Quadratic Discriminant Analysis (QDA)
Bivariate Binary Classification

MA515: Foundations of Data Science

Arun Kumar

Department of Mathematics, IIT Ropar

August 14, 2025

- QDA is a classification method based on **Bayes' theorem**.
- Assumes each class k has:

$$\mathbf{x} \mid \mathcal{C}_k \sim \mathcal{N}(\boldsymbol{\mu}_k, \Sigma_k)$$

- Unlike LDA, QDA allows **different** covariance matrices: $\Sigma_1 \neq \Sigma_2$.
- This results in a **quadratic decision boundary**.

Bayes decision rule:

$$\text{Assign } \mathbf{x} \text{ to } \mathcal{C}_k \text{ if } \pi_k f_k(\mathbf{x}) > \pi_j f_j(\mathbf{x})$$

where:

$$f_k(\mathbf{x}) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_k)^\top \Sigma_k^{-1}(\mathbf{x} - \boldsymbol{\mu}_k)\right)}{\sqrt{(2\pi)^p |\Sigma_k|}}$$

Taking logs and simplifying gives the discriminant:

$$\delta_k(\mathbf{x}) = -\frac{1}{2}\ln|\Sigma_k| - \frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_k)^\top \Sigma_k^{-1}(\mathbf{x} - \boldsymbol{\mu}_k) + \ln \pi_k$$

Set $\delta_1(\mathbf{x}) = \delta_2(\mathbf{x})$:

$$(\mathbf{x} - \boldsymbol{\mu}_1)^\top \Sigma_1^{-1} (\mathbf{x} - \boldsymbol{\mu}_1) - (\mathbf{x} - \boldsymbol{\mu}_2)^\top \Sigma_2^{-1} (\mathbf{x} - \boldsymbol{\mu}_2) = \ln \frac{|\Sigma_2|}{|\Sigma_1|} + 2 \ln \frac{\pi_1}{\pi_2}$$

This can be expanded into:

$$\mathbf{x}^\top (\Sigma_1^{-1} - \Sigma_2^{-1}) \mathbf{x} + (-2\Sigma_1^{-1}\boldsymbol{\mu}_1 + 2\Sigma_2^{-1}\boldsymbol{\mu}_2)^\top \mathbf{x} + \boldsymbol{\mu}_1^\top \Sigma_1^{-1} \boldsymbol{\mu}_1 - \boldsymbol{\mu}_2^\top \Sigma_2^{-1} \boldsymbol{\mu}_2 = \text{RHS constant}$$

This is a quadratic curve in $\mathbb{R}^2$.

We use:

$$\boldsymbol{\mu}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{\mu}_2 = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\Sigma_1 = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix}, \quad \Sigma_2 = \begin{bmatrix} 1 & -0.3 \\ -0.3 & 1.5 \end{bmatrix}$$

$$\pi_1 = \pi_2 = 0.5$$

From this:

$$\Sigma_1^{-1} = \begin{bmatrix} 0.5714 & -0.2857 \\ -0.2857 & 1.1429 \end{bmatrix}, \quad \Sigma_2^{-1} = \begin{bmatrix} 1.0638 & 0.2128 \\ 0.2128 & 0.7092 \end{bmatrix}$$

Numerical Decision Boundary Equation

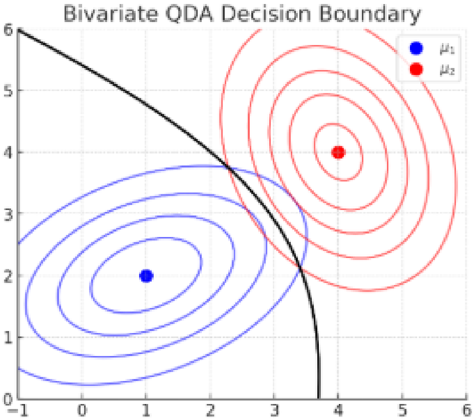
Plugging values:

$$\mathbf{x}^\top \begin{bmatrix} -0.4924 & -0.4985 \\ -0.4985 & 0.4337 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 2.697 & -2.101 \end{bmatrix} \mathbf{x} - 3.504 = 0.$$

Which further, simplifies to

$$-0.4924\, x_1^2 \;-\; 0.997\, x_1 x_2 \;+\; 0.4337\, x_2^2 \;+\; 2.697\, x_1 \;-\; 2.101\, x_2 \;-\; 3.504 \;=\; 0.$$

This quadratic equation defines the curved black boundary in our plot.



- Blue/Red ellipses: contours of bivariate normal distributions
- Black curve: quadratic decision boundary
- Dots: class means