Chaotic Motion in a Double Pendulum System and Convective Transport in Phase Space

Tushar Muralidharan

31 May 2022

1 Introduction

The study of dynamical systems is of great import in many branches of mathematics and physics. In simple terms, a *dynamical system* may be described as a system that changes over time [3]. *Chaos* is then generally defined as a dynamical system that is highly sensitive to initial conditions and produces states of disorder [1].

A pendulum with another pendulum attached to its end, the double pendulum is one of the simplest dynamical systems that exhibit chaotic behavior. The motion of particles in the presence of an electrostatic wave also comprises a dynamical system, with which the study of convective transport is much concerned. Extensive inroads have been made on each of these topics by Shinbrot et al. and Koch, but, in this paper, we aim to model and investigate these phenomena through the theory of ordinary differential equations and numerical methods.

2 Methods

A dynamical system consists of time-dependent *state variables*, along with a rule that allows us to determine the values of these variables based on their specified values at some point in time [3]. The values of the state variables are collectively referred to as the *state* of the system, and the initial values of these variables are called the *specified state* of the system. The behaviour of the system corresponding to a specified state is called the *response* of the system. Dynamical systems where state variables are defined over a continuous range of time are called *continuous-time* dynamical systems [3]. In this paper, we will only be studying continuous-time dynamical systems.

In continuous-time dynamical systems, the rule accompanying the dynamical system often takes the form of *ordinary differential equations.* An ordinary differential equation is an equation containing only ordinary derivatives of one or more unknown functions, which are called *dependent variables*, with respect to a single *independent variable* [3]. An nth-order *initial-value problem* consists of obtaining a solution $y(t)$ of an nth-order ordinary differential equation

$$\frac{d^n y}{dt^n} = f(x, y, y', \ldots, y^{(n-1)})$$

subject to n side conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \ldots, \quad y^{(n-1)}(t_0) = y_{n-1}$$

specified at some t_0, which are called *initial conditions* [3]. Finally, a *system of ordinary differential equations* consists of two or more equations involving the derivatives of two or more unknown functions with respect to a single independent variable [3].

A double pendulum exists in the setting of classical mechanics. Naturally, mathematical formulations of classical mechanics are various. Most familiar to us is the *Newtonian* formulation, which uses Cartesian coordinates and Newton's Second Law of Motion. However, there exist the more widely-used *Lagrangian* and *Hamiltonian* formulations, the latter of which is particularly well-suited to numerical solving. Therefore, we narrow our focus to the Hamiltonian formulation.

In order to proceed, we must first understand the idea of a phase space. A *phase space* of is a multidimensional space consisting of the state variables of a dynamical system such that a point in the space represents a state of a system. In phase space coordinates $\mathbf{q}$ and $\mathbf{p}$, Hamilton's equations are given by

$$\begin{aligned} \frac{d\mathbf{q}}{dt} &= \frac{\partial H}{\partial \mathbf{p}} \\ \frac{d\mathbf{p}}{dt} &= -\frac{\partial H}{\partial \mathbf{q}}, \end{aligned} \tag{1}$$

where $H(\mathbf{q}, \mathbf{p}, t)$ is a function known as a *Hamiltonian.*

Definition 1 (Hamiltonian system)**.** A *Hamiltonian system* is a continuous-time dynamical system with a Hamiltonian function $H(\mathbf{q}, \mathbf{p}, t)$ that is governed by Hamilton's equations.

The response of a Hamiltonian system is given by the initial value problem consisting of Hamilton's equations together with the initial conditions $\mathbf{q}(t_0) = \mathbf{q}_0$ and $\mathbf{p}(t_0) = \mathbf{p}_0$. In the multidimensional case, the variables $\mathbf{q}$ and $\mathbf{p}$ are each placeholders for multiple variables, which are called position and momentum variables, respectively. For example, in a two-dimensional Hamiltonian system, we have two position variables q_1, q_2 and two momentum variables p_1, p_2. Thus, the phase space of such a system would consist of four variables, which is difficult to visualise.

In order to visualise a two-dimensional dynamical system, we may create a *Poincaré plot*. We may think of solutions to the various responses of a Hamiltonian system as trajectories in phase space, which are parameterised by time. The process of creating a Poincaré plot involves choosing an oriented two-dimensional surface $S \subset \mathbb{R}^4$ and recording successive intersections of trajectories with this surface. An illustration is given in Figure 1.

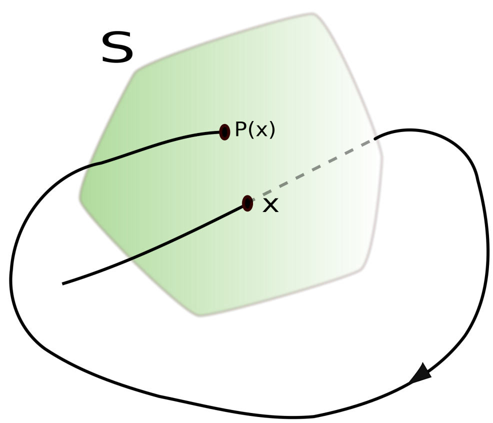

Figure 1: The intersections of a trajectory with a surface S [2]

The standard way of choosing a two-dimensional surface for a Poincaré plot is to fix a constant value for the Hamiltonian $H(q_1, q_2, p_1, p_2)$ as well as one other variable. If a trajectory intersects the surface S at a point x, then $P(x)$ is the next intersection of the trajectory with the surface.

Definition 2 (Fixed point of a Poincaré plot)**.** A point x^* on the surface of a Poincaré plot is called a fixed point if there exists an $n \in \mathbb{N}$ such that $P^n(x^*) = x^*$.

It follows that locating fixed points allows us to locate periodic orbits in the phase space.

In many cases, solutions to initial value problems of Hamiltonian systems are almost impossible to find analytically. Thus, if we are to understand solutions to such problems, then we must resort to numerical methods. There is still rich theory in the area of numerical solving, for which we will use the numerical computing software and programming language MATLAB to assist.

3 Results

3.1 Chaotic Motion in a Double Pendulum System

A double pendulum is a pendulum with another pendulum attached to its end. An illustration of a double pendulum system may be seen in Figure 2.

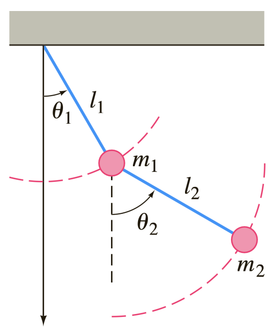

Figure 2: Double pendulum [3]

The Hamiltonian governing the dynamics of a double pendulum system is given by

$$\begin{aligned} H(\theta_1, \theta_2, p_1, p_2) = {} & \frac{l_2^2 m_2 p_1^2 + l_1^2 p_2^2 (m_1 + m_2) - 2 l_1 l_2 m_1 p_1 p_2 \cos(\theta_1 - \theta_2)}{2 l_1^2 l_2^2 m_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]} \\ & - m_2 g l_2 \cos\theta_2 - (m_1 + m_2) g l_1 \cos\theta_1, \end{aligned} \tag{2}$$

where g is the gravitational constant, $m_1, m_2, l_1, l_2, \theta_1, \theta_2$ are in accordinace with Figure 2, and p_1, p_2 are the canonical momenta of the corresponding masses.

Here we are dealing with a two-dimensional Hamiltonian system with two position variables θ_1, θ_2 and two momentum variables p_1, p_2. Hamiltonian's equations then allow us to derive the system of ordinary differential equations governing the evolution of the system, where we use the equations for each pair θ_i, p_i. Using Newton's dot notation, we have

$$\begin{aligned} \dot{\theta}_1 = & \quad \frac{\partial H}{\partial p_1} && = \frac{l_2 p_1 - l_1 p_2 \cos(\theta_1 - \theta_2)}{l_1^2 l_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]} \\ \dot{\theta}_2 = & \quad \frac{\partial H}{\partial p_2} && = \frac{l_1 p_2 (m_1 + m_2) - l_2 m_2 p_1 \cos(\theta_1 - \theta_2)}{l_1 l_2^2 m_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]} \\ \dot{p_1} = & -\frac{\partial H}{\partial \theta_1} && = -g l_1 (m_1 + m_2) \sin\theta_1 - C_1 \\ \dot{p_2} = & -\frac{\partial H}{\partial \theta_2} && = -g l_2 m_2 \sin\theta_2 + C_1, \end{aligned} \tag{3}$$

where

$$C_1 = \frac{\sin(\theta_1 - \theta_2)[l_1 p_2 \cos(\theta_1 - \theta_2) - l_2 p_1][l_2 m_2 p_1 \cos(\theta_1 - \theta_2) - l_1 p_2 (m_1 + m_2)]}{l_1^2 l_2^2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]^2}.$$

To numerically solve initial value problems that use this system of ordinary differential equations, we will utilize a numerical technique called the fourth-order Runge-Kutta method. Our method may be described as follows. Let y be a vector-valued function whose components are $\theta_1, \theta_2, p_1, p_2$, respectively. We then discretise the solution across time based on a step size h. If y_i is the value of the solution after i steps, then the approximate values for the solution y are given by

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

where

$$
\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f\left(t_n + \frac{1}{2}h, y_n + + \frac{1}{2}hk_1\right) \\
k_3 &= f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right) \\
k_4 &= f(t_n + h, y_n + hk_3).
\end{aligned}
$$

It is important to test this procedure with simpler problems before proceeding. As a first test, we consider the simple pendulum. The Hamiltonian of this system is given by

$$
H(\theta, p) = \frac{p^2}{2ml^2} - mlg\cos\theta
$$

Then Hamilton's equations give

$$
\begin{aligned}
\dot{\theta} &= \frac{\partial H}{\partial p} &&= \frac{p}{ml^2} \\
\dot{p} &= -\frac{\partial H}{\partial \theta} &&= -mlg\sin\theta.
\end{aligned}
$$

We may now compare the solutions found by our fourth-order Runge-Kutta method and the built-in ode45 solver in MATLAB. The comparison is shown in Figure 3.

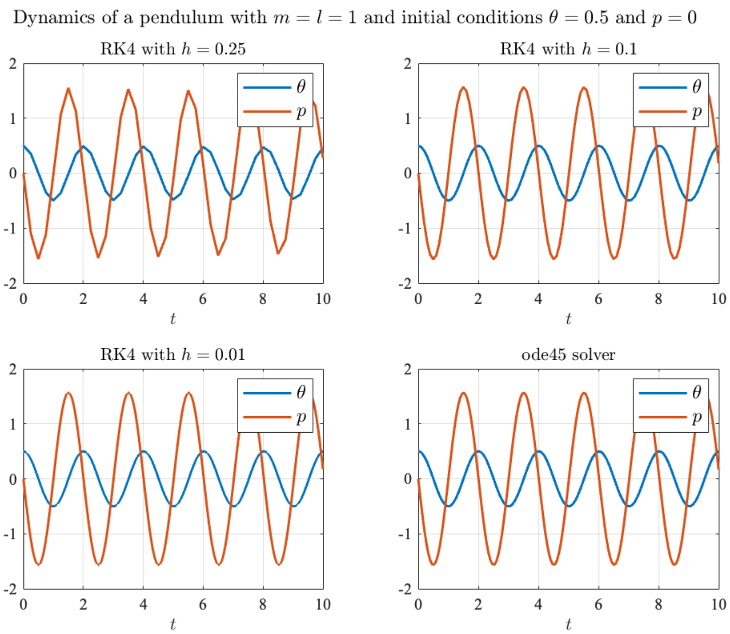

Figure 3: RK4 scheme vs ode45 solver for simple pendulum

We see that the fourth-order Runge-Kutta method is extremely accurate, and the difference between its solution and the exact solution becomes imperceptible as the step size decreases. The truncation error in our fourth-order Runge-Kutta method means that our numerical solver does not preserve energies, but the mean-squared error of the numerical solution quickly becomes insignificant as the step size decreases below 0.1.

Therefore, we may wholeheartedly trust our fourth-order Runge-Kutta solver for the double pendulum system. The comparison between our fourth-order Runge-Kutta solver using the system of equations in (3) and the ode45 solver is shown in Figure 4.

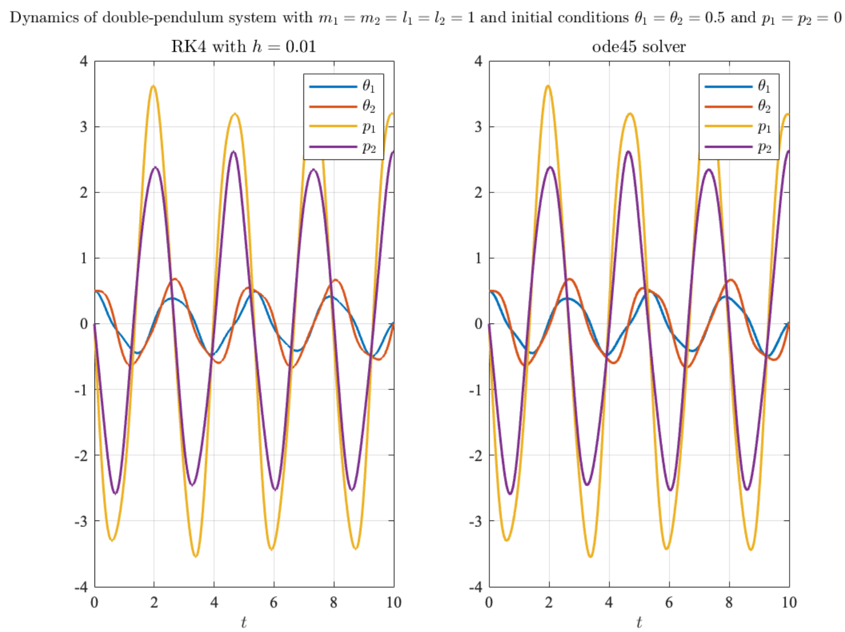

Figure 4: RK4 scheme vs ode45 solver for double pendulum

Here too we support our conclusion that the fourth-order Runge-Kutta method is extremely accurate and maintains fourth-order convergence, yet does not compromise in simplicity.

To gather more information about the behavior of the double-pendulum system, we examine the equilibrium points of the system. An *equilibrium point* is a point at which all the rates of the system are zero. That is, a point is an equilibrium point if it makes all of the equations in (3) equal zero. This defines a system of four nonlinear equations in four unknowns, for which numerical solutions yield the four equilibrium points in Table 1.

θ_1	θ_2	p_1	p_2
0	0	0	0
0	π	0	0
π	0	0	0
π	π	0	0

Table 1: Equilibrium points of the double pendulum system

These points conform with our intuition, because our intuition tells us that a pendulum in any of these configurations would remain still as long as it is undisturbed. However, each equilibrium points may have drastically different properties with respect to surrounding trajectories [2].

We now seek to create Poincaré plot for the double pendulum system. We will assume that $m_1 = m_2 = l_1 = l_2 = 1$. If we choose a constant energy $H = H_{\text{min}} + 10$ and $\theta_1 = 0$, we create a two-dimensional surface. The Poincaré plot in the θ_2-p_2 plane using these constraints is shown in Figure 5.

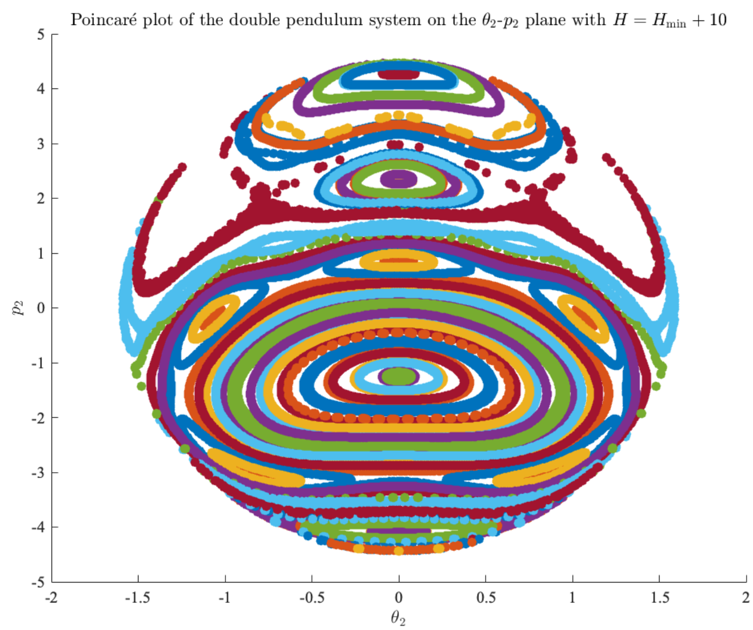

Figure 5: Poincaré plot of the double pendulum system on the θ_2-p_2 plane with $H = H_{\text{min}} + 10$.

The Poincaré plot contains many loops, which are invariant tori caused by our two angle variables, and eye-shaped regions called island chains. If we increase the constant energy to $H = H_{\text{min}} + 15$, we obtain the plot shown in Figure 6.

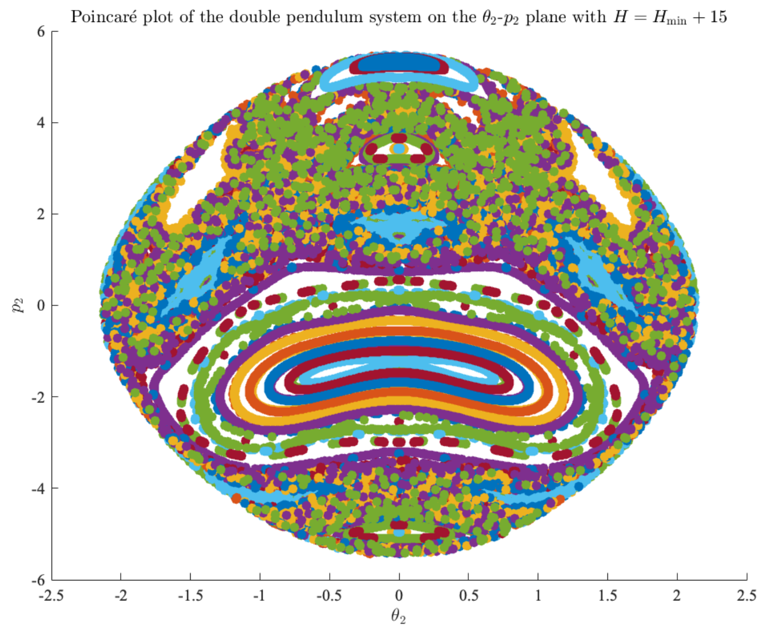

Figure 6: Poincaré plot of the double pendulum system on the θ_2-p_2 plane with $H = H_{\min} + 15$.

We begin to see fewer invariant tori, and much of the upper region plot has become densely covered with points. We call this region the *chaotic* region. Increasing the energy even further to $H = H_{\min} + 20$, the chaotic dynamics are fully apparent. This is seen in Figure 7.

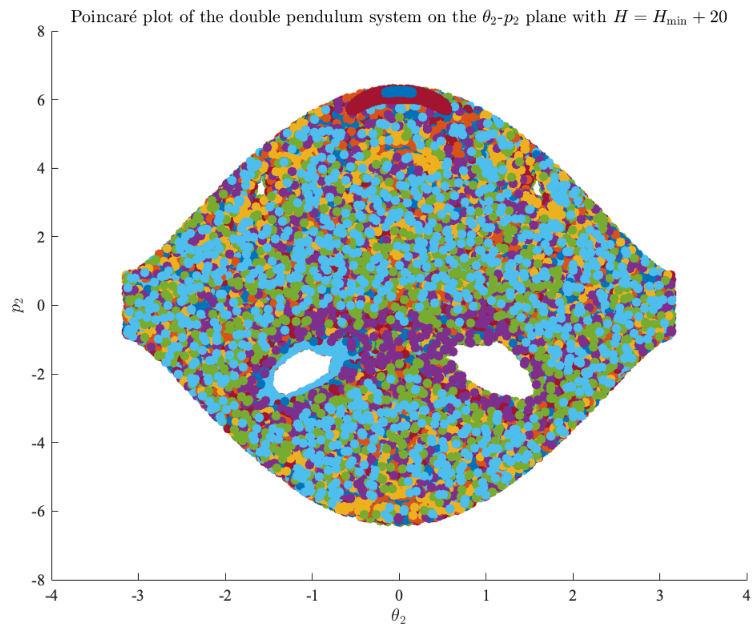

Figure 7: Poincaré plot of the double pendulum system on the θ_2-p_2 plane with $H = H_{\min} + 20$.

In this plot, the chaotic region encompasses a majority of the plot and the invariant tori and island chains have almost completely disappeared.

Finally, we are interested in determining the fixed points of the plot in Figure 5. By the definition of a fixed point, we are seeking a point x^* such that there exists an $n \in \mathbb{N}$ with $P^n(x^*) = x^*$. Since θ_2 is mod 2π, this defines an equation that we may solve numerically in Matlab using the `fsolve` command. The fixed points of the system are shown in Figure 8

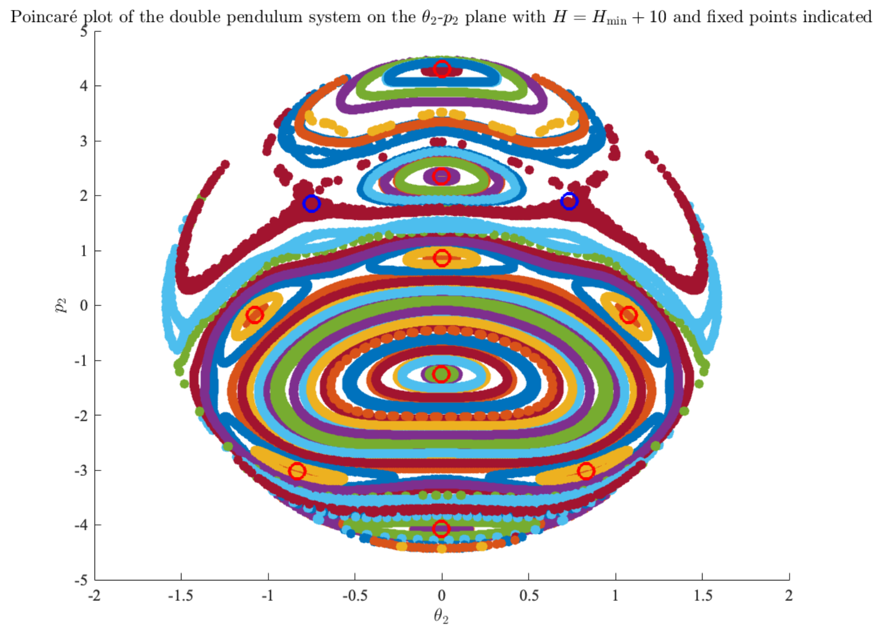

Figure 8: Fixed points of the Poincaré plot with $H = H_{\text{min}} + 10$

Some fixed points are located in the middle of islands while others are located between islands. Fixed points with different rotation numbers are indicated in different colors.

3.2 Convective Transport in a Phase Space

We will now model the convective transport of particles in the presence of an electrostatic wave, which sheds light on the phenomenon of wave-particle interaction. The frame moves with the fast oscillatory component of the wave but remain stationary with respect to the slow frequency component. The Hamiltonian of such a system is given by

$$H(p, \Theta, t) = \frac{[p - p_s(t)]^2}{2} + \phi(t) \cos \Theta \tag{4}$$

where $p \leftarrow \mathbb{R}$ and $\Theta \leftarrow \mathbb{R} \mod 2\pi$. We are given that $m = e = 1$ and $\phi(t) = 1$.

Using Hamilton's equations, the system of ordinary differential equations that govern the evolution of this system are

$$\begin{aligned} \dot{p} = & -\frac{\partial H}{\partial \Theta} & = \sin\Theta \\ \dot{\Theta} = & \quad \frac{\partial H}{\partial p} & = p - p_s(t). \end{aligned} \tag{5}$$

The function $p_s(t)$ may be thought of as the frequency of the wave.

Let us assume that $p_s = 1$. As in the double-pendulum case, the Hamiltonian becomes time-independent. The system of differential equations in (5) then becomes

$$\dot{p} = \sin\Theta.$$
$$\dot{\Theta} = p - 1$$

Solving this system with various initial conditions using the fourth-order Runge-Kutta method described in Section 3.1 gives the phase dynamics seen in Figure 9.

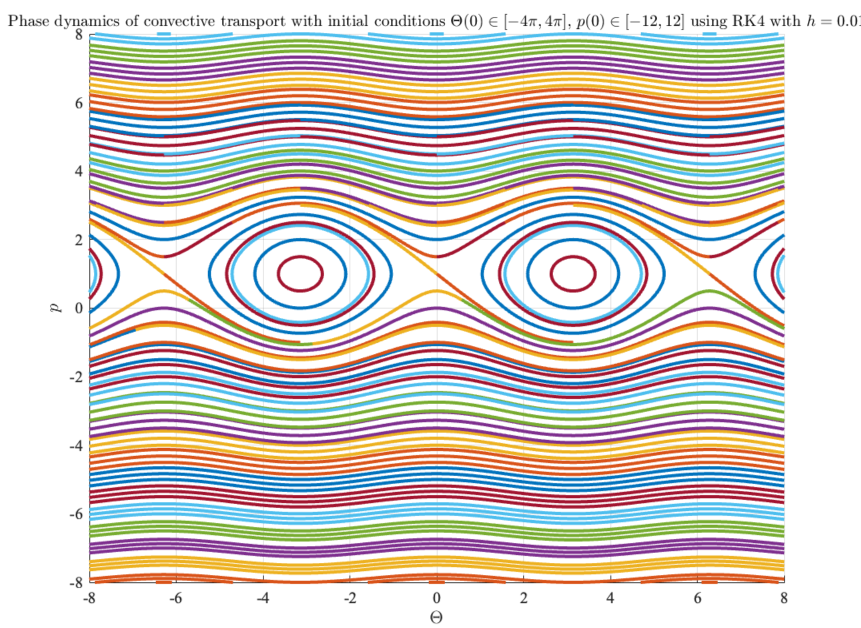

Figure 9: Phase dynamics of convective transport with initial conditions $\Theta(0) \in [-4\pi, 4\pi]$, $p(0) \in [-12, 12]$

The phase dynamics of this system are strikingly similar to those of a single pendulum system. The trajectories inside the eye-shaped regions are closed, whereas those outside are not. Like a single pendulum system, this system shows distinct separatrices with periodic attractor points at $\Theta = (2k-1)\pi$ for some integer k.

To study time-dependent Hamiltonian systems, we now relax our assumption that $p_s = 1$ and introduce in p_s a term proportional to the time t. Let $p_s(t) = ct + 1$. Using the system of equations in (5), we now have

$$\dot{p} = \sin\Theta.$$

$$\dot{\Theta} = p - ct - 1$$

We are interested in the behavior of a particle trapped inside the wave. In other words, we want to choose a point from inside a separatrix. Therefore, we choose the initial conditions $p(0) = 3$ and $\Theta(0) = 1$. Solving this system numerically for while increasing the value of c gives the results in Figure 10

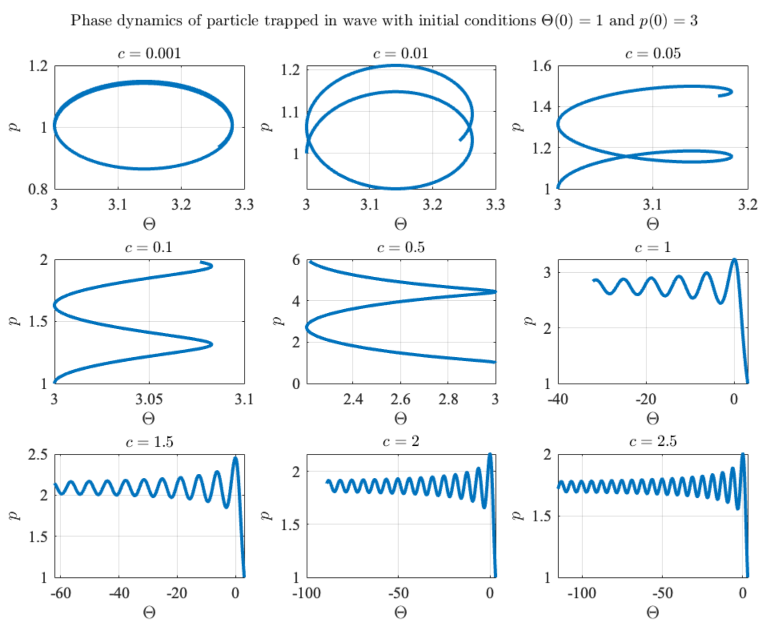

Figure 10: Phase dynamics of a particle trapped in the wave for increasing values of c

When c is small, the system largely maintains its time-independent characteristics and the dynamics of the particle are similar to that of a loop in an island. This is because if the time-dependent term $p_s(t)$ of the Hamiltonian changes slowly enough, the area of the ellipse in phase space remains approximately constant throughout time because the area is an adibatic invariant. This speaks to the fact that the separatrices in the phase dynamics change whenever c changes. This may be seen in the first plot of Figure 10. As c increases, the ellipse becomes increasingly distorted and the frequency of oscillation increases because the energy of each trajectory changes more drastically with time.

4 Discussion

In studying chaotic motion in a double pendulum system, we have been able to interpret the behavior of a Hamiltonian system with a Poincaré plot and detect chaos through visual means. As for convective transport in phase space, we carefully examined the phase dynamics of a time-dependent Hamiltonian system and saw principles of adiabatic invariance reflected in phase space.

Both experiments were conducted with use of a custom numerical solver, which is practical and applicable to other problems. The results presented are limited, of course, by numerical precision, and many further simulations were impossible due to hardware constraints. The study would benefit from a more thorough development of rotations of fixed points, descriptions of the behavior of equilibrium points, and analysis of orbit types.

References

[1] The Editors of Encyclopaedia Britannica. chaos theory, 2021.

[2] Zhisong Qu. *Regular and Chaotic Dynamics Lecture Notes.* 2020.

[3] Dennis Zill. *A First Course in Differential Equations with Modeling Applications.* Cengage Learning, 11th edition, 2018.