Representation Theory of the Finite Group $GL_2(\mathbb{F}_p)$

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1 Introduction

In this paper, we study the representations of the general linear group $GL_2(\mathbb{F}_p)$ over a finite field up to isomorphism. We do so in a constructive manner relying on basic results in representation theory.

2 $GL_2(\mathbb{F}_p)$

The group $GL_2(\mathbb{F}_p)$ of invertible 2×2 matrices with entries in the finite field $\mathbb{F}_p$ with p elements, where p is a prime, is an important object of study in representation theory. For starters, the group is finite and non-abelian. There are $p^2 - 1$ possibilities for the first column whose entries are both nonzero and $p^2 - p$ possibilities for the second column that are not multiples of the first column. Let $G = GL_2(\mathbb{F}_p)$. Hence we have $|G| = (p^2 - 1)(p^2 - p) = p(p+1)(p-1)^2$.

3 Conjugacy Classes of $GL_2(\mathbb{F}_p)$

It is well known that there is a bijective correspondence between the conjugacy classes of a group and its irreducible representations, so we start our investigation there.

Each of the $p - 1$ matrices $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is fixed under conjugation and defines a conjugacy class with one element. These matrices have one repeated eigenvalue and are diagonalizable.

Next we consider the matrices that are conjugate to $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$. These matrices have one repeated eigenvalue but are not diagonalizable. Since conjugation preserves eigenvalues, each such matrix defines a distinct conjugacy class. The centralizer of this matrix consists of the $p(p-1)$ matrices $\begin{pmatrix} b & c \\ 0 & b \end{pmatrix}$. The counting formula then tells us that the order of the conjugacy class is $|G|/p(p-1) = (p+1)(p-1)$. We have $p - 1$ such conjugacy classes, one for each choice of a.

Now we consider the matrices that are conjugate to $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ for $a \neq b$. These matrices have two distinct eigenvalues and are always diagonalizable. The centralizer of this matrix consists of the $(p-1)^2$ diagonal matrices in G. The counting formula then tells us that the order of the conjugacy class is $|G|/(p-1)^2 = p(p+1)$. Since $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ is conjugate to $\begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$, there are $\binom{p-1}{2} = (p-1)(p-2)/2$ such conjugacy classes, one for each choice of $a \neq b$.

Since $\mathbb{F}_p$ need not be algebraically closed, the roots of a characteristic polynomial may not be in $\mathbb{F}_p$. Fix some x in $\mathbb{F}_p$ that is not a perfect square (the existence of which is guaranteed for all $p > 2$), and consider the matrices that are conjugate to $\begin{pmatrix} a & xb \\ b & a \end{pmatrix}$ for $b \neq 0$. The eigenvalues of this matrix are $a \pm b\sqrt{x}$, which is not in $\mathbb{F}_p$ but the quadratic extension thereof. The number of elements in this conjugacy class is $p(p-1)$, and there are $p(p-1)/2$ such conjugacy classes [1].

Since the classes are disjoint and account for $|G|$ elements, these are all of the conjugacy classes of G. In total, we find $p^2 - 1$ conjugacy classes, which must be equal to the number of irreducible representations of G.

4 Representations of $GL_2(\mathbb{F}_p)$

4.1 Type I

For any character χ of $\mathbb{F}_p^\times$, we obtain a one-dimensional and hence irreducible character $\chi \circ \det\colon G \to \mathbb{C}^\times$. This character acts on each type of conjugacy class as follows.

	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{a\neq b}$	$\begin{pmatrix} a & xb \\ b & a \end{pmatrix}_{b\neq 0}$
$\chi \circ \det$	$\chi(a)^2$	$\chi(a)^2$	$\chi(ab)$	$\chi(a^2 - xb^2)$

There are $p-1$ characters of this form because each character is uniquely determined by $\chi(1)$, which must be a $(p-1)$th root of unity.

4.2 Type II (Steinberg)

We identify the subgroups

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right\} \supset N = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\}, \quad B/N = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right\} \cong {\mathbb{F}_p}^\times \times {\mathbb{F}_p}^\times.$$

For any character χ of $\mathbb{F}_p^\times$, consider the character $\mathrm{Ind}_B^G \mathrm{Res}_B^G(\chi \circ \det)$ of G. However, this character is not irreducible since the Frobenius reciprocity gives

$$\langle \mathrm{Ind}_B^G \mathrm{Res}_B^G(\chi\circ\det), \chi\circ\det\rangle_G = \langle Res_B^G(\chi\circ\det), Res_B^G(\chi\circ\det)\rangle_B = \frac{1}{|B|}\sum_{b\in B} |\chi\circ\det(b)|^2 = 1.$$

We see that $\mathrm{Ind}_B^G \mathrm{Res}_B^G(\chi \circ \det) - \chi \circ \det$ is an irreducible character by computing its inner product with itself [2]. The values of this character are computed using the induced character formula [3] and subtracting the values of the corresponding characters.

$\mathrm{Ind}_B^G \mathrm{Res}_B^G(\chi \circ \det) - \chi \circ \det$	$p\chi(a)^2$	0	$\chi(ab)$	$-\chi(a^2 - xb^2)$

We again have $p-1$ such irreducible characters, one for each χ.

4.3 Type III (Principal Series)

Let $\pi \colon B \to B/N$ be a quotient map. For any two characters χ_1, χ_2 of ${\mathbb{F}_p}^\times$, let consider the character of B that sends

$$B \xrightarrow{\pi} B/N \cong \mathbb{F}_p^\times \times \mathbb{F}_p^\times \xrightarrow{\chi_1,\chi_2} \mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{\cdot} \mathbb{C}^\times.$$

Then the induced representation $\chi_{1,2}$ on G is irreducible [1] and acts according to:

$\chi_{1,2}$	$(p+1)\chi_1(a)\chi_2(a)$	$\chi_1(a)\chi_2(a)$	$\chi_1(a)\chi_2(b) + \chi_1(b)\chi_2(a)$	0

There are $(p-1)(p-2)/2$ such characters, one for each distinct χ_1 and χ_2.

4.4 Type IV (Cuspidal)

Consider the subgroup

$$K = \left\{ \begin{pmatrix} a & xb \\ b & a \end{pmatrix} \right\}$$

of G. For any character $\varphi \colon K^\times \to \mathbb{C}^\times$, we have the induced character $\chi_{\mathrm{Ind}\,\varphi}$ on G. We may also invoke a type II character with the trivial representation and a type III character with χ_1 arbitrary and χ_2 trivial to obtain the characters $\chi_{II(\mathrm{triv})}$ and $\chi_{III(\_,\mathrm{triv})}$, respectively. A class of irreducible representations is then given by $\chi_{II(\mathrm{triv})} - \chi_{III(\_,\mathrm{triv})} - \chi_{\mathrm{Ind}\,\varphi}$, of which there are $p(p-1)/2$ [1].

$\chi_{II(\mathrm{triv})} - \chi_{III(\_,\mathrm{triv})} - \chi_{\mathrm{Ind}\,\varphi}$	$(p-1)\varphi(a)$	$-\varphi(a)$	0	$-(\varphi(a + b\sqrt{x}) + \varphi(a + b\sqrt{x})^p)$

5 Summary

In a lengthier paper, it might be illuminating to verify the irreducibility of characters of types II, III, and IV via Schur's orthogonality relations and the conditions on χ_1, χ_2 that lead to irreducibility in the type III case. Also, additional motivation for the constructions could be of immense utility to the reader. In total, we have accounted for $p^2 - 1$ irreducible representations, so the complete character table for the irreducible representations of $GL_2(\mathbb{F}_p)$ is provided below.

	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{a \neq b}$	$\begin{pmatrix} a & xb \\ b & a \end{pmatrix}_{b \neq 0}$
$\chi \circ \det$	$\chi(a)^2$	$\chi(a)^2$	$\chi(ab)$	$\chi(a^2 - xb^2)$
$\operatorname{Ind}_B^G \operatorname{Res}_B^G(\chi \circ \det) - \chi \circ \det$	$p\chi(a)^2$	0	$\chi(ab)$	$-\chi(a^2 - xb^2)$
$\chi_{1,2}$	$(p+1)\chi_1(a)\chi_2(a)$	$\chi_1(a)\chi_2(a)$	$\chi_1(a)\chi_2(b) + \chi_1(b)\chi_2(a)$	0
$\chi_{II(\mathrm{triv})} - \chi_{III(\_,\mathrm{triv})} - \chi_{\operatorname{Ind}\varphi}$	$(p-1)\varphi(a)$	$-\varphi(a)$	0	$-(\varphi(a + b\sqrt{x}) + \varphi(a + b\sqrt{x})^p)$

Table 1: Character table of $GL_2(\mathbb{F}_p)$

References

[1] Fulton, W. & Harris, J. (1996) *Representation Theory: A First Course*, Springer-Verlag.

[2] Lang, S. (2002) *Algebra*, Springer-Verlag.

[3] Serre, J.-P. (1977) *Linear Representations of Finite Groups*, Springer-Verlag.