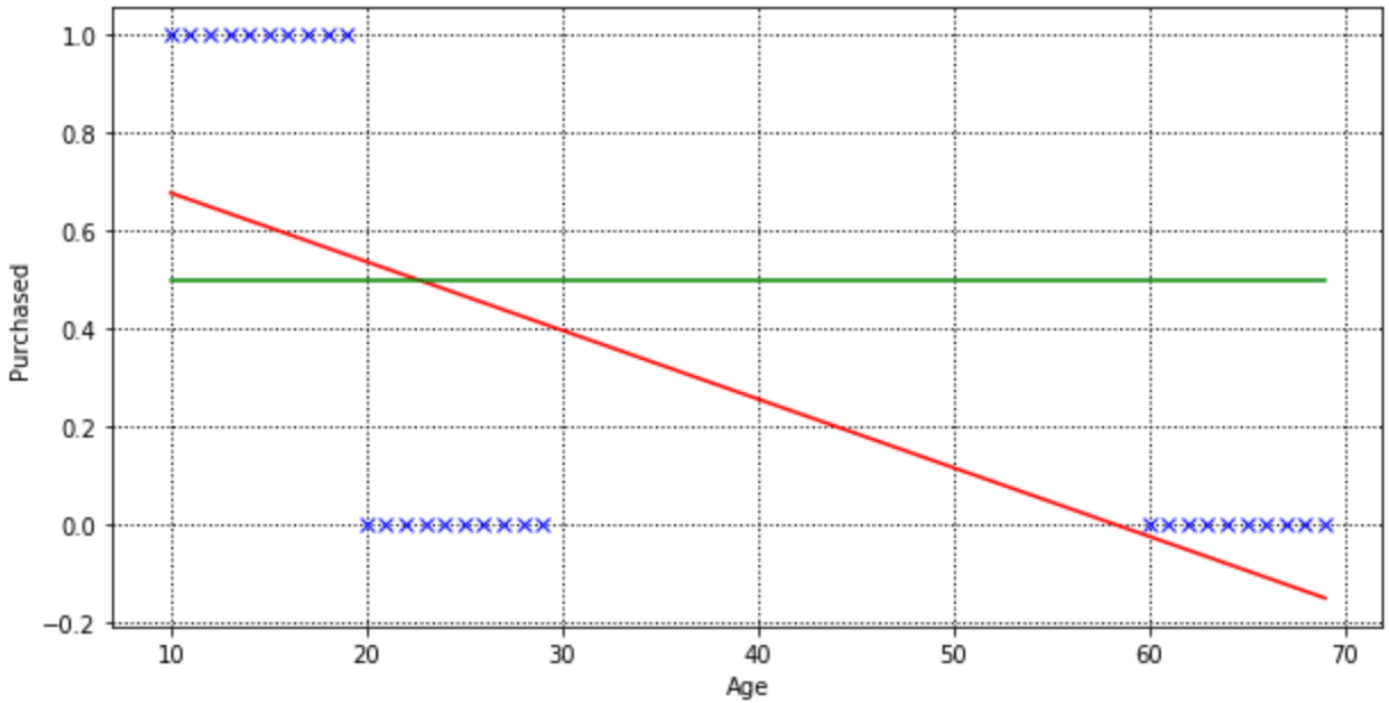


Figure: Linear regression sensitivity

Logistic Regression as Classifier

Definition

A sigmoid function is any mathematical function whose graph is *S*-shaped or sigmoid curve. A sigmoid function is a bounded, differentiable, real function and has a positive derivative at each point.

Example (Logistic function)

$$f(x) = \frac{1}{1 + e^{-x}}, \ x \in \mathbb{R}.$$

Example (Error function)

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \ x > 0.$$

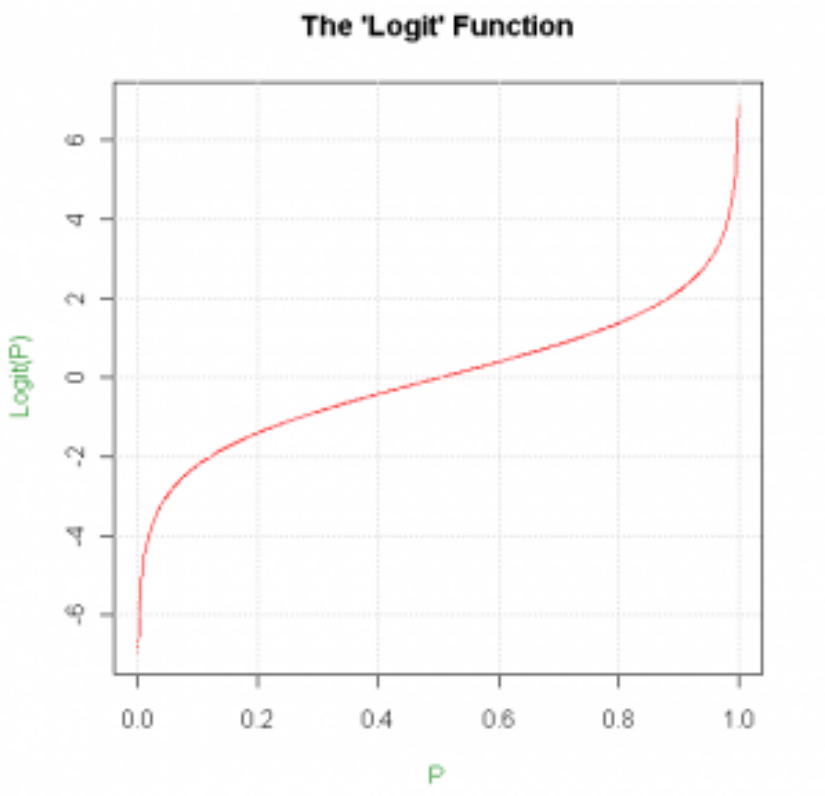
Example (Hyperbolic tangent or Tan Hyperbolic)

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \ x \in \mathbb{R}.$$

Definition (Logit Function)

The logit function takes p values in the range $(0, 1)$ and transform them to y values in the interval $(-\infty, \infty)$ and is defined by

$$logit(p) = \log\left(\frac{p}{1-p}\right).$$



Definition (Inverse-logit Function)

The inverse-logit (or logistic) function does the reverse, and takes x values along the real line and transform them to y values in the interval $(0, 1)$ and is defined by

$$logit^{-1}(x) = logistic(x) = \frac{e^x}{1 + e^x}.$$

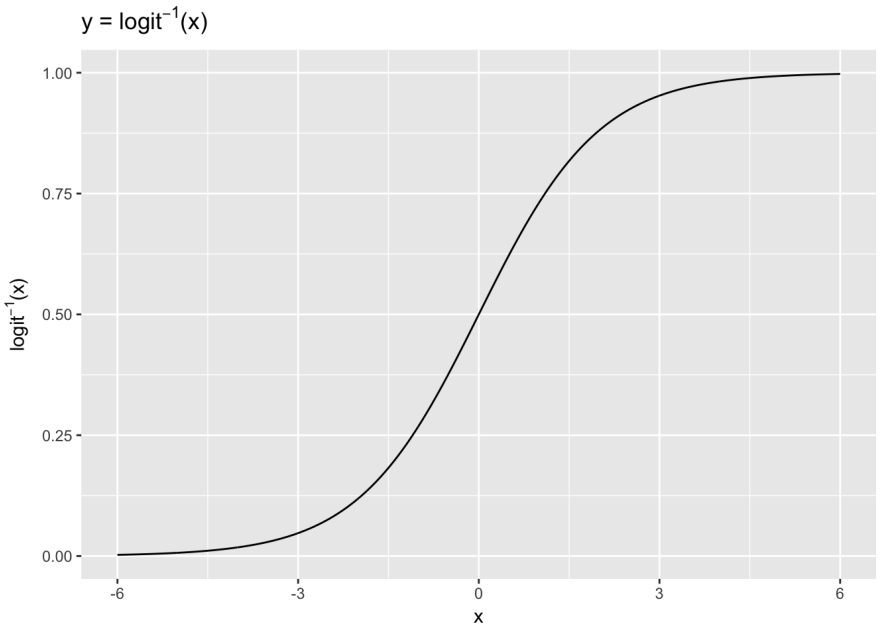


Figure: Logistic function

- In logistic regression rather than predicting 1 and 0, we predict the probability of the response variable.
- The logistic regression model is given by

$$\log\left(\frac{p(x)}{1-p(x)}\right) = \beta_0 + \beta_1 x,$$

or equivalently

$$p(x) = \frac{e^{\beta_0+\beta_1 x}}{1+e^{\beta_0+\beta_1 x}}.$$

- Here x is the independent variable and $p(x)$ is the probability of response variable taking value 1.

Estimating β_0 and β_1

- Our aim is to use the training data $\{(x_i, y_i), i = 1, 2, \ldots, n\}$, to estimate the parameters β_0 and β_1.
- We can use the maximum likelihood estimation technique.
- The likelihood function is

$$L(\beta_0, \beta_1 | x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} p(x_i)^{y_i} (1 - p(x_i))^{1-y_i}.$$

- The log-likelihood function is

$$\begin{aligned}
\log L(\beta_0, \beta_1 | x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} [y_i \log(p(x_i)) + (1 - y_i) \log(1 - p(x_i))] \\
&= \sum_{i=1}^{n} \log(1 - p(x_i)) + \sum_{i=1}^{n} y_i \log\left(\frac{p(x_i)}{1 - p(x_i)}\right) \\
&= -\sum_{i=1}^{n} \log\left(1 + e^{\beta_0 + \beta_1 x_i}\right) + \sum_{i=1}^{n} y_i (\beta_0 + \beta_1 x_i).
\end{aligned}$$

Estimating β_0 and β_1 contd...

- The log-likelihood function is

$$\log L(\beta_0, \beta_1 | x_1, x_2, \ldots, x_n) = -\sum_{i=1}^{n} \log\left(1 + e^{\beta_0 + \beta_1 x_i}\right) + \sum_{i=1}^{n} y_i(\beta_0 + \beta_1 x_i).$$

- If we take derivatives with respect to β_0 and β_1 of the log-likelihood function and put these to equal to 0 to get the critical points.
- We see that it is not possible to solve these equations in term of β_0 and β_1.
- We choose β_0 and β_1, which maximizes the log-likelihood function.
- We need some **iterative method** and generally **Newton's method** or **stochastic gradient method** is used.

Linear Discriminant Analysis

Example

Suppose you have 4 coins. Three of them are fair coins but the fourth is a weighted coin which has 80% chance of landing heads up. You take a coin randomly out of your pocket and toss it. Suppose it lands heads up. What is the probability that the coin is the weighted coin ?

Prior: $\pi(\text{fair}) = 0.75$, $\pi(\text{weighted}) = 0.25$.

Likelihood: $\mathbb{P}(\text{head}|\text{fair}) = 0.5$ and $\mathbb{P}(\text{head}|\text{weighted}) = 0.8$.

Posterior:

$$\mathbb{P}(\textit{fair}|\textit{head}) = \frac{\mathbb{P}(\textit{head}|\textit{fair})\mathbb{P}(\textit{fair})}{\mathbb{P}(\textit{head}|\textit{fair})\mathbb{P}(\textit{fair}) + \mathbb{P}(\textit{head}|\textit{weighted})\mathbb{P}(\textit{weighted})} = 0.652.$$

$$\mathbb{P}(\textit{weighted}|\textit{head}) = 0.348.$$

Bayes' theorem - different representations

Discrete parameter discrete data

$$p_{\Theta|X}(\theta|x) = \frac{p_{X|\Theta}(x|\theta)p_{\Theta}(\theta)}{p_X(x)}$$

Discrete parameter continuous data

$$p_{\Theta|X}(\theta|x) = \frac{f_{X|\Theta}(x|\theta)p_{\Theta}(\theta)}{f_X(x)}$$

Continuous parameter discrete data

$$f_{\Theta|X}(\theta|x) = \frac{p_{X|\Theta}(x|\theta)f_{\Theta}(\theta)}{p_X(x)}$$

Continuous parameter continuous data

$$f_{\Theta|X}(\theta|x) = \frac{f_{X|\Theta}(x|\theta)f_{\Theta}(\theta)}{f_X(x)}$$

Example

Find the posterior distribution of probability of heads denoted by θ on a coin. Suppose x heads are observed in n tosses.

Solution: Suppose we assume a uniform prior for Θ i.e. $f_\Theta(\theta) = 1,\ 0 < \theta < 1$. Also

$$p_{X|\Theta}(x|\theta) = \binom{n}{x}\theta^x(1-\theta)^{n-x}.$$

We then have

$$\begin{aligned} f_{\Theta|X}(\theta|x) &= \frac{\binom{n}{x}\theta^x(1-\theta)^{n-x}}{\int_0^1 \binom{n}{x}\theta^x(1-\theta)^{n-x}d\theta} \\ &= \frac{(n+1)!\theta^x(1-\theta)^{n-x}}{x!(n-x)!}. \end{aligned}$$

- Suppose that, we wish to classify an observation into one of K classes, where $K \geq 2$.
- Let $\pi_k = \mathbb{P}(Y = k)$ represent the **prior probability** that a randomly chosen observation comes from the kth class.
- Let

$$f_k(x) = \mathbb{P}(X = x | Y = k),$$

  denote the density function of X for an observation that come from the class k.
- In other words, $f_k(x)$ represent the **likelihood** that an observation in the kth class has $X \approx x$.

- Using Bayes' theorem

$$\mathbb{P}(Y = k|X = x) = \frac{f_k(x)\pi_k}{\sum_{l=1}^{K} f_l(x)\pi_l}.$$

- To get $\mathbb{P}(Y = k|X = x)$, we put the estimate of π_k and $f_k(x)$.
- These estimates are observed from the training data.

- We assume that $p = 1$, i.e. we have one predictor only.
- Assume further that $f_k(x)$ is Gaussian i.e.

$$f_k(x) = \frac{1}{\sigma_k\sqrt{2\pi}} e^{-\frac{(x-\mu_k)^2}{2\sigma_k^2}},$$

  where μ_k and σ_k^2 are the mean and variance parameters for the kth class.
- Let us further assume that the variance among K classes is same i.e. $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_K^2 = \sigma^2$.
- Thus

$$\mathbb{P}(Y = k|X = x) = \frac{\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu_k)^2}{2\sigma^2}} \pi_k}{\sum_{l=1}^{K} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu_l)^2}{2\sigma^2}} \pi_l}.$$

- The Bayes classifier assign an observation to the class for which $\mathbb{P}(Y = k|X = x)$ is higher.
- Since the denominator is same, hence the classifier assign $X = x$ to the class where numerator is higher i.e.

$$g_k(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu_k)^2}{2\sigma^2}} \pi_k$$

is highest.

- Equivalently, $X = x$ is assigned to the class for which

$$\log g_k(x) = \log(\pi_k) - \log(\sigma\sqrt{2\pi}) - \frac{(x-\mu_k)^2}{2\sigma^2}$$

is highest.

- Alternatively, $X = x$ is assigned to the class for which

$$\delta_k(x) = x\frac{\mu_k}{\sigma^2} - \frac{\mu_k^2}{2\sigma^2} + \log(\pi_k)$$

is largest.

- For simplicity assume that $k = 2$ and $\pi_1 = \pi_2$.
- The classifier assign an observation $X = x$ to the class 1 if

$$x\frac{\mu_1}{\sigma^2} - \frac{\mu_1^2}{2\sigma^2} + \log(\pi_1) > x\frac{\mu_2}{\sigma^2} - \frac{\mu_2^2}{2\sigma^2} + \log(\pi_2).$$

- Equivalently, $X = x$ is assigned to the class 1, if

$$x(\mu_1 - \mu_2) > \frac{1}{2}(\mu_1^2 - \mu_2^2) \implies x > \frac{\mu_1 + \mu_2}{2},$$

  provided $\mu_1 > \mu_2$.
- The Bayes' decision boundary in this case will be

$$x = \frac{\mu_1 + \mu_2}{2}.$$

Estimation of parameters in LDA

- We estimate the parameters from the training data.
- The mean for kth class is estimated by

$$\hat{\mu}_k = \frac{1}{n_k} \sum_{\{i:y_i=k\}} x_i.$$

- The common variance is estimated by

$$\hat{\sigma}^2 = \frac{1}{N-K} \sum_{k=1}^{K} \sum_{\{i:y_i=k\}} (x_i - \hat{\mu}_k)^2.$$

- Further, $\hat{\pi}_k = \frac{n_k}{N}$.
- Here N is the total number of observations and n_k counts the number of observations in the kth class.

LDA for Bivariate Binary Classification

Setup of LDA

- Goal: Classify an observation $\mathbf{x}$ into one of two classes $\mathcal{C}_1$ or $\mathcal{C}_2$
- Assume each class is normally distributed with:

$$\mathbf{x} \mid \mathcal{C}_k \sim \mathcal{N}(\boldsymbol{\mu}_k, \Sigma), \quad k = 1, 2$$

- **Key assumption:** Both classes share the same covariance matrix Σ
- LDA finds a **linear boundary** that best separates the classes

Bayes decision rule:

$$\text{Assign } \mathbf{x} \text{ to } \mathcal{C}_1 \text{ if } p(\mathcal{C}_1 \mid \mathbf{x}) > p(\mathcal{C}_2 \mid \mathbf{x})$$

Using Gaussian likelihoods with same Σ, the rule simplifies to:

$$\delta_k(\mathbf{x}) = \mathbf{x}^\top \Sigma^{-1} \boldsymbol{\mu}_k - \frac{1}{2} \boldsymbol{\mu}_k^\top \Sigma^{-1} \boldsymbol{\mu}_k + \ln \pi_k$$

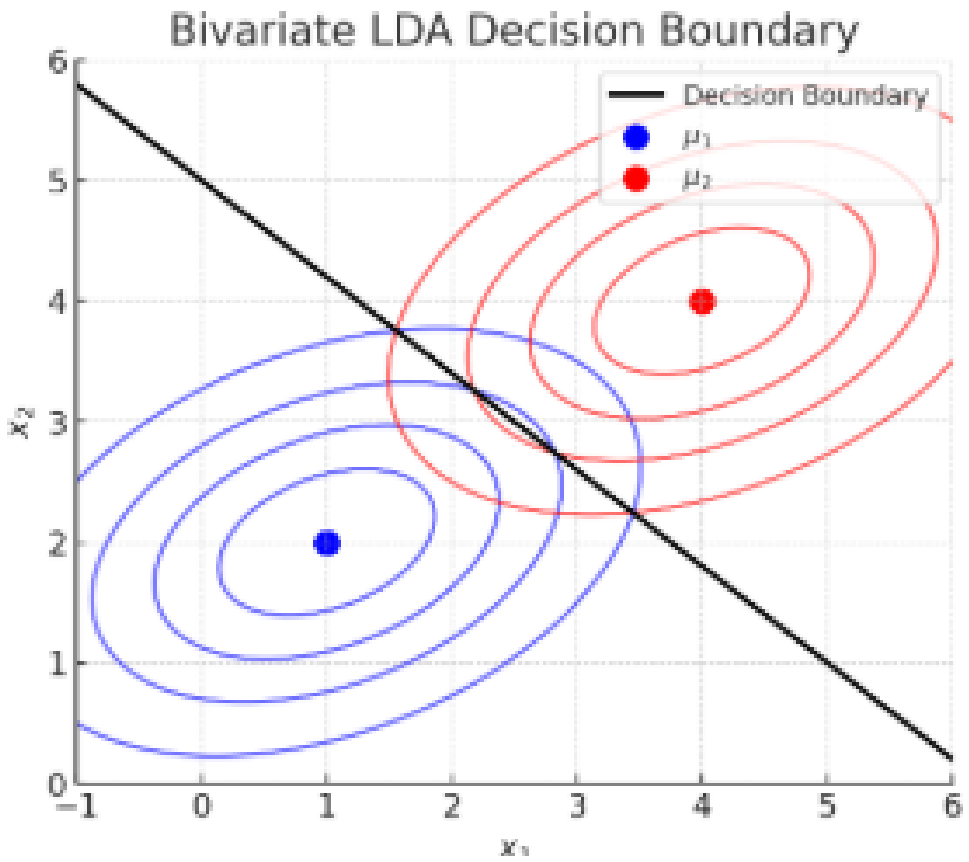
Decision Boundary Equation

Set $\delta_1(\mathbf{x}) = \delta_2(\mathbf{x})$:

$$(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)^\top \Sigma^{-1} \mathbf{x} = \frac{1}{2}\left(\boldsymbol{\mu}_1^\top \Sigma^{-1} \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2^\top \Sigma^{-1} \boldsymbol{\mu}_2\right) - \ln \frac{\pi_1}{\pi_2}$$

- Linear equation in $\mathbf{x}$
- Normal vector: $\Sigma^{-1}(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)$

LDA in Action



- Ellipses: Contours of the bivariate normal distributions
- Solid line: Decision boundary
- Points: Sample observations

THANK YOU!