Theory of Matrices

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Introduction

Linear Algebra includes the application of linear systems of equations, linear transformation, Eigenvalue problems, curve fitting and optimization problems, etc. Linear Algebra makes systematic use of vectors and matrices.

In 1857, Arthur Cayley discovered the matrices. It has a wide application in various field of sciences. They play a very crucial role not only in mathematics but also in communication theory, network analysis, theory of structures, quantum mechanics, biology, sociology, economics, psychology, statistics, etc.

A matrix is a rectangular array of numbers (or functions) enclosed in brackets. These numbers (or functions) are called **entries or elements** of the matrix. For example

$$[1], \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

We denote matrices by Capital letters A, B, C by writing the general entry in brackets. Thus $A = [a_{ij}]$ and so on. By an $m \times n$ matrix (read m *by* n matrix). We mean a matrix with m rows and n columns. Thus an $m \times n$ matrix is of the form

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}_{m \times n} \qquad (1)$$

If $m = n$; we call A an $n \times n$ **square matrix**. Then its diagonal containing the entries $a_{11}, a_{22}, \dots a_{nn}$ is called the **main diagonal** or **principal diagonal** or **diagonal** of matrix A. All the elements of the diagonal of a matrix is called **diagonal elements** of the matrix. For example

$$\begin{bmatrix} 1 & 2 & e^x \\ x^2 & e^{2x} & 0 \\ -5 & 9 & logx \end{bmatrix}_{3 \times 3}$$

The diagonal is $1, e^{2x}, logx$.

A matrix that is not square is called the **rectangular matrix.**

Few important Definitions

Row Matrix: A matrix having only one Row. Example: $\begin{bmatrix} 5 & 9 & 7 & 6 \end{bmatrix}$

Column Matrix: A matrix having only one Column. Example: $\begin{bmatrix} 5 \\ 9 \\ 7 \\ 6 \end{bmatrix}$

Null or zero matrix: A Matrix in which all elements are zero. Example: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $[0\ 0\ 0]$

Equality: Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are equal if they are of same order and $a_{ij} = b_{ij}$ for every i, j.

Diagonal elements: If $A = [a_{ij}]_{n \times n}$ is a square matrix, then the elements a_{ii} of the matrix is called diagonal element.

Trace of a matrix: If $A = [a_{ij}]_{n \times n}$ is a square matrix, then sum of the diagonal elements is knows as the trace of the matrix. That is

$$\text{Trace} = \sum_{i=1}^{n} a_{ii}$$

Singular matrix and Non-Singular matrix: A square matrix A is known as singular if $|A| = 0$ and non-singular if $|A| \neq 0$

Upper and Lower Triangular matrix: A square matrix $A = [a_{ij}]_{n \times n}$ is called an

Upper triangular if $A: a_{ij} = 0\ for\ i > j$ and

Lower triangular if $A: a_{ij} = 0\ for\ i < j$

Sub-matrix: A matrix which is obtained from a given matrix by deleting any number of rows or columns or both is called a sub-matrix of the given matrix. For example

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \end{bmatrix} \text{ is a subset of } \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & 2 & 3 & 2 \end{bmatrix}$$

Addition of matrices: Addition of two matrix $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ of same size is obtained by adding the corresponding entries. *i.e.* $A + B = a_{ij} + b_{ij}\ \forall\ i, j$.

Note: Matrices of different sizes cannot be added.

For example:

$$A = \begin{bmatrix} 4 & -6 & 3 \\ 2 & 0 & 1 \end{bmatrix};\ B = \begin{bmatrix} 5 & -9 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 9 & -15 & 6 \\ 2 & 0 & 2 \end{bmatrix}$$

Scalar Multiplication: (Multiplication by a number)

The product of any $m \times n$ matrix $A = [a_{ij}]_{m \times n}$ and any scalar c $(number\ c)$ is the $m \times n$ matrix obtained by multiplying each element in A by c. That is

$$cA = [ca_{ij}]$$

Note: $(-1)A$ is simply written as $-A$ and is called the ***negative* of *A***. Also, $A + (-B)$ is written as $A - B$ and is called the difference of A and B (both A and B must have the same order). For example

$$A - B = A + (-1)B = \begin{bmatrix} 4 & -6 & 3 \\ 2 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -5 & 9 & -3 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

Diagonal matrix: A square matrix $A = [a_{ij}]_{n\times n}$ is called diagonal if $A: a_{ij} = 0$ when $i \neq j$. For example

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Scalar matrix: A diagonal matrix with $a_{ii} = k \;\forall\; i$ is called a scalar matrix *(where k is any constant)*. For example

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Identity matrix: A scalar matrix is called identity matrix if $k = 1$. For example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

Note:

1. An identity matrix of order $n \times n$ can be denoted by I_n.
2. Let I be an identity matrix and A be a matrix of same order then $AI = IA = A$.
3. Let S be a scalar matrix and A be of same order, then $AS = SA = kA$.

Power of a matrix: A is a square matrix. A^k is again a matrix obtained by multiplying A by itself k times.

Transposition of a matrix: The transpose of an $m \times n$ matrix $A = [a_{ij}]_{m\times n}$ is the $n \times m$ matrix which is obtained by interchanging the rows and the corresponding columns. The transpose of a matrix is denoted by A^T or A'

Transpose of the matrix A given in equation (1) is:

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{nm} \end{bmatrix} = [a_{ji}]_{n\times m}$$

Symmetric and Skew-symmetric matrix: A square matrix $A = [a_{ij}]_{n\times n}$ will be called

symmetric if $A = A^T$ [or $a_{ij} = a_{ji} \;\forall\; i, j$] and

skew-symmetric if $A = -A^T$ [or $a_{ij} = -a_{ji} \;\forall\; i, j$].

Note:

1. For a skew symmetric matrix, all the diagonal elements are $zero$.
2. $(AB)^T = B^T A^T$
3. $(A + B)^T = A^T + B^T$
4. $(CA)^T = CA^T; c$ is a scalar .

Matrix Multiplication: (Multiplication of a matrix by a matrix)

The product $C = AB$ of a $m \times n$ matrix $A = [a_{ij}]_{m\times n}$ and a $r \times p$ matrix $B = [b_{jk}]_{r\times p}$ is defined if and only if $n = r$. That is

Number of Columns of 1st matrix A = Number of rows of 2nd matrix B

and is defined as the $m \times p$ matrix $C = [a_{ik}]$ with entries

$$C_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}\,; i = 1,2,3, \dots, m \text{ and } k = 1,2,3, \dots, p$$

Note:

1. $AB \neq BA$.
2. $AB = 0$ does not necessarily imply $A = 0$ or $B = 0$ or $BA = 0$.
3. $AC = AD$ does not necessarily imply $C = D$.

Orthogonal Matrix: A square matrix $A = [a_{ij}]_{n\times n}$ is called an orthogonal matrix when the product of the matrix and its transpose matrix is an identity matrix. That is:

$$AA^T = A^T A = I$$

Conjugate of a Matrix: Conjugate of a *Complex Matrix* $A = [a_{ij}]_{m\times n}$ is denoted by $\bar{A}$ and obtained by replacing each entry of the matrix A by its conjugate. That is

$$\bar{A} = [\overline{a_{ij}}]_{m\times n}$$

For example:

$$\text{If } A = \begin{bmatrix} 1+i & 2-3i & 4 \\ 7+2i & -i & 3-2i \end{bmatrix} \text{ then } \bar{A} = \begin{bmatrix} 1-i & 2+3i & 4 \\ 7-2i & i & 3+2i \end{bmatrix}$$

Note: $\overline{(\bar{A})} = A$.

Conjugate transpose or Tranjugate of a matrix: Transpose of the conjugate of a matrix $A = [a_{ij}]_{m\times n}$ is called the Tranjugate (or conjugate transpose) of the matrix and is denoted by A^θ. That is

$$A^\theta = (\bar{A})^T = [\overline{a_{ji}}]_{n\times m}.$$

For example:

$$\text{If } A = \begin{bmatrix} 3+i & 5 & -2i \\ 2-2i & i & -7-13i \end{bmatrix} \text{ then } A^{\theta} = \begin{bmatrix} 3-i & 2+2i \\ 5 & -i \\ 2i & -7+13i \end{bmatrix}.$$

Unitary Matrix: A Square matrix A is said to be unitary if

$$AA^{\theta} = A^{\theta}A = I$$

For example:

$$A = \begin{bmatrix} \frac{1+i}{2} & \frac{-1+i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix} \text{ is a unitary matrix as } AA^{\theta} = A^{\theta}A = I.$$

Hermitian and Skew-Hermitian Matrix: A square matrix $A = [a_{ij}]_{n\times n}$ is called

Hermitian matrix if $A = A^{\theta}$ [or $a_{ij} = \overline{a_{ji}}\ \forall\, i, j$] and

Skew-Hermitian matrix if $A = -A^{\theta}$ [or $a_{ij} = -\overline{a_{ji}}\ \forall\, i, j$].

For example:

$$A = \begin{bmatrix} 1 & 2+3i & 3+i \\ 2-3i & 2 & 1-2i \\ 3-i & 1+2i & 5 \end{bmatrix} \text{ is a Hermitian matrix.}$$

$$B = \begin{bmatrix} 0 & 2+3i & -3-i \\ -2+3i & 2i & 1-2i \\ 3-i & -1-2i & -7i \end{bmatrix} \text{ is a Skew-Hermitian matrix.}$$

Note:

1. For Hermitian matrix, the diagonal element should be a pure real number or zero.
2. All the diagonal elements of a Skew-Hermitian matrix are either zeroes or purely imaginary.

Idempotent Matrix: A square matrix A is called Idempotent Matrix if $A^2 = A$. For example:

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Nilpotent Matrix: A square matrix A will be called a Nilpotent Matrix $if\ A^k = 0$; where k is a $+ve$ integer. If, however k is the least $+ve$ integer for which $A^k = 0$; then k is called the index of the nilpotent matrix.

For example:

$$A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \text{then } A^2 = 0$$

Here A is a nilpotent matrix whose index is 2.

Involuntary Matrix: A matrix A will be called an Involuntary matrix if $A^2 = I$.

Note: Since $I^2 = I$, so unit matrix is always involuntary

Periodic Matrix: A matrix A will be called a periodic matrix if

$$A^{k+1} = A$$

where k is a $+ve$ integer. If k is the least $+ve$ integer, for which $A^{k+1} = A$, then k is said to be the period of A.

Note: If we choose $k = 1$; we get $A^2 = A$ and we call it an idempotent matrix. So, idempotent matrix is a periodic matrix with period 1.

Normal Matrix: A square matrix A is called normal if $A^\theta A = AA^\theta$. For example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Unimodular Matrix: A square matrix A is called unimodular if $|A| = 1$.

Matrix Polynomial: Let A be a square matrix of order n. An expression of the form $a_0A^m + a_1A^{m-1} + \cdots + a_mI_n$; where $a_0, a_1, \ldots, a_m$ are scalars is called a matrix polynomial.

Property Set I: Properties of special matrices

1. If any matrix A and B are of same order. Then
   a. $A + B = B + A$ (Addition is Commutative)
   b. $(A + B) + C = A + (B + C)$ (Addition is Associative)
2. A is unitary $\Leftrightarrow$ A^θ is unitary.
3. Every Hermitian matrix is a normal matrix.
4. Sum of two Hermitian matrices is Hermitian.
5. Inverse of invertible Hermitian matrix is Hermitian.
6. Product of two Hermitian matrices A and B is Hermitian if and only if $AB = BA$.
7. For any integer n, A^n is Hermitian if A is Hermitian.
8. Sum of a Square matrix and its Conjugate Transpose $(A + A^\theta)$ is Hermitian.
9. The difference of a Square matrix and tis Conjugate Transpose $(A - A^\theta)$ is skew-Hermitian.
10. A is symmetric and non-singular then A^{-1} is also symmetric.
11. Determinant of a Hermitian matrix is real.
12. If A is a square matrix; then

a. $A + A^T$ is symmetric and
b. $A - A^T$ is skew-symmetric.

13. U is unitary $\Leftrightarrow$ U is invertible with $U^{-1} = U^\theta$.
14. If A and B are skew-Hermitian; then $aA + bB$ is skew-Hermitian for all scalars a and b.
15. If A is a skew-Hermitian then both iA and $-iA$ are Hermitian.
16. If A is a skew-Hermitian; then A^k is Hermitian if k is an even integer and skew-Hermitian if k is an odd integer.
17. $(A+B)^\theta = A^\theta + B^\theta$ for any two matrices A and B of the same dimension.
18. $(AB)^\theta = B^\theta A^\theta$ for any $m \times n$ matrix A and any $n \times p$ matrix B.
19. $(A^\theta)^\theta = A$ for any matrix A.
20. If A is a square matrix then
    a. $\det(A^\theta) = (\det A)^\theta$
    b. $trace = (\text{trace}\, A)^\theta$.
21. A is invertible iff A^θ is invertible and in that case $\left(A^\theta\right)^{-1} = (A^{-1})^\theta$.
22. $A(BC) = (AB)C$ (Matrix multiplication is associative)
23. $A(B+C) = AB + AC$ (Matrix multiplication is distributive)
24. Determinant of a skew-symmetric matrix of odd-order is always zero.
25. $trace(aA + bB) = a(traceA) + b(traceB)$; a, b are scalars and A, B are matrix of same order.

Differentiation and Integration of Matrices

If the elements of a matrix A are functions of x, the matrix is called a matrix function of x i.e.

$$A = A(x) = [a_{ij}(x)]$$

The **differential co-efficient** or **derivative** of A w.r.t. x is defined as:

$$\frac{d}{dx}A = \left[\frac{d}{dx}(a_{ij})\right]$$

Here, the elements of the differentiated matrix $\frac{dA}{dx}$ are the derivative of the corresponding elements of A

$$\frac{d}{dx}A(x) = \begin{bmatrix} \frac{da_{11}}{dx} & \frac{da_{12}}{dx} & \cdots & \frac{da_{1n}}{dx} \\ \frac{da_{21}}{dx} & \frac{da_{22}}{dx} & \cdots & \frac{da_{2n}}{dx} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{da_{m1}}{dx} & \frac{da_{m2}}{dx} & \cdots & \frac{da_{mn}}{dx} \end{bmatrix}_{m\times n}$$

Note: $\frac{d}{dx}(AB) = A\frac{dB}{dx} + B\frac{dA}{dx}$

The **integral** of a matrix A is defined as

$$\int A\,dx = \left[\int (a_{ij})\,dx\right] + cI$$

Thus, the integral of A is obtained by integrating each element of A

Adjoint of a Matrix

Minors of a Matrix: $A = [a_{ij}]_{n\times n}$, then the minor of the entry in the i^{th} row and j^{th} column is the determinant of the submatrix formed by deleting the i^{th} row and j^{th} column. It is denoted by M_{ij} or $M(i,j)$.

Co-factor: The ij^{th} Co-factor is obtained by multiplying the minor by $(-1)^{i+j}$. For a matrix $A = [a_{ij}]$, the ij^{th} Co-factor can be denoted by A_{ij}. That is,

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Adjoint of a Matrix: Let $A = [a_{ij}]_{n\times n}$ be a square matrix. Then the transpose of matrix $[A_{ij}]$ is known as the adjoint of the matrix A and is write as $adjA$. Thus,

If $$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}_{n\times n}$$

then $$[A_{ij}] = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \dots & \dots & \dots & \dots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{bmatrix}_{n\times n}$$

$$\therefore \quad adjA = [A_{ij}]^T = \begin{bmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \dots & \dots & \dots & \dots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{bmatrix}_{n\times n}.$$

Working Rule:

Step 1. Replace each element of A by its Co-factors in A
Step 2. Take the transpose of the Co-factor matrix

Property Set II: Properties of Adjoint of a Matrix

1. If $A = [a_{ij}]_{n\times n}$ is a square matrix, then $A(adjA) = (adjA)A = |A|I_n$; I_n is a unit matrix of order n.
2. If $A = [a_{ij}]_{n\times n}$ is non-singular square matrix ($i.e. |A| \neq 0$) then
$$A\left(\frac{adjA}{|A|}\right) = \left(\frac{adjA}{|A|}\right)A = I_n.$$

3. If $A = [a_{ij}]_{n\times n}$ is non-singular square matrix $(i.e. |A| \neq 0)$ then $|adjA| = |A|^{n-1}$.
4. If A and B are two $n \times n$ matrices, then
   a. $adj(AB) = (adjB)(adjA)$
   b. $adj(A^T) = (adjA)^T$.
5. If $A = [a_{ij}]_{n\times n}$ then $adj(adjA) = |A|^{n-2}A$.
6. If $A = [a_{ij}]_{n\times n}$ and $|A| \neq 0$; then $|adj(adjA)| = |A|^{(n-1)^2}$.
7. Adjoint of a Hermitian matrix is Hermitian.
8. Adjoint of a Symmetric matrix is symmetric.
9. Adjoint of a digital matrix is diagonal.
10. Adjoint of a triangular matrix is triangular.
11. If $A = [a_{ij}]_{n\times n}$ and k is a scalar, then $adj(Ak) = (adjA)k^{n-1}$

Inverse (or Reciprocal) of a Matrix

Let A be a square matrix of order $n \times n$. Let B be another square matrix of same order such that

$$AB = BA = I_n$$

where I_n is the identity matrix of order $n \times n$; then the matrix B is said to be the inverse of the matrix A and is denoted by A^{-1}. Thus

$$AA^{-1} = I = A^{-1}A$$

Note:

1. Condition for a square matrix A to process an inverse is that the matrix A is non-singular $(i.e. |A| \neq 0)$.
2. A non-square matrix does not possess an inverse.
3. If B is the inverse of A, then A is the inverse of B.
4. A matrix possessing an inverse is called an **invertible** matrix.

To find the inverse of a matrix with the help of Adjoint of the matrix:

Form the property 2 of property Set II, we know that $A\left(\frac{adjA}{|A|}\right) = I_n$; Provided $|A| \neq 0$. Thus

$$A^{-1} = \left(\frac{adjA}{|A|}\right); |A| \neq 0.$$

Property Set III: Properties of inverse of a matrix

1. $|A^{-1}| = |A|^{-1} = \frac{1}{|A|}$.
2. The inverse of a matrix is unique.
3. A square matrix A is invertible if and only if $|A| \neq 0$.
4. $(AB)^{-1} = B^{-1}A^{-1}$.

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5. If $A = [a_{ij}]_{n\times n} (\neq 0)$ and p is any $+ve$ integer then $(A^p)^{-1} = (A^{-1})^p$.
6. If $A = [a_{ij}]_{n\times n}$ is non-singular square matrix. Then
   a. $(A^{-1})^{-1} = A$
   b. $(A^T)^{-1} = (A^{-1})^T$
   c. $(A^\theta)^{-1} = (A^{-1})^\theta$
   d. $(adjA)^{-1} = adj(A^{-1})$
   e. $AX = AY \Rightarrow X = Y$.
7. If A and B are any two $n \times n$ matrices such that $AB = 0$ where 0 is a null matrix; then at least one of them is singular.
8. If a non-singular matrix A is symmetric then A^{-1} is also symmetric.
9. If $A = diag(a_{11}, a_{22}, \ldots, a_{nn})$ then $A^{-1} = diag(a_{11}^{-1}, a_{22}^{-1}, \ldots, a_{nn}^{-1})$provided $a_{11}, a_{22}, \ldots, a_{nn} \neq 0$.
10. If $|A| \neq 0$ *and* $|B| \neq 0$ and both are symmetric and commute under multiplication, then $A^{-1}B$, AB^{-1} and $A^{-1}B^{-1}$ are symmetric.
11. If $adjB = A$ and $|P| = |Q| = 1$; then
$$adj(Q^{-1}BP^{-1}) = PAQ$$
12. Inverse of a non-singular symmetric (Hermitian) matrix is symmetric (Hermitian).

Example 1: Find the inverse of $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$.

Solution: Let, $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$$\text{Now, } |A| = 1\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ -4 & -2 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix}$$

$$= 1(-24) + 1(10) + 3(2) = -8 \ (which\ is\ \neq 0)$$

$$Now, A_{11} = (-1)^{1+1}M_{11} = (-1)^2\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} = -24$$

$$A_{12} = 10$$
$$A_{13} = 2$$
$$A_{21} = (-1)^{2+1}M_{21} = -1\begin{vmatrix} 1 & 3 \\ -4 & -4 \end{vmatrix} = -8$$

Similarly,

$A_{22} = 2$ $\quad$ $A_{23} = 2$

$A_{31} = -12$ $\quad$ $A_{32} = 6$ $\quad$ $A_{33} = 2$

$$\therefore \text{adjA} = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}^T = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \left(\frac{adjA}{|A|}\right) = \frac{1}{-8}\begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3/2 \\ -5/4 & -1/4 & -3/4 \\ -1/4 & -1/4 & -1/4 \end{bmatrix}$$

(Try yourself)

Exercise 1: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$, **Find** A^{-1}.

Answer: $adjA = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$, $A^{-1} = \frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$

Elementary Transformations

Any one of the following on a matrix is called an elementary transformation.

1. Interchanging any two rows (or columns). The transformation is indicated by R_{ij} or with $R_i \leftrightarrow R_j$ if the i^{th} and j^{th} rows are interchanged.
2. Multiplication of the elements of any rows (or columns) by a non-zero scalar quantity k to get (can be denoted by kR_i or $R_i \to kR_i$).
3. Addition of constant multiplication of the elements of any row to the corresponding elements of any other row (can be denoted by $R_i + kR_j$ or $R_i \to R_i + kR_j$)

Equivalent Matrices

Two matrices are said to be equivalent if one can be obtained from the other by applying a finite number of elementary transformations. If the matrices A and B are equivalent, then it is denoted by $A \sim B$

Elementary Matrices

A matrix obtained from a unit matrix by an elementary transformation is called elementary matrix.

For example

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Consider the matrix obtained by the elementary transformation $R_2 + 3R_1$, that is

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an elementary matrix.

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Property Set IV: Properties of equivalent matrices

1. If A and B be equivalent matrix, then ∃ non-singular matrix R and C such that:
   a. $B = RAC$
   b. $A = R^{-1}BC^{-1}$
2. Every non-singular square matrix can be expressed as the **Product of elementary matrices**.
3. If A is a non-singular square matrix of order n, then ∃ elementary matrices $E_1, E_2, \dots, E_k$ such that:
   a. $(E_k E_{K-1} \dots E_2 E_1)A = I_n$
   b. $A^{-1} = (E_k E_{K-1} \dots E_2 E_1)I_n$

Method of finding the inverse from elementary transformation (Gauss-Jordan method):

From the property 3 of property set IV; we can say that if a sequence of elementary operation on a non-singular square matrix of order n transforms to a unit matrix I_n, then the same sequence when applied on I_n transforms I_n to A^{-1}. This property is quite useful in finding the inverse of a non-singular square matrix A.

Working rule: Write $A = IA$. Perform elementary raw transformation on A of the left side and on I of the right side so that A (of the left side) is reduced to I and I (of right side) is reduced to some another matrix P getting $I = PA$. Then P is the inverse of A.

Example 2: Find the inverse of $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ Gauss-Jordan method.

Solution: Let us right

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $R_2 + R_1$, we have

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $R_2 + 2R_3$, we have

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A$$

Apply $R_3 + 2R_2$, we have

$$\begin{bmatrix}1 & 2 & -2\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 2\\2 & 2 & 5\end{bmatrix} A$$

Apply $R_1 + 2R_3$, we have

$$\begin{bmatrix}1 & 2 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}5 & 4 & 10\\1 & 1 & 2\\2 & 2 & 5\end{bmatrix} A$$

Apply $R_1 - 2R_2$, we have

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}3 & 2 & 6\\1 & 1 & 2\\2 & 2 & 5\end{bmatrix} A$$

Hence,

$$A^{-1} = \begin{bmatrix}3 & 2 & 6\\1 & 1 & 2\\2 & 2 & 5\end{bmatrix}.$$

(Try yourself)

Exercise 2: With the help of Gauss-Jordan method, find the inverse of the following

a) $A = \begin{bmatrix}1 & 2 & 1\\3 & 2 & 3\\1 & 1 & 2\end{bmatrix}$

b) $B = \begin{bmatrix}-1 & -3 & 3 & 3\\1 & 1 & -1 & 0\\2 & -5 & 2 & -3\\-1 & 1 & 0 & 1\end{bmatrix}$

Answer: a) $A^{-1} = \begin{bmatrix}-\frac{1}{4} & \frac{3}{4} & -1\\\frac{3}{4} & -\frac{1}{4} & 0\\-\frac{1}{4} & -\frac{1}{4} & 1\end{bmatrix}$ b) $B^{-1} = \begin{bmatrix}0 & 2 & 1 & 3\\1 & 1 & -1 & -2\\1 & 2 & 0 & 1\\-1 & 1 & 2 & 6\end{bmatrix}$

Reduction of a matrix of triangular from:

Every square matrix can be reduced to triangular form by elementary row operations

For example:

$$A = \begin{bmatrix}3 & 1 & 4\\1 & 2 & -5\\0 & 1 & 5\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & 2 & -5\\3 & 1 & 4\\0 & 1 & 5\end{bmatrix} \qquad R_1 \leftrightarrow R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & -5 & 19 \\ 0 & 1 & 5 \end{bmatrix} \qquad R_2 \to R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 5 \\ 0 & -5 & 19 \end{bmatrix} \qquad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -5 \\ 0 & 1 & 5 \\ 0 & 0 & 44 \end{bmatrix} \qquad R_3 \to R_3 - 5R_2$$

which is in triangular form.

Exercise 3: Reduce the matrix to triangular form

a) $A = \begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$

Answer: a) $\begin{bmatrix} -1 & 2 & -2 \\ 0 & 4 & -1 \\ 0 & 0 & 5 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$

Canonical Form or Normal Form

By performing elementary transformation (Both row and column transformation); any non-zero matrix A can be reduced to one of the following four forms, called normal form of A.

(i)I_r (ii)$[I_r \quad 0]$ (iii) $\begin{bmatrix} I_r \\ 0 \end{bmatrix}$ (iv) $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The form (iv) is called first canonical form of a matrix A

Example: (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Echelon Form of a Matrix

Echelon matrix comes in two forms:

1. Row Echelon Form (REF)
2. Reduced Row Echelon Form (RREF)

Row Echelon form:

A matrix is in row echelon form when it satisfies the following conditions:

1. The first non-zero element in each row, called the leading entry is 1.

2. Each leading entry is in a column to the right of the leading entry in the previous row.
3. Rows will all zero elements, if any, are below rows have a non-zero elements.

Example: (a) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

Reduce Row Echelon Form:

A matrix is in reduced row echelon form when it satisfies the following condition

1. The matrix is in row-echelon form.
2. Each leading 1 is the only non-zero entry in its column.

Example: (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, (c) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Example 3: Reduce to normal form

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$$

Solution: Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 \\ 0 & -6 & -4 & -22 \end{bmatrix} \qquad \begin{matrix} C_2 - 2C_1 \\ C_3 - 3C_1 \\ C_4 - 4C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \\ 0 & 1 & \frac{2}{3} & \frac{11}{3} \end{bmatrix} \qquad \begin{matrix} -\frac{1}{3}R_2 \\ -\frac{1}{6}R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{2}{3} & \frac{5}{3} \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \qquad \begin{matrix} C_3 - \frac{2}{3}C_2 \\ C_4 - \frac{5}{3}C_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \qquad C_{34}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \frac{1}{2}C_3$$

$$= [I_3 \quad 0]$$

which is the normal form.

Exercise 4: Using elementary row transformation reduce the following matrices to triangular form, REF, RREF, Normal form.

a) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{bmatrix}$

b) $B = \begin{bmatrix} -1 & 2 & -2 & 4 \\ 1 & 2 & 1 & -2 \\ -1 & -1 & 0 & 0 \\ 1 & -1 & 2 & 1 \end{bmatrix}$

c) $C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 1 & 2 \end{bmatrix}$

Rank of a Matrix

The rank of a matrix is said to r if:

1. There exist at least one non-zero mirror of order r
2. Every minor of order higher than r is zero.

The rank of a matrix A is denoted by $\rho(A)$ or $rank(A)$.

Nullity of a Matrix

Let $A = [a_{ij}]_{n\times n}$. Then $n - \rho(A)$ is called the nullity of the matrix A and denoted by $N(A)$.

Note: The rank of a non-singular square matrix of order n is n and therefore its nullity is equal to zero.

Property Set V: Properties of rank

1. Rank=Number of nonzero rows in the echelon form of a matrix.
2. In the definition of Normal Matrix, r denotes the rank of the matrix.
3. If A is a null matrix; then $\rho(A) = 0$.
4. $\rho(A) \geq 1$ when $A \neq 0$
5. $\rho(A) \leq r$ when every minor of order $(r + 1)$ vanishes
6. $\rho(A) \geq r$ when $\exists$ a non-zero minor of order r
7. $\rho(A) \leq m$ when A is an $m \times n$ matrix such that $m \leq n$
8. $\rho(A) \leq n$ when A is an $m \times n$ matrix such that $n \leq m$
9. If A is a square matrix of order n such that, $|A| \neq 0$ then $\rho(A) = n$

10. If I_n is an identity matrix of order n then $\rho(I_n) = n$
11. If A is a diagonal matrix of order n with non-zero diagonal elements, then $|A| \neq 0$ and therefore $\rho(A) = n$.
12. $\rho(A) = \rho(A^T)$
13. $\rho\left(A^\theta\right) = \rho\left(A^\theta\right)$
14. If A is a non-zero Column matrix and B is a non-zero row matrix, then $\rho(AB) = 1$
15. If A is any $m \times n$ matrix of rank r ; then $\exists$ non-singular matrices R and C such that: $$RAC = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$
16. $\rho(AB) \leq \min(rankA, rankB)$
17. If $A = \left[a_{ij}\right]_{n\times n}$ with $\rho(A) = n-1$ then $\rho(adjA) = 1$
18. If $|P| \neq 0$ then $\rho(AP) = \rho(PA) = \rho(A)$
19. $\rho(AA^T) = \rho(A) = \rho(A^T) = \rho(A^T A)$
20. Only a zero matrix has rank zero
21. If $A = \left[a_{ij}\right]_{n\times n}$ then A is invertible if and only if A has rank n
22. $\rho(A) = \rho(\bar{A}) = \rho(A^T) = \rho\left(A^\theta\right) = \rho\left(A^\theta A\right)$
23. $\rho(A+B) \leq rank(A) + rank(B)$.

Exercise 5: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{bmatrix}$

Exercise 6: Reduce the matrix $A = \begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$ to normal form and find its rank

Exercise 7: Reduce the matrix to normal form and hence find its rank:

(i) $\begin{bmatrix} 0 & 1 & -3 & 1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ $(\textbf{Ans}: 2)$ (ii) $\begin{bmatrix} 3 & -1 & 2 \\ -6 & 2 & 4 \\ -3 & 1 & 2 \end{bmatrix}$ $(\textbf{Ans}: 2)$

Exercise 8: Find the rank of the matrix using elementary transformation

(i) $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ $(\textbf{Ans}: 3)$ (ii) $\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & 1 & 2 & 3 \end{bmatrix}$ $(\textbf{Ans}: 2)$

System of Linear Equations

Consider a system of m linear equations with n unknowns $(m > n,\ m = n \text{ or } m < n)$ given below:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \qquad (2)$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \ldots + a_{mn}x_n = b_m$$

In matrix notation, this system of equation can be written in the form

$$AX = B \qquad (3)$$

If in the system of equation (2), all b_i's are not zero (or the matrix $B \neq 0$ in equation (3)), then the system is known as **linear non-homogenous system of equations**. Otherwise if all b_i's are zero in the system of equation (2) (or the matrix $B = 0$ in equation (3)), then the system is known as **linear homogenous system of equations**.

The matrix A is called the **co-efficient matrix**, the matrix X of of n unknowns $x_1, x_2, x_3, \ldots, x_n$ is known as **unknown matrix** and the matrix B of m constants $b_1, b_2, \ldots, b_m$ is known as **constant matrix**.

System of linear non-homogenous equation

Define a matrix $C = [A\!:\!B]$ obtained by placing the constant matrix B to the right of the matrix A is called **augmented matrix**. Thus, the matrix

$$C = [A\!:\!B] = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} & : & b_2 \\ \vdots & & & & & : & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn} & : & b_m \end{bmatrix}$$

is called the **augmented matrix**.

Consistent Equation: If $\rho(A) = \rho(C)$ then the system of equations is called consistent. It has:

a) Unique Solution if $\rho(A) = \rho(C) = n$
b) Infinite Solutions if $\rho(A) = \rho(C) = r; r < n$

Inconsistent Equation: If $\rho(A) \neq \rho(C)$ then the system of solution is called inconsistent. In that case, the solution does not exist.

Example 4: Solve the equations:

$$2x + 6y = -11$$

$$6x + 20y - 6z = -3$$

$$6y - 18z = -1$$

Solution: For the above system, the augmented matrix can be written as

$$C = [A:B] = \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 6 & 20 & -6 & : & -3 \\ 0 & 6 & -18 & : & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 6 & -18 & : & -1 \end{bmatrix} \qquad R_2 - 3R_1$$

$$\sim \begin{bmatrix} 2 & 6 & 0 & : & -11 \\ 0 & 2 & -6 & : & 30 \\ 0 & 0 & 0 & : & -91 \end{bmatrix} R_3 - 3R_2$$

Here $\rho(c) = 3$ and $\rho(A) = 2$.

Hence $\rho(A) \neq \rho(C)$.

Therefore the system is inconsistent.

Example 5: Discuss the consistency of the following system of equations and hence solve it, if exist.

$$2x + 3y + 4z = 11$$

$$x + 5y + 7z = 15$$

$$3x + 11y + 13z = 25$$

Solution: For the above system, the augmented matrix can be written as

$$C = [A:B] = \begin{bmatrix} 2 & 3 & 4 & : & 11 \\ 1 & 5 & 7 & : & 15 \\ 3 & 11 & 13 & : & 25 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & : & 15 \\ 2 & 3 & 4 & : & 11 \\ 3 & 11 & 13 & : & 25 \end{bmatrix} R_{12}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & : & 15 \\ 0 & -7 & -10 & : & -19 \\ 0 & -4 & -8 & : & -20 \end{bmatrix} \qquad \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & : & 15 \\ 0 & 1 & \frac{10}{7} & : & \frac{19}{7} \\ 0 & 1 & 2 & : & 5 \end{bmatrix} \quad \begin{matrix} -\frac{1}{7}R_2 \\ -\frac{1}{4}R_3 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 5 & 7 & : & 15 \\ 0 & 1 & \frac{10}{7} & : & \frac{19}{7} \\ 0 & 0 & \frac{4}{7} & : & \frac{16}{7} \end{bmatrix} \quad R_3 - R_2$$

Here $\rho(c) = 3$ and $\rho(A) = 3$.

Hence $\rho(A) = \rho(C) = 3 =$ Number of unknowns.

Therefore, the system is consistent and has unique solution. Now to find the solution, we can re-write the system as

$$\begin{bmatrix} 1 & 5 & 7 \\ 0 & 1 & \frac{10}{7} \\ 0 & 0 & \frac{4}{7} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ \frac{19}{7} \\ \frac{16}{7} \end{bmatrix}$$

Or
$$\begin{aligned} x + 5y + 7z &= 15 \\ y + \frac{10}{7}z &= \frac{19}{7} \\ \frac{4}{7}z &= \frac{16}{7} \end{aligned} \qquad (4)$$

Hence by solving (4), we have $z = 4, y = -3$ and $x = 2$.

Thus $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}$.

Example 6: Solve the equations:

$$5x + 3y + 7z = 4$$

$$3x + 26y + 2z = 9$$

$$7x + 2y + 10z = 5$$

Solution: For the above system, the augmented matrix can be written as

$$C = [A:B] = \begin{bmatrix} 5 & 3 & 7 & : & 4 \\ 3 & 26 & 2 & : & 9 \\ 7 & 2 & 10 & : & 5 \end{bmatrix}$$

$$\sim\begin{bmatrix}1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5}\\ 3 & 26 & 2 & : & 9\\ 7 & 2 & 10 & : & 5\end{bmatrix}\quad \frac{R_1}{5}$$

$$\sim\begin{bmatrix}1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5}\\ 0 & \frac{121}{5} & -\frac{11}{5} & : & \frac{33}{5}\\ 0 & -\frac{11}{5} & \frac{1}{5} & : & -\frac{3}{5}\end{bmatrix}\quad \begin{matrix}R_2-3R_1\\ R_3-7R_1\end{matrix}$$

$$\sim\begin{bmatrix}1 & \frac{3}{5} & \frac{7}{5} & : & \frac{4}{5}\\ 0 & \frac{121}{5} & -\frac{11}{5} & : & \frac{33}{5}\\ 0 & 0 & 0 & : & 0\end{bmatrix} R_3+\frac{R_2}{11}$$

Here $\rho(c) = 2$ and $\rho(A) = 2$.

Hence $\rho(A) = \rho(C) = 2 <$ Number of unknowns.

Therefore the system is consistent but has infinite number of solutions. Now to find the solutions, we can re-write the system as

$$\begin{bmatrix}1 & \frac{3}{5} & \frac{7}{5}\\ 0 & \frac{121}{5} & -\frac{11}{5}\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}\frac{4}{5}\\ \frac{33}{5}\\ 0\end{bmatrix}$$

Or
$$x+\frac{3}{5}y+\frac{7}{5}z=\frac{4}{5}$$
$$\frac{121}{5}y-\frac{11}{5}z=\frac{33}{5}\Rightarrow 11y-z=3$$

Now let $z = k$; then

$$11y-k=3\Rightarrow y=\frac{3+k}{11}$$

and $x+\frac{3}{5}\left(\frac{3+k}{11}\right)+\frac{7}{5}k=\frac{4}{5}\Rightarrow x=-\frac{16}{11}k+\frac{7}{11}$

Thus $\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-\frac{16}{11}k+\frac{7}{11}\\ \frac{3+k}{11}\\ k\end{bmatrix}$.

Exercise 9: Determine the values of a and b for which the system

$$x+2y+3z=6$$

$$x + 3y + 5z = 9$$

$$2x + 5y + az = b$$

Has (i) No solution (ii) Unique Solution (iii) infinite numbers of solution. Hence find the solution, if exist.

System of linear homogenous equation

A system of equation is said to be homogenous if $B = 0$ in the system of equations (3) i.e.

$$AX = 0$$

Note:

1. $\boldsymbol{X} = \mathbf{0}$ is always a solution. This solution is called **null or trivial solution**. Thus a homogenous system is always consistent.
2. If $\rho(A) =$ number of unknowns, the system has only the trivial solution.
3. If $\rho(A) <$ number of unknowns, the system has a infinite number of non-trivial solution.

Exercise 10: Determine *b* such that the system of homogenous equations

$$2x + y + 2z = 0$$

$$x + y + 3z = 0$$

$$4x + 3y + bz = 0$$

has (i) Trivial solution (ii) Non-trivial Solution. Hence find the solutions.

Exercise 11: Solve

$$x + y - 2z + 3w = 0$$

$$x - 2y + z - w = 0$$

$$4x + y - 5z + 8w = 0$$

$$5x - 7y + 2z - w = 0$$

Eigen Values and Eigen Vectors (Characteristic values/roots and Characteristic vectors):

Characteristic Matrix

If A is a square matrix, then the matrix $A - \lambda I$ is known as the characteristic matrix of A where λ is the scalar and I is the unit matrix

For example:

If $A = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$, then

$$A - \lambda I = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 4-\lambda & -5 \\ 1 & -2-\lambda \end{pmatrix}$$

is the characteristic matrix of A.

Characteristic Polynomial

The determinant (when expanded) of the matrix $A - \lambda I$ *i.e.* $|A - \lambda I|$ is known as the characteristic polynomial of A and is denoted by $\phi(\lambda)$

$$\phi(\lambda) = \begin{vmatrix} 4-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3$$

or $$\phi(\lambda) = \lambda^2 - 2\lambda - 3$$

is the characteristic polynomial of A.

Characteristic Equation

The equation $|A - \lambda I| = 0$ is called the characteristic equation of the matrix A i.e.

$$\phi(\lambda) = 0$$

or $$\lambda^2 - 2\lambda - 3 = 0$$

is the characteristic equation of A.

Characteristic Roots or Eigen Values

The roots of the characteristic equation $|A - \lambda I| = 0$ are called characteristic root or eigen value of the matrix A

For example

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\Rightarrow \lambda = -1, 3$$

are the characteristic roots or eigen values of A.

Characteristic Vectors or Eigen Vectors

Let $\lambda = \lambda_1$ be any characteristic root of A. Then the **non-zero** vector X which satisfies the following equation is called characteristic vector of A corresponding to the characteristic root $\lambda = \lambda_1$.

$$(A - \lambda_1 I)X = 0 \text{ or } AX = \lambda_1 X.$$

For example

For $\lambda = -1$, the equation $(A - \lambda_1 I)X = 0$ can be written as

$$(A - (-1)I)X = 0$$

or $$\left(\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

This gives a system of equations, which is

$$\begin{aligned} 5x_1 - 5x_2 &= 0 \\ x_1 - \ \ x_2 &= 0 \end{aligned}$$

This is a system of linear homogeneous equations. By solving this, we have

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k \\ k \end{pmatrix}, \text{ where } k \text{ is any } \textbf{non-zero} \text{ scalar.}$$

Hence this is the eigen vector of A corresponding to the eigen value $\lambda = -1$. Similarly one can also find the eigen vector of A corresponding to the eigen value $\lambda = 3$.

Property Set VI: Properties of Eigen Value

1. The sum of the eigen values of a matrix is equal to the trace of the matrix
2. The product of the eigen values of a matrix A is equal to the determinant of A
3. If $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigen values of A, then the eigen values of
   a. kA are $k\lambda_1, k\lambda_2, \dots, k\lambda_n$; k is any scalar
   b. A^m are $\lambda_1^m, \lambda_2^m, \dots, \lambda_n^m$; m is any scalar
   c. A^{-1} are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}$.
4. At least on eigen value of every singular matrix is zero
5. Two matrices A and $P^{-1}AP$(where P is any non-singular matrix of same order as A) has the same characteristic root.
6. A, B be two square matrix, then AB and BA have the same eigen values.
7. If α is a eigen value of a non-singular matrix A, then $\frac{|A|}{\alpha}$ is a eigen value of $adjA$.
8. Eigen values of every orthogonal matrix of odd order is either 1 or -1.
9. If λ is an eigen value of an orthogonal matrix, then $\frac{1}{\lambda}$ is also eigen value.
10. Eigen values of a diagonal matrix are same as its diagonal elements.

11. Eigen values of a triangular matrix are just the diagonal elements of the matrix
12. Any Square Matrix A and its transpose A^T have the same eigen values
13. Eigen values of A^θ are the conjugate of the eigen values of A.
14. The eigen values of a real symmetric matrix are all real.
15. The eigen values of a skew-symmetric matrix is always zero or pure imaginary.
16. Hermitian matrix has only real eigen values.
17. **A** eigen value of a skew-Hermitian matrix is either zero or a purely imaginary number.

Cayley-Hamilton Theorem

Statement: Every square matrix satisfies its own characteristic equation

OR

Alternative statement: If $|A-\lambda I| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + \cdots + a_n)$ be the characteristic polynomial of the matrix $A = (a_{ij})_{n\times n}$ then the equation $\lambda^n + a_1\lambda^{n-1} + \cdots + a_n I = 0$ is satisfied by A. That is

$$A^n + a_1A^{n-1} + \cdots + a_nI = 0. \tag{5}$$

For example

Since $\lambda^2 - 2\lambda - 3 = 0$ is the characteristic equation of $A = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$. Hence $A = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$ will satisfy the equation $\lambda^2 - 2\lambda - 3I = 0$. That is

$$A^2 - 2A - 3I = 0.$$

In some other words, $\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}^2 - 2\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a zero matrix. (Readers can verify this)

Finding Inverse of a Matrix Using Cayley-Hamilton Theorem

Cayley-Hamilton theorem can be used to find the inverse of a matrix. For that, one need to pre-multiply the equation (5) by A^{-1}. Then we have

$$A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I + a_nA^{-1} = 0$$

therefor $$A^{-1} = -\frac{1}{a_n}(A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I).$$

For example

The characteristic equation of $A = \begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix}$ is $\lambda^2 - 2\lambda - 3 = 0$.

Then according to Cayley-Hamilton theorem, A will satisfy its characteristic equation $\lambda^2 - 2\lambda - 3 = 0$. That is

$$A^2 - 2A - 3I = 0$$

Now pre-multiplying the equation by A^{-1}, we have

$$A - 2I - 3A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{3}(A - 2I)$$

$$= \frac{1}{3}\left(\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

$$= \frac{1}{3}\left(\begin{pmatrix} 4 & -5 \\ 1 & -2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right)$$

$$= \frac{1}{3}\begin{pmatrix} 2 & -5 \\ 1 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{2}{3} & -\frac{5}{3} \\ \frac{1}{3} & -\frac{4}{3} \end{pmatrix}.$$

Which is the inverse of A.

Example 7: Find the characteristic values of the matrix and hence find A^{-1}

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{pmatrix}$$

Sol^n: The characteristic equations of A is given by:

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda^2 - 4)(1 - \lambda) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda^2 - 4) = 0 \qquad (6)$$

$$\Rightarrow \lambda = 1, 2, -2 \text{ are the eigen values of } A$$

Now, (6) can be written as

$$(\lambda - 1)(\lambda^2 - 4) = 0$$

$$\Rightarrow \lambda^3 - \lambda^2 - 4\lambda + 4 = 0$$

Now by Cayley-Hamilton's theorem

$$A^3 - A^2 - 4A + 4I = 0$$

$$\Rightarrow \mathrm{A}^{-1} = -\frac{1}{4}(-A^2 + A + 4I)$$

$$\Rightarrow \mathrm{A}^{-1} = \frac{1}{4}\begin{bmatrix} 4 & 4 & -4 \\ -2 & -1 & 3 \\ -2 & 1 & 1 \end{bmatrix}.$$

Exercise 12: If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$, Find the characteristic roots and A^{-1}.

Exercise 13: If $a + b + c = 0$, find the characteristic roots of the matrix

$$A = \begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix}$$

Exercise 14: Verify Cayley-Hamilton theorem for the following matrix and hence find A^{-1} and A^4.

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Exercise 15: Find the eigen values and eigen vectors of the following matrices and also verify the Cayley-Hamilton theorem and hence find their inverses.

a) $A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix}$,

b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

c) $C = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 2 \\ -1 & 1 & 2 \end{bmatrix}$

d) $D = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

e) $E = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

f) $F = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

g) $G = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

h) $H = \begin{pmatrix} 1 & 3 \\ -2 & 6 \end{pmatrix}$.