Data Science Advanced

Lesson01–Matrix and Vectors

Objective

After completing this lesson you will be able to:

- Differentiate Matrix and Vectors
- Perform Matrix addition and subtraction
- Perform Matrix – Vector, Matrix – Matrix multiplication
- Describe Matrix Inverse and Matrix Transpose
- Understand Vectors in Algebra

Matrix and Vectors

$$A = \begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix}$$

- Matrix is a two dimensional array.
- Generally denoted by uppercase letter.
- A_{ij} refers to the element in the i^{th} row and j^{th} column.

$$v = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

- Vector is a matrix with one column and many rows. Generally denoted by lowercase.
- Vector with n rows is referred to as n dimensional vector.
- v^i refers to the element in the i^{th} row of the vector.

- "*Scalar*" means that an object is a single value, not a vector or matrix.
- **R** refers to a scalar real numbers.
- $\mathbf{R}^{\mathbf{n}}$ refers to a set of n-dimensional vectors of real numbers

Addition and Scalar Multiplication

- Addition or Subtraction is element wise:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} + \begin{bmatrix} l & m & n \\ o & p & q \\ r & s & t \end{bmatrix} = \begin{bmatrix} a+l & b+m & c+n \\ d+o & e+p & f+q \\ g+r & h+s & i+t \end{bmatrix}$$

- Scalar Multiplication:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} * x = \begin{bmatrix} a*x & b*x & c*x \\ d*x & e*x & f*x \\ g*x & h*x & i*x \end{bmatrix}$$

To add or subtract two matrices, their dimensions must be the same.

Matrix Vector Multiplication

- Matrix - vector multiplication: Map the column of the vector onto each row of the matrix, multiplying each element and summing the result.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a*x+b*y+c*z \\ d*x+e*y+f*z \\ g*x+h*y+i*z \end{bmatrix}$$

- Matrix - matrix multiplication: multiply two matrices by breaking it into several vector multiplications and concatenating the result

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} * \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} a*w+b*y & a*x+b*z \\ c*w+d*y & c*x+d*z \\ e*w+f*y & e*x+f*z \end{bmatrix}$$

- Number of columns of the first matrix must equal the number of rows of the second.
- Matrix Multiplication properties: **Not commutative**: A∗B≠B∗A. **Associative:** (A∗B)∗C=A∗(B∗C)

Inverse and Transpose

- The inverse of a matrix A is denoted A^{-1}. Multiplying by the inverse results in the identity matrix.
- A non square matrix does not have an inverse matrix. Matrices that don't have an inverse are called singular or degenerate matrix.
- The transposition of a matrix is like rotating the matrix 90° in clockwise direction and then reversing it:

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c & e \\ b & d & f \end{bmatrix}$$

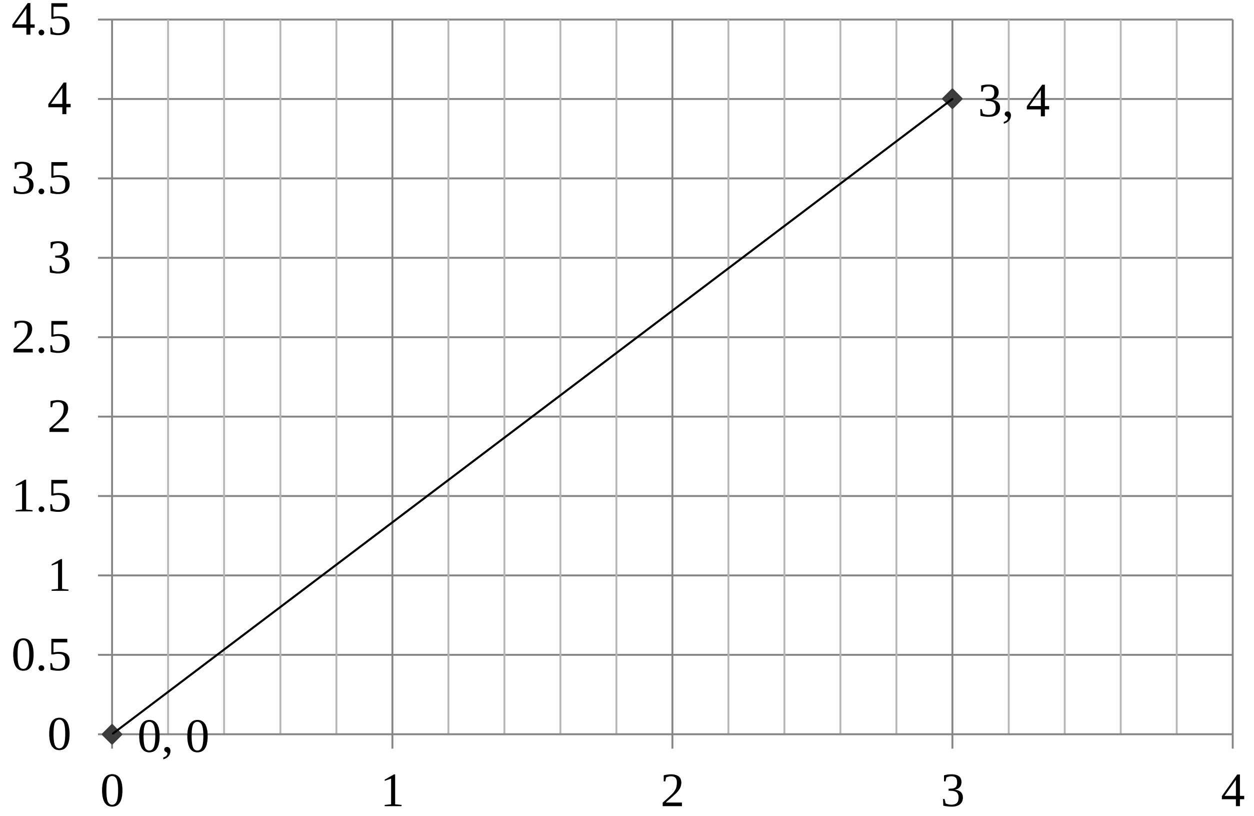
The identity matrix has 1's on the diagonal and 0's elsewhere.

The identity matrix, when multiplied by any matrix of the same dimensions, results in the original matrix.

Algebra–Vectors

Vectors

$Let\ A(3,4)\ be\ a\ pair\ in\ R^2\ i.e.set\ of\ ordered\ pair\ of\ real\ numbers.$



$x(x1, x2)\ x \neq 0\ specifies\ a\ vector\ in\ a\ plane\ starting\ at\ origin(0,0) and\ ending\ at\ x.$

Vector–Magnitude and Direction

- If point at origin is O, then $\overrightarrow{OA}$ *or* $\boldsymbol{u}$ represents a vector.

Vector is an object which has both magnitude and direction.

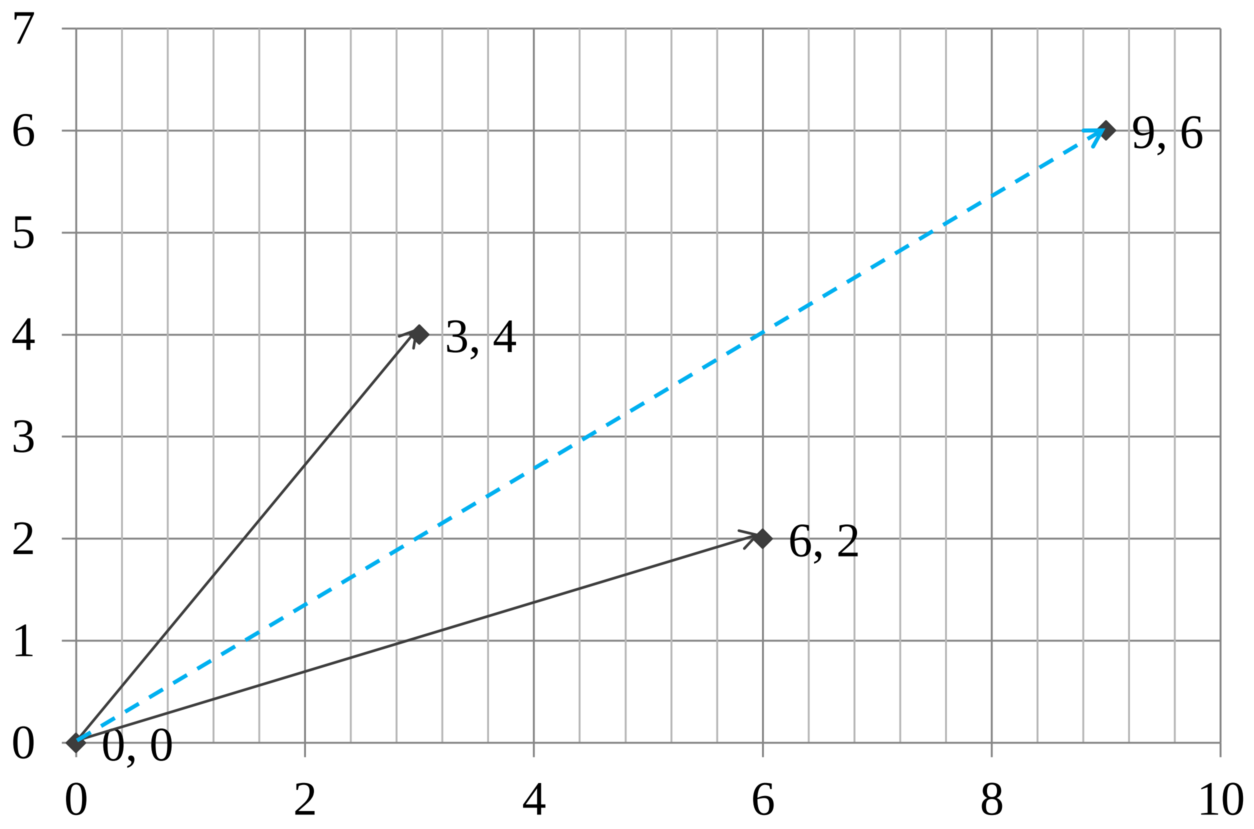
- Magnitude of a vector $\boldsymbol{u}$ *is* $||u||$ *and is called its norm*.
  - $For\ \overrightarrow{OA}$, $||OA||$ *is the lenght of the segment OA*.
  - By Pythagoras theorem, it will be $\sqrt{3^2 + 4^2} = 5 = ||OA||$
  - This also can be thought of as Euclidian distance between origin and vector **u**

- Direction of vector $\boldsymbol{u}$ *is a vector* $\boldsymbol{w}\left(\frac{\boldsymbol{u1}}{||\boldsymbol{u1}||}, \frac{\boldsymbol{u2}}{||\boldsymbol{u2}||}\right)$ which basically is the cosine of the angle formed with the axes.
  - $For\ \overrightarrow{OA}$, the direction is $\left(\frac{3}{5}, \frac{4}{5}\right) = (0.6,0.8\ )$

Norm of w will be 1 and thus direction vector are also unit vectors.

Vector – Addition

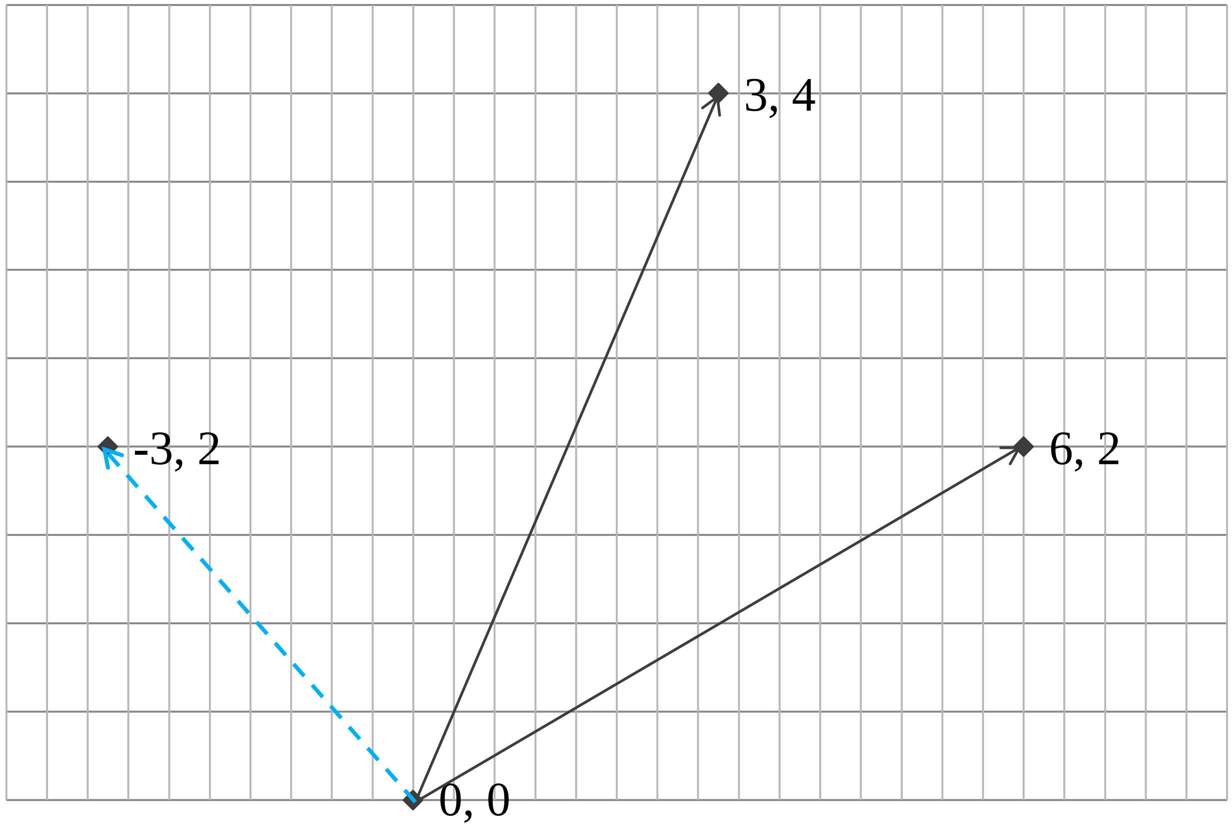
- $Given\ \boldsymbol{u}(u1, u2)\ and\ \boldsymbol{v}(v1, v2), the\ sum\ is\ a\ vector\ \boldsymbol{z}(u1 + v1, u2 + v2)$



Adding two vectors gives us a third vector whose coordinate are the sum of the coordinates of the original vectors.

Vector – Subtraction

- $Given\ \boldsymbol{u}(3,4)\ and\ \boldsymbol{v}(6,2), the\ subtraction\ is\ a\ vector\ \boldsymbol{z}(u1-v1, u2-v2)$



Vector – Subtraction

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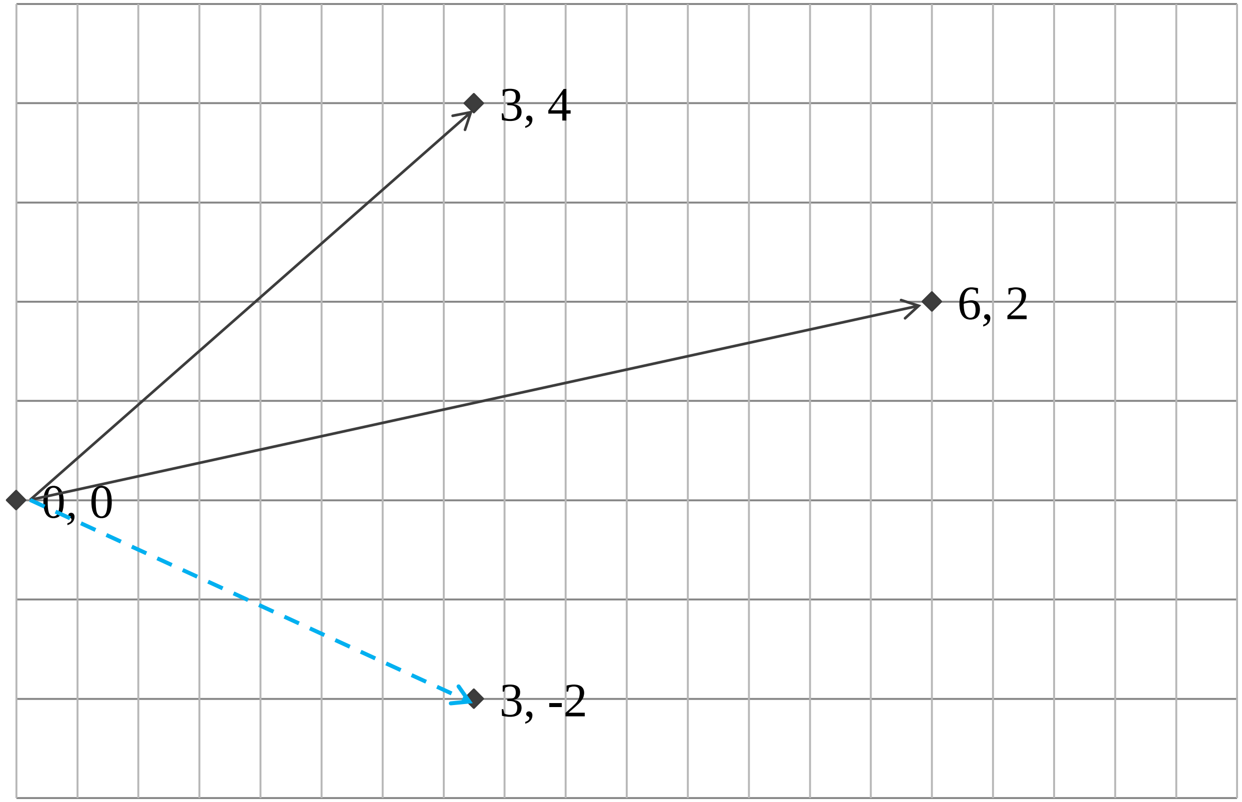
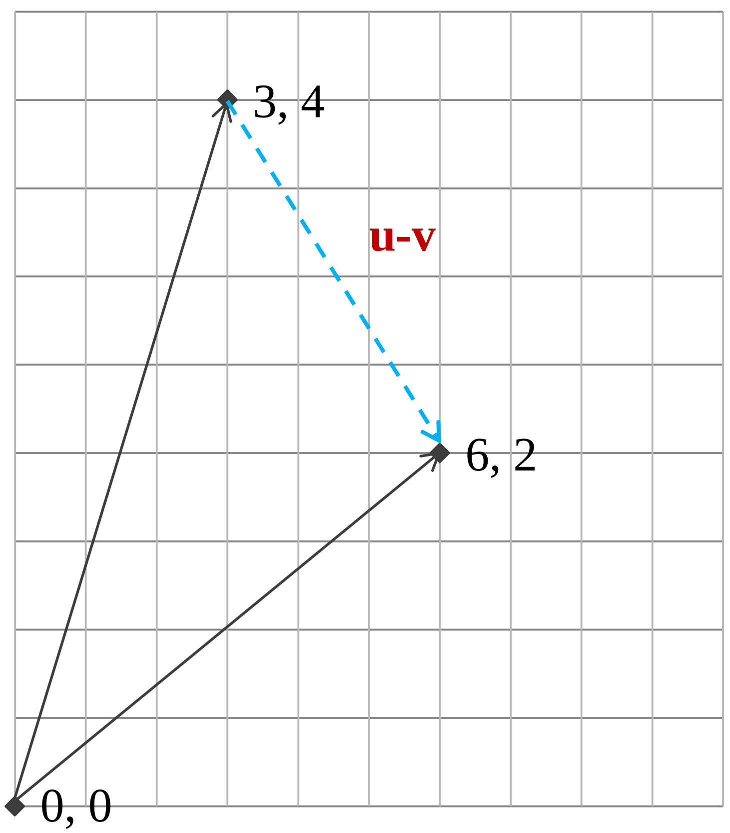
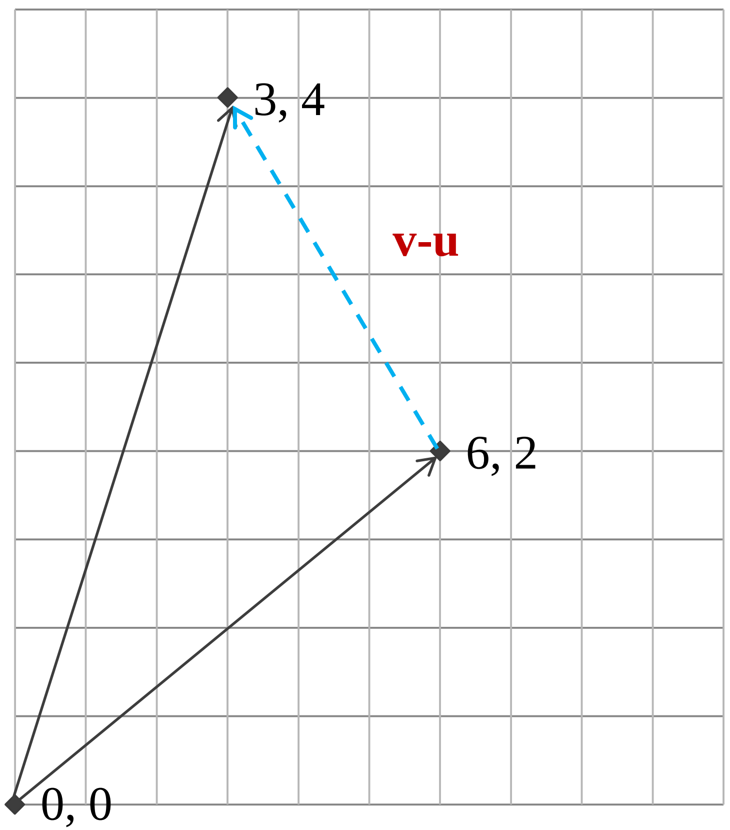


Vector – Subtraction

Parallel translate of a vector is the same vector drawn in different place in space.



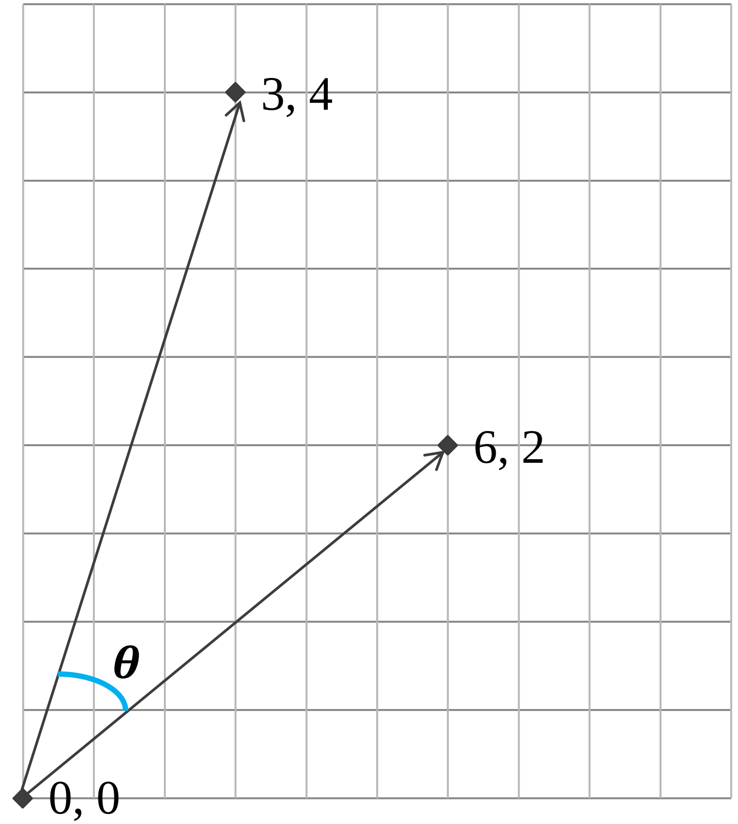


Dot Product

- Geometrically, it is the product of the Euclidian magnitudes of the two vectors and the cosine of the angle between them. It's the same as inner product in linear algebra.

- If we have two vectors **x**(3,4) and **y**(6,2) and there is an angle θ (theta) between them, their dot product is :

$$\mathbf{x}\cdot\mathbf{y} = \|x\|\|y\|\cos(\theta) = x1y1+x2y2$$



Orthogonal Projection of a vector

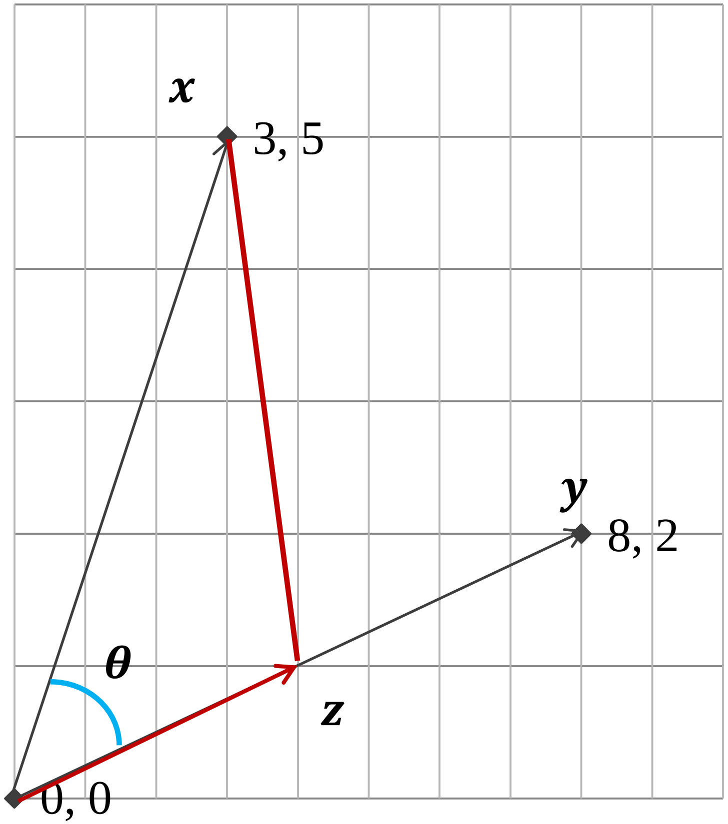
Let **z** be the orthogonal projection of **x** on **y**

$$cos(\theta) = \frac{||z||}{||x||}$$

$$cos(\theta) = \frac{\boldsymbol{x.y}}{||x||||y||} \; by\ dot\ product$$

$$||z|| = \left(\frac{\boldsymbol{x.y}}{||y||}\right)$$

$$||z|| = \boldsymbol{x.u}\,; where\ u\ is\ the\ direction\ of\ y$$



Here **u** is $\left(\frac{8}{\sqrt{68}}, \frac{2}{\sqrt{68}}\right)$. So $||z|| = \left(3 * \frac{8}{\sqrt{68}} + 5 * \frac{2}{\sqrt{68}}\right) = \frac{34}{\sqrt{68}}$

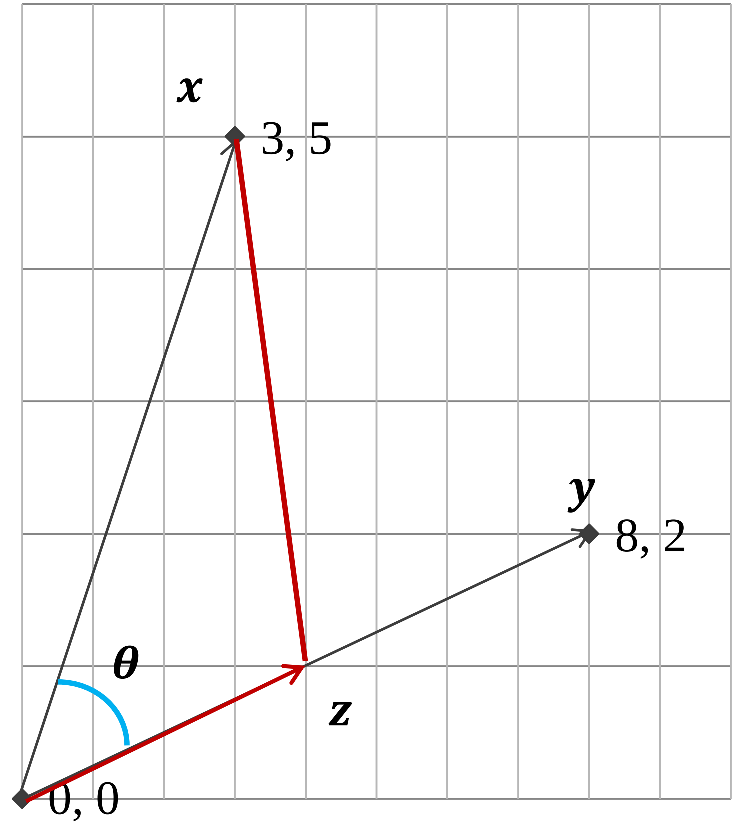
Orthogonal Projection of a vector

z has the same direction as **y,** so

$$\boldsymbol{u} = \frac{\boldsymbol{z}}{||z||}$$

$$\boldsymbol{z} = ||z||\boldsymbol{u}$$

So vector $\boldsymbol{z} = (\boldsymbol{x}.\boldsymbol{u})\boldsymbol{u}$ is the orthogonal projection of x on y.

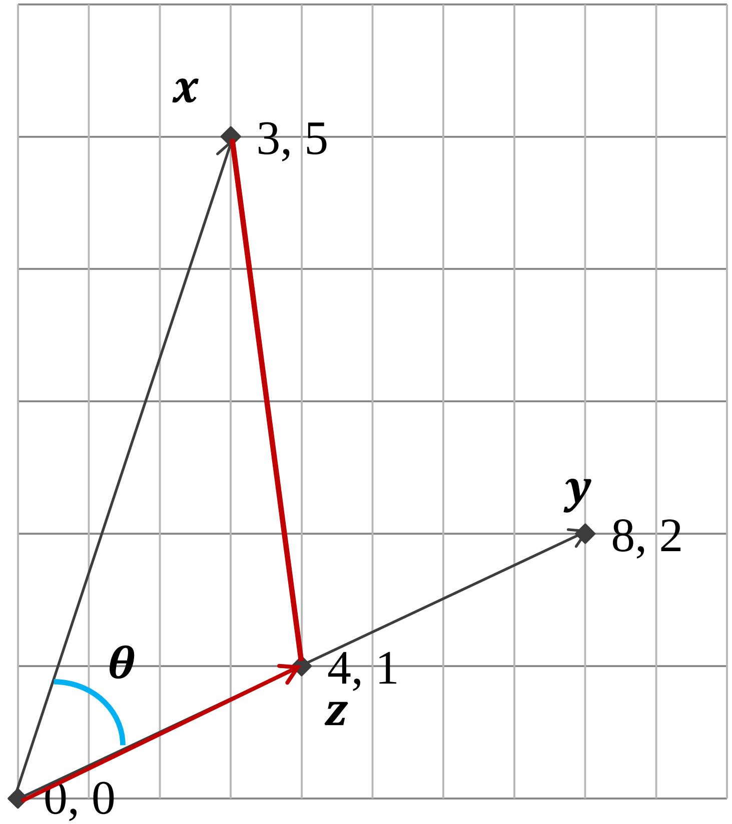


So $\mathbf{z} = \frac{34}{\sqrt{68}} * \left(\frac{8}{\sqrt{68}}, \frac{2}{\sqrt{68}}\right) = (4,1)$

Why Orthogonal Projection

It helps compute the distance between **x** and the line which goes through **y**

$$||\boldsymbol{x} - \boldsymbol{z}|| = \sqrt{(\mathbf{3} - \mathbf{4})^{\mathbf{2}} + (\mathbf{5} - \mathbf{1})^{\mathbf{2}}} = \sqrt{\mathbf{17}}$$



End of Lesson01–Matrix and Vectors