Non-Positive Curvature and the Word Problem in Group Theory

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Declaration

The work in this thesis is my own except where otherwise stated.

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Abstract

This thesis investigates the interplay between algebraic and geometric approaches to group theory through the lens of the word problem and non-positive curvature. We begin by introducing Dehn functions as a tool for measuring the complexity of the word problem. Next, we present van Kampen diagrams as geometric tools for analyzing the word problem and prove van Kampen's Lemma. We give an account of key results in small cancellation theory, including including isoperimetric inequalities for groups satisfying the conditions $C'(1/6)$, $C(6)$, and $C(4)$–$T(4)$.

To study groups as metric spaces and examine their curvature, we model them using Cayley graphs and introduce quasi-isometries to compare their large-scale geometry, with a brief discussion of commensurability. In the setting of non-positive curvature, we define hyperbolic and CAT(0) groups and establish isoperimetric inequalities for both. Finally, we study the relationship between subgroup distortion and Dehn functions, and present the Bieri doubling trick—a technique for constructing subgroups of CAT(0) groups with arbitrarily large Dehn functions.

Contents

Notation

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$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$	The natural numbers (including 0), the integers, the rational numbers, the real numbers, the complex numbers
$\mathbb{E}^2$	The Euclidean plane
G, Γ	A group
$1 \in G$	The identity element of G
$G \cong K$	G is isomorphic to K
$F(A)$	The free group on the set A
$\lvert w \rvert$	The length of a word w
$\equiv$	Equivalence in the free group
$\mathbb{P}$	The isoperimetric spectrum
$G_1 * G_2$	The free product of two groups G_1 and G_2
$G_1 *_H G_2$	The amalgamated free product of groups G_1 and G_2 along a subgroup H
$G \rtimes H$	The semidirect product of groups G and H
$\mathrm{Cay}(G, A)$	The Cayley graph of a group G with respect to a generating set A
$\delta_G(n)$	The Dehn function of G
$\delta_H^G(n)$	The distortion of the subgroup H in the group G

Chapter 1

Introduction

> The study of decision problems in group theory is a subject that does not impinge on most geometers' lives – for many it remains an apparently arcane region of mathematics near the borders of group theory and logic, echoing with talk of complexity and undecidability, devoid of the light of geometry... Despite this sharp contrast in emotions, the study of the large scale geometry of least-area discs in Riemannian manifolds is intimately connected with the study of the complexity of word problems in finitely presented groups.
>
> Martin R. Bridson [Bri02]

1.1 Historical Background

In 1911, the German mathematician Max Dehn [Deh11] posed the following three problems (courtesy of Stillwell's translation [Deh87]).

> *The word problem*: An element of the group is given as a product of generators. One is required to give a method whereby it may be decided in a finite number of steps whether this element is the identity or not.
>
> *The conjugacy problem*: Any two elements S and T of the group are given. A method is sought for deciding the question whether S and T can be transformed into each other, i.e. whether there is an element U of the group satisfying the relation
>
> $$S = UTU^{-1}$$

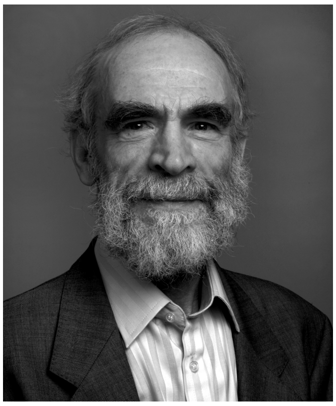
> *The isomorphism problem*: Given two groups, one is to decide whether they are isomorphic or not (and further, whether a given correspondence between the generators of one group and elements of the other group is an isomorphism or not).

During the next year, Dehn [Deh12] used *geometric* ideas to produce an algorithm to solve the word problem for canonical presentations of fundamental groups of closed 2-manifolds.

In 1955, the Soviet mathematician Pyotr Novikov constructed a finitely presented group for which there does not exist an algorithm to solve the word problem [Nov55]. Soon after, Adian [Adi55] and Rabin [Rab58] (independently) proved that most "reasonable" properties of finitely presentable groups are algorithmically undecidable. In 1959, a different proof of the undecidability of the word problem was obtained by William Boone [Boo59]. Not to be deterred by its (general) undecidability, mathematicians set out on expanding their repository of groups for which the word problem was known to be solvable. As we shall see in this thesis, the word problem is in fact solvable for many large classes of groups.

Max Dehn (1878–1952), circa 1945 [Jac]

In 1987, Mikhael Gromov introduced a class of groups called *hyperbolic groups* by abstracting properties from hyperbolic geometry and proved that the word problem is solvable in hyperbolic groups [Gro87]. Gromov was able to show that, in a precise probabilistic sense, "almost every" finitely presented group is hyperbolic. His groundbreaking insights revealed that the word problem was fundamentally more geometric in nature than previously understood. Mathematicians have since determined that the word problem is solvable in automatic groups, such as Coxeter groups and braid groups, polycyclic groups, and non-positively curved groups [BH99].



Mikhael Gromov in 2009 [Fla09]

Figure 1.1 shows a simplified version of Bridson's universe of (finitely generated) groups [Bri06]. The boundary regions represent well-understood classes—

amenable groups on one side and non-positively curved groups on the other, the latter of which will serve as our point of entry. In contrast, the vast interior represents largely uncharted territory.

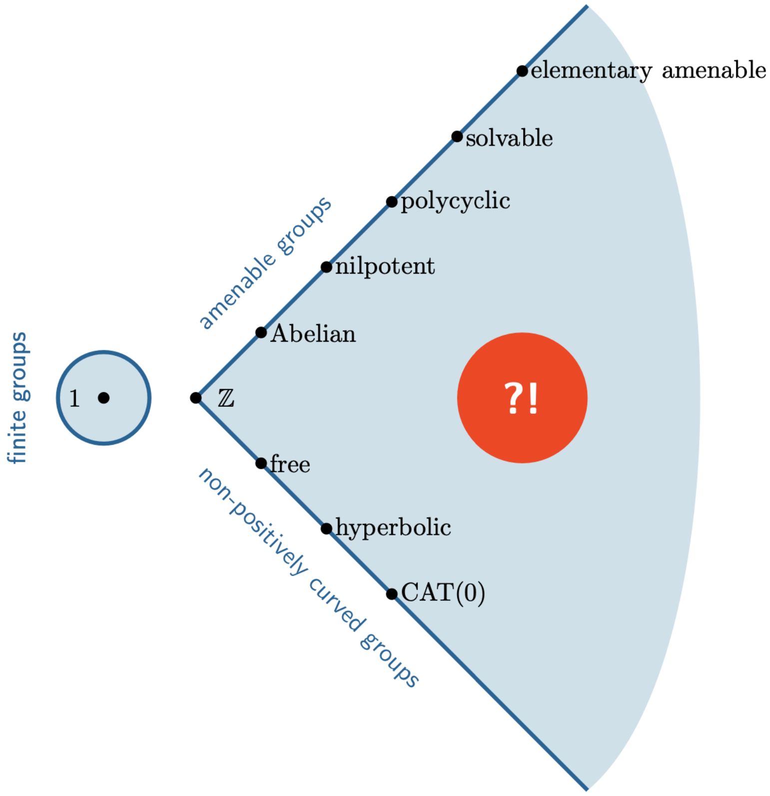


Figure 1.1: A simplified version of Bridson's universe of groups [Bri06] found in [Lö17].

Approaches to solving the word problem vary. Contemporary mathematicians have used geometric approaches to the word problem that involve fundamental groups, covering spaces, and Morse theory [BRS07].

The word problem has found surprising applications in other fields. The undecidability of the word problem implies the undecidability of other problems in geometry and has even been investigated as a basis for a public-key cryptosystem [WM85].

1.2 Structure of the Thesis

This thesis is intended to be accessible to readers with a background in basic group theory and familiarity with elementary concepts of computability theory.

We provide a broad overview of the chapters.

In Chapter 2, we introduce free groups (their construction and universal property), group presentations, and Cayley graphs as geometric representations of groups.

In Chapter 3, we introduce Tietze transformations to show that solvability of the word problem is independent of the chosen presentation. We then survey algorithmic approaches to solving the word problem in various classes of groups and introduce Dehn functions as a measure of complexity, exploring their key properties and equivalence. We conclude with a discussion of the isoperimetric spectrum.

In Chapter 4, we introduce van Kampen diagrams as a combinatorial tool for studying the word problem and prove van Kampen's Lemma, connecting the algebraic and geometric perspectives. We then discuss applications of these diagrams and define the presentation complex and the Cayley complex associated to a presentation. The chapter concludes with an account of small cancellation theory, including isoperimetric inequalities for groups satisfying the $C'(1/6)$ and $C(4)$–$T(4)$ conditions.

In Chapter 5, we study groups and their Cayley graphs as metric spaces and introduce the notion of quasi-isometry as a way to compare their large-scale geometric structures. We then explore how group properties behave under quasi-isometries and examine commensurability, a related algebraic notion that also preserves many geometric features.

In Chapter 6, we explore group theory from the perspective of non-positive curvature. We introduce δ-hyperbolic spaces and hyperbolic groups, along with CAT(0) spaces and CAT(0) groups, and establish isoperimetric inequalities in each context. We then explore the concept of subgroup distortion and its impact on Dehn functions, focusing on the double of a group G over a subgroup H, namely the amalgamated free product $G *_H G$. The chapter concludes with the Bieri doubling trick—a construction that embeds the double of a CAT(0) group into a larger CAT(0) group. This method shows that by increasing the distortion of H, one can construct subgroups of CAT(0) groups with arbitrarily large Dehn functions.

Appendix A provides a brief overview of the standard group-theoretic constructions used throughout the thesis. We review free products, amalgamated free products, semidirect products, and HNN extensions as basic methods for building new groups from existing ones.

Chapter 2

Preliminaries

So, my interest in symmetry has not been misplaced.

H. S. M. Coxeter, reflecting on the unusual level of
bilateral symmetry in his brain [Rob24].

This chapter introduces the fundamental concepts and tools of geometric group theory that will be used throughout the rest of the thesis.

2.1 Free Groups

The free group is one of the most fundamental objects in combinatorial group theory. Before constructing free groups, we require the following definitions.

Definition 2.1. Let A be a set, which we call an *alphabet*. A *word* is a finite sequence of elements from A, written as

$$w = a_1 a_2 \cdots a_n, \quad \text{where } a_i \in A.$$

The *length* of a word w, denoted $|w|$, is the number of symbols it contains. The unique word of length zero is called the *empty word*.

Given two words $u = a_1 a_2 \cdots a_m$ and $v = b_1 b_2 \cdots b_n$, we can concatenate them to form a new word:

$$uv = a_1 a_2 \ldots a_m b_1 b_2 \ldots b_n.$$

This operation is associative, meaning that if u, v, w are words, then

$$(uv)w = u(vw).$$

The set of all words over A, including the empty word, is denoted A^*.

To build a group from the alphabet A, we need inverses. For each symbol $a \in A$, we introduce a *formal inverse* a^{-1}, and let

$$A^{-1} = \{a^{-1} \mid a \in A\}, \quad \text{with } (a^{-1})^{-1} = a.$$

We then define the extended alphabet as

$$A^{\pm 1} = A \cup A^{-1}.$$

Words over $A^{\pm 1}$ can now include both symbols from A and their formal inverses. The set of all such words is denoted $(A^{\pm 1})^*$

In a group, an element and its inverse cancel each other. Hence, not every word in $(A^{\pm 1})^*$ should represent a distinct element in our group. To reflect this, we introduce a reduction process for words.

Definition 2.2. A word in $(A^{\pm 1})^*$ is said to be *reducible* if it contains a subword of the form aa^{-1} or $a^{-1}a$ for some $a \in A$. These pairs correspond to an element and its inverse and should be removed. A word is called *reduced* if no such cancellations are possible.

A reduced word $a_1 a_2 \cdots a_n$ is called *cyclically reduced* if $a_1 \neq a_n^{-1}$. A *cyclic reduction* consists of removing an initial a_1 and a terminal a_n when $a_1 = a_n^{-1}$. This process corresponds to conjugating a word by one of its letters.

Any word in $(A^{\pm 1})^*$ can be transformed into a *unique* reduced word by repeatedly removing adjacent pairs of the form aa^{-1} or $a^{-1}a$. This reduction process motivates the definition of an equivalence relation.

We define an equivalence relation $\sim$ on $(A^{\pm 1})^*$ by declaring that two words are equivalent if one can be transformed into the other by a finite sequence of reductions and insertions of such cancelling pairs.

Definition 2.3. Let A be a set. The *free group* on A, denoted $F(A)$, is the set of equivalence classes of words in $(A^{\pm 1})^*$ under the relation $\sim$. That is,

$$F(A) = (A^{\pm 1})^* / \sim .$$

Multiplication in $F(A)$ is defined by concatenating representatives and then reducing the result. Given two elements $[u], [v] \in F(A)$, their product is

$$[u][v] = [uv],$$

where uv is the concatenation of u and v, followed by reduction if necessary.

The identity element is the equivalence class of the empty word. The inverse of a word $u = a_1 a_2 \cdots a_n$ is given by:

$$u^{-1} = a_n^{-1} a_{n-1}^{-1} \dots a_1^{-1},$$

and this operation respects the equivalence relation.

In practice, we often identify a word $u \in (A^{\pm 1})^*$ with its equivalence class $[u] \in F(A)$, when there is no confusion. The elements of A serve as generators of $F(A)$, and no relations other than those required by the group axioms are imposed.

The free group $F(A)$ possesses the following important characterization, which justifies the term "free."

Theorem 2.4 (Universal Property of Free Groups)**.** *Let A be a set, and let $F(A)$ be the free group on A. Given any group G and any function $f\colon A \to G$, there exists a unique group homomorphism $\varphi\colon F(A) \to G$ such that*

$$\varphi(a) = f(a) \quad \textit{for all } a \in A.$$

This property implies a natural bijection between functions from A to a group G and group homomorphisms from $F(A)$ to G. In a sense, $F(A)$ is the "freest" group generated by A, since no relations among generators are assumed beyond those necessary to satisfy the group axioms.

To construct the unique homomorphism $\varphi\colon F(A) \to G$, we first send each generator $a \in A$ to $f(a) \in G$, then extend this map to formal inverses via $\varphi(a^{-1}) = f(a)^{-1}$, and finally to all reduced words by requiring that $\varphi(uv) = \varphi(u)\varphi(v)$ for concatenation of words. The identity is mapped to the identity of G, and inverses are preserved.

This universal property uniquely characterizes $F(A)$ up to isomorphism, and in many contexts, it is taken as the defining property of free groups.

2.2 Group Presentations

In a free group, the only possible cancellations come from subwords of the form aa^{-1} or $a^{-1}a$, reflecting the group axioms alone. Two words in a free group represent the same element if and only if their equality follows from these axioms, no additional relations are imposed.

To describe more general groups, we introduce the idea of specifying additional relations among a set of generators. Suppose Γ is a group generated by a set A, and let $R \subseteq F(A)$ be a collection of words which we declare to represent the identity in Γ. These words, called relators, define constraints on how the generators interact.

It is important to note that relations in a group are preserved under conjugation. That is, if a word $r \in F(A)$ represents the identity, then so does any conjugate grg^{-1} for $g \in F(A)$. This motivates the need to consider not just the elements of R, but also all their conjugates when defining the group.

Definition 2.5. Let G be a group, and let $A \subseteq G$. The *normal closure* of A in G, denoted $\langle\langle A\rangle\rangle$, is the smallest normal subgroup of G containing A.

Explicitly, it can be described as:

$$\langle\langle A\rangle\rangle = \langle\{gag^{-1} \mid a \in A, g \in G\}\rangle.$$

In other words, it consists of all finite products of conjugates of elements of A.

This leads to the formal definition of a group presentation. A presentation specifies a group by a generating set and a set of relators.

Let S be a set of generators and $R \subseteq F(S)$ a set of relators. The group defined by the presentation is the quotient

$$\langle S \mid R\rangle = F(S)/\langle\langle R\rangle\rangle,$$

where $F(S)$ is the free group on S, and $\langle\langle R\rangle\rangle$. This quotient forces each word in R (and all of its conjugates) to become the identity, ensuring that the relations are respected in the resulting group.

Definition 2.6. A *presentation* of a group Γ is an isomorphism

$$\Gamma \cong \langle S \mid R\rangle$$

A *finite presentation* is a presentation where both S and R are finite sets. A group is *finitely presented* if it admits a finite presentation.

For convenience, we often write presentations using a compact notation:

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m\rangle$$

where the x_i are generators and the r_j are relators. Each relator corresponds to a word that is declared to be the identity in the group. Note that an equation such as $ab = ba$ is a *relation*, while $aba^{-1}b^{-1}$ is the corresponding *relator* in the free group.

Example 2.7. The following are finitely presented groups:

- If no relations are imposed (i.e., $R = \emptyset$), then

$$\langle S \mid \emptyset \rangle \cong F(S).$$

  For example, $\langle a \mid \emptyset \rangle \cong \mathbb{Z}$ and $\langle a, b \mid \emptyset \rangle \cong F_2$.

- The cyclic group of order n, denoted $\mathbb{Z}_n$, has the presentation

$$\langle a \mid a^n \rangle.$$

  The relator a^n ensures that a has order n.

- The $\mathbb{Z} \oplus \mathbb{Z}$ is presented as

$$\langle a, b \mid ab = ba \rangle.$$

  This relation enforces that a and b commute, making the group abelian.

- More generally, the group $\mathbb{Z}^n$ has the presentation

$$\langle e_1, \ldots, e_n \mid e_i e_j = e_j e_i,\ 1 \leq i < j \leq n \rangle.$$

  Alternatively, one may use commutator relators: $e_i e_j e_i^{-1} e_j^{-1} = 1$ for $1 \leq i < j \leq n$.

2.3 Cayley Graphs

A presentation expresses a group in terms of generators and relations, but it does not immediately convey the group's structure. The Cayley graph offers a geometric perspective: it visualizes group elements as vertices and multiplication by generators as edges. This construction provides a powerful tool for studying the geometry and algorithmic properties of a group, such as the word problem.

Definition 2.8. Let Γ be a group with generating set A. The *Cayley graph* $\mathrm{Cay}(\Gamma, A)$ is a directed, labeled graph defined as follows:

- Each vertex corresponds to an element $g \in \Gamma$.
- For each $g \in \Gamma$ and each generator $a \in A$, there is a directed edge from g to ga labeled by a.

Cayley introduced this diagrammatic view in 1878, remarking on its symmetry and the group-theoretic behavior of paths [Cay78]:

> The diagram is drawn, in the first instance, with the arrows but without the letters, which are then affixed *at pleasure*; viz: the *form of group* is quite independent of the way in which this is done, though the group itself is of course dependent upon it.
>
> ...
>
> The diagram has a remarkable property, *in virtue whereof it in fact represents a group*; any route leading from one point to itself leads from every other point to itself... A route is thus, in effect, a substitution.

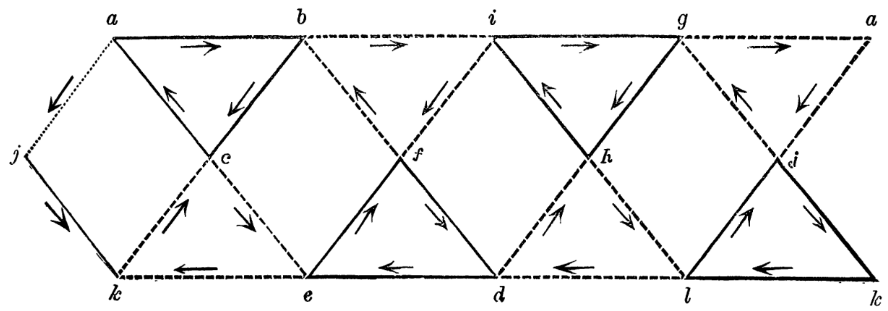
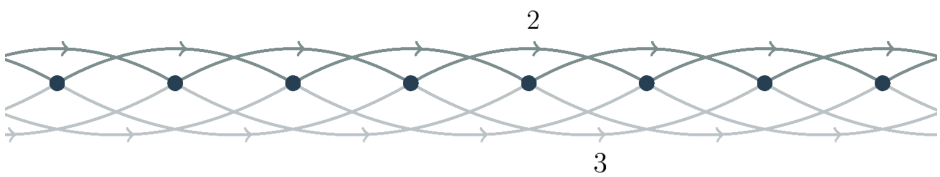


Figure 2.1: Cayley's original group diagram, as published in 1878 [Cay78].

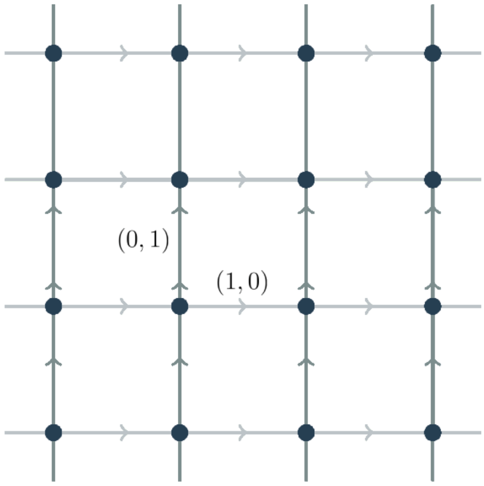
Example 2.9. Examples of Cayley graphs:

1. The Cayley graph of $\mathbb{Z}$ with generator 1 is an infinite line.

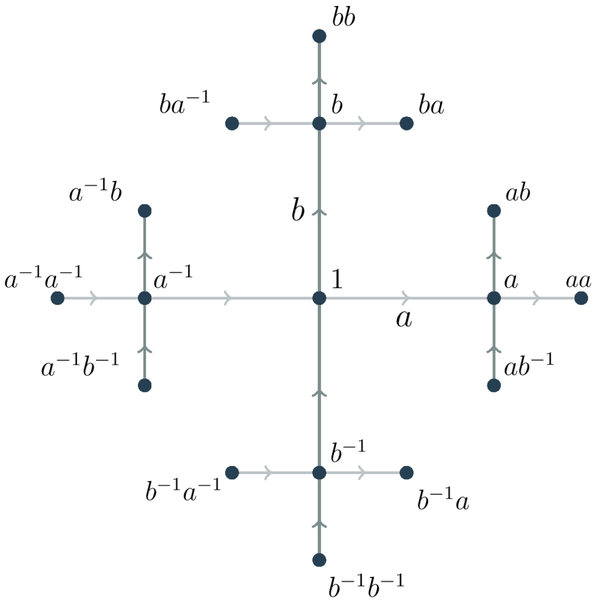
   With generators $\{2, 3\}$, the Cayley graph is again a line, but edges now correspond to steps of size 2 and 3.



2. The Cayley graph of $\mathbb{Z}^2$ with generators $\{(1, 0), (0, 1)\}$ is an infinite grid.



3. The Cayley graph of the free group F_2 on two generators is a regular infinite tree.



Cayley graphs have the following key properties [BRS07]:

- Since A generates Γ, every vertex in $\mathrm{Cay}(\Gamma, A)$ is reachable from the identity. Hence, the graph is connected.

- By assigning each edge a length of 1, we obtain a metric space. The *word metric* $d_A(g, h)$ is the length of the shortest path from g to h. Equivalently, $d_A(g, h)$ is the minimum length of a word in $A^{\pm 1}$ representing $g^{-1}h$. This metric turns the Cayley graph into a geometric object, although it depends on the choice of generating set A (see Chapter 5).

- Paths in the Cayley graph correspond to words in $F(A)$: each edge contributes a generator, and reversing an edge corresponds to inverting a generator. A word represents the identity in Γ if and only if the corresponding path is a loop based at the identity vertex.
- The group Γ acts on $\mathrm{Cay}(\Gamma, A)$ by left multiplication: $g \cdot h = gh$. This action is:
  - *Free*: if $g \cdot v = v$, then $g = 1$.
  - *Transitive*: any vertex can be mapped to any other vertex by the action of some group element.
  - *Isometric*: the word metric is preserved under the action.
- The Cayley graph of a finitely generated group is a covering space of the wedge of loops $K(A)$: a graph with one vertex and one loop for each generator in A. The fundamental group of $K(A)$ is $F(A)$, and each vertex in the Cayley graph has the same local structure, one incoming and one outgoing edge for each generator.
- If Γ has a presentation $\langle A \mid R \rangle$, then $\mathrm{Cay}(\Gamma, A)$ is the covering space of $K(A)$ corresponding to the normal subgroup $\langle\langle R \rangle\rangle$ of $F(A)$.
- The Cayley graph offers a geometric approach to the word problem. If we can construct the ball of radius $n/2$ around the identity in the Cayley graph, we can determine whether a word of length n represents the identity, since any loop of length n must lie within this ball.

Chapter 3

The Word Problem

Given a finitely presented group,

$$\Gamma = \langle S \mid R \rangle,$$

the *word problem* asks whether there exists an algorithm that can decide, for any word w in the generators $S^{\pm 1}$, whether or not w represents the identity element in Γ. Formally, the word problem is the decision problem: given $w \in F(S)$, decide whether $w \in \langle\langle R \rangle\rangle$. If such an algorithm exists, we say that Γ has a *solvable* word problem.

This problem is foundational in combinatorial and computational group theory. A striking and profound result by Novikov [Nov55] and Boone [Boo59] shows that there exist finitely presented groups with unsolvable word problem. That is, no algorithm can determine whether a given word represents the identity in such a group.

Why restrict attention to finitely presented groups? This focus is not only natural, but also supported by three foundational perspectives:

- Geometrically, every finitely presented group arises as the fundamental group of a closed 4-manifold.
- Topologically, as Bridson explains in [Bri09], finite presentation corresponds to compactness: a group is finitely presented if and only if it is the fundamental group of a compact space with a reasonable CW-complex structure (see Definition 4.5).
- Computationally, Higman's embedding theorem [Hig61] shows that any recursively presented, finitely generated group can be embedded in a finitely

presented group. This allows finitely presented groups to simulate arbitrary computation, including Turing machines.

In this way, finite presentation serves as a kind of Goldilocks condition: sufficiently restrictive to allow deep structural theorems, yet broad enough to capture a wide spectrum of groups and phenomena.

3.1 Presentation Invariance

If a group Γ is finitely presentable, then different finite presentations of Γ should yield the same answer to the word problem. In particular, if one presentation admits an algorithm that solves the word problem, then any other finite presentation of Γ must also admit such an algorithm. This invariance arises from a foundational result: different finite presentations of the same group are connected by a finite sequence of Tietze transformations, elementary moves that relate isomorphic presentations.

Let $\Gamma = \langle S \mid R\rangle$ be a finite presentation. There are four types of *Tietze transformations*:

I. If a word $w \in F(S)$ already lies in the normal closure $\langle\langle R\rangle\rangle$, that is, w represents the identity in Γ, we may add it to the relators:

$$\Gamma = \langle S \mid R \cup \{w\}\rangle.$$

II. Conversely, if a relator $w \in R$ follows from the others, we may remove it:

$$\Gamma = \langle S \mid R \setminus \{w\}\rangle.$$

III. For any word $w \in F(S)$, we may introduce a new generator y with defining relation $y = w$, giving:

$$\Gamma = \langle S \cup \{y\} \mid R \cup \{yw^{-1}\}\rangle.$$

This amounts to naming an existing element, without changing the group.

IV. If a generator $y \in S$ appears in a relation $y = w$ for some $w \in F(S \setminus \{y\})$, we may eliminate y by replacing all its occurrences with w in the relators:

$$\Gamma = \langle S \setminus \{y\} \mid R'\rangle.$$

Tietze [Tie08] proved a fundamental result connecting these moves:

Proposition 3.1. *Two finite presentations define isomorphic groups if and only if one can be transformed into the other by a finite sequence of Tietze transformations.*

Proof. We follow [LS01].

The forward direction is immediate: each transformation preserves the group up to isomorphism. For the converse, suppose that both presentations define the same group Γ, with quotient maps $\varphi_i \colon F(S_i) \to \Gamma$, where $\ker \varphi_i = \langle\langle R_i \rangle\rangle$.

Without loss of generality, assume $S_1 \cap S_2 = \emptyset$ (by re-labeled if needed), and let $S = S_1 \cup S_2$. By the universal property of free groups (Theorem 2.4), there is a unique homomorphism $\varphi \colon F(S) \to \Gamma$ that extends both φ_1 and φ_2.

Now for each $x \in S_1$, choose a word $w_x \in F(S_2)$ with $\varphi(x) = \varphi(w_x)$, and for each $y \in S_2$, choose $v_y \in F(S_1)$ so that $\varphi(y) = \varphi(v_y)$. Define:

$$W_1 = \{x^{-1} w_x \mid x \in S_1\}, \qquad W_2 = \{y^{-1} v_y \mid y \in S_2\}.$$

We now construct a sequence of Tietze transformations:

- Add generators S_2 and the relations W_2:
$$\langle S_1 \mid R_1 \rangle \to \langle S \mid R_1 \cup W_2 \rangle.$$
- Add the relators $R_2 \cup W_1$:
$$\langle S \mid R_1 \cup W_2 \rangle \to \langle S \mid R_1 \cup R_2 \cup W_1 \cup W_2 \rangle.$$
- Remove the now-redundant $R_1 \cup W_2$:
$$\langle S \mid R_1 \cup R_2 \cup W_1 \cup W_2 \rangle \to \langle S_2 \mid R_2 \rangle.$$

Reversing this sequence recovers the original presentation. Hence, the two are connected by a finite sequence of Tietze transformations. $\square$

Corollary 3.2. *If a finitely presented group Γ has a finite presentation for which the word problem is solvable, then the word problem is solvable for any finite presentation of Γ.*

Proof. Let $\Gamma = \langle S_1 \mid R_1 \rangle$ be a presentation with an algorithm to solve the word problem. For any other presentation $\langle S_2 \mid R_2 \rangle$, Proposition 3.1 guarantees a finite sequence of Tietze transformations connecting the two. These transformations induce an isomorphism between the two presentations, allowing any word $w \in F(S_2)$ to be converted to a word $w' \in F(S_1)$ that represents the same element of the group. Apply the known algorithm to w' to decide whether w represents the identity. $\square$

Thus, solvability of the word problem is a group-theoretic property, independent of presentation.

3.2 Solving the Word Problem

Corollary 3.2 offers a practical strategy for solving the word problem: If we can find an isomorphism between a given group and one for which the word problem is already known to be solvable, we can transfer that solution between presentations. Among the simplest examples are free groups. These contain no nontrivial relations, so we can decide whether a word represents the identity by iteratively canceling adjacent inverses (i.e., subwords of the form aa^{-1} or $a^{-1}a$). This reduction process terminates with the empty word if and only if the original word is trivial.

The following result collects several standard cases in which the word problem is solvable:

Proposition 3.3.

(i) Every finite group has solvable word problem.

(ii) If G and H have solvable word problem, then so do their free product $G * H$ and direct product $G \times H$.

(iii) Every finitely generated abelian group has solvable word problem.

(iv) If G has solvable word problem and H is a finitely generated subgroup of G, then H also has solvable word problem.

Proof.

(i) Any finite group embeds into some symmetric group S_n by Cayley's theorem. Given a word w, we can compute its image as a permutation in S_n and test whether it fixes all points $1, \ldots, n$. This algorithm terminates, as S_n is finite.

(ii) For $G \times H$, a word (w_G, w_H) represents the identity if and only if both components do, so the problem reduces to solving it in each factor.

For $G * H$, the normal form theorem (Theorem A.3) tells us that a reduced word $g_1 h_1 \cdots g_k h_k$ represents the identity if and only if all individual pieces are trivial in their respective groups. Hence, solvability in G and H yields solvability in $G * H$.

(iii) Every finitely generated abelian group decomposes as $\mathbb{Z}^r \oplus T$, where T is finite. The word problem is solvable in $\mathbb{Z}^r$ and in finite groups, so by part (ii), it is solvable in their direct sum.

(iv) Let $H = \langle h_1, \ldots, h_k \rangle$ and suppose that we are given a word w in the generators of H. We can express each h_i as a word in the generators of G, which produces a word $w' \in F(G)$ that represents the same element in G. Then $w =_H 1$ if and only if $w' =_G 1$, which is decidable. □

Another important class of groups with solvable word problems are the residually finite groups. A group G is residually finite if for every non-trivial element $g \in G$, there exists a homomorphism $\phi\colon G \to H$ to a finite group H such that $\phi(g) \neq 1$. Free groups, finite groups, and finitely generated abelian groups are all residually finite. McKinsey [McK43] proved that any finitely presented residually finite group has a solvable word problem.

Beyond algebraic methods, geometric properties of Cayley graphs can also imply decidability of the word problem [Mei08]. Jim Cannon introduced the notion of almost convexity [Can87]: A group G is *almost convex* with respect to a generating set S if there exists a constant K such that, for all n, any two elements in the sphere of radius n that are within distance ≤ 2 in the Cayley graph can be connected by a path of length $\leq K$ that stays entirely within the ball of radius n. This property ensures that the geometry of spheres in the Cayley graph is well behaved and allows one to construct the Cayley graph inductively, layer by layer. Cannon showed that this inductive construction leads to a solution of the word problem. Free groups and free abelian groups are almost convex with respect to their standard generators.

From the standpoint of formal language theory, we may ask whether the set of words representing the identity element can be recognized by finite-state machines. Fix a finite generating set S for a group G, and let $\mathrm{WP}(G, S) \subseteq F(S)$ denote the set of words equal to the identity in G. We then treat $\mathrm{WP}(G, S)$ as a formal language and investigate its complexity.

Surprisingly, this language is rarely regular:

Theorem 3.4 (Anisimov, 1971 [Ani71])**.** *For any finitely generated group G, the language $\mathrm{WP}(G, S)$ is regular if and only if G is finite.*

Proof. If G is finite, its Cayley graph $\mathrm{Cay}(G, S)$ is finite and can be viewed as a finite automaton: vertices are states, edges are labeled by generators, and the identity vertex serves as both the start and the accept state. This automaton accepts precisely those words that represent the identity.

Conversely, suppose that a finite automaton M recognizes $\mathrm{WP}(G, S)$. Remove any states from M that do not lie on a path to an accept state (this does not change the recognized language). If two words w and w' terminate at the same state and there exists a path v from that state to an accept state, then both wv and $w'v$ lie in $\mathrm{WP}(G, S)$, so $w =_G w'$. Thus, distinct group elements must map to distinct states. Since the number of states is finite, G must be finite. □

This result shows that regular languages are too restrictive to describe the word problem in infinite groups. A more flexible class is the context-free languages. The Muller–Schupp theorem [MS83] characterizes groups with a context-free word problem: a finitely generated group has a context-free word problem if and only if it has a free subgroup of finite index.

In general, solving the word problem in a group $G = \langle S \mid R \rangle$ amounts to deciding whether a given word $w \in F(S)$ lies in the normal closure $\langle\langle R \rangle\rangle \subset F(S)$. There is a natural way to enumerate this normal closure.

We begin by enumerating all the words $x_1, x_2, \dots \in F(S)$ by increasing length. For each relator $r \in R^{\pm 1}$, we form all conjugates $x_i r x_i^{-1}$ and their inverses. The normal closure consists of all finite products of these conjugates, so we can enumerate them using a *normal closure*: we first list all conjugates, then all products of two, then three, etc.

This allows us to recognize whether a word *does* represent the identity: if $w \in \langle\langle R \rangle\rangle$, it will eventually appear in the enumeration. However, if w does *not* represent the identity, there is no way to confirm this via finite computation. This asymmetry underscores the difficulty of the word problem: detecting *non-triviality* is the challenge.

If $w \in \langle\langle R \rangle\rangle$, then we eventually obtain an expression of the form:

$$w \equiv \prod_{i=1}^{N} x_i r_i x_i^{-1}, \tag{3.1}$$

where $x_i \in F(S)$, $r_i \in R^{\pm 1}$, and $\equiv$ denotes equality in the free group (i.e., equality of reduced words). There is a natural interpretation of this process: applying a relator replaces a subword u_2 of w (where $r = u_1 u_2 u_3$) with $(u_3 u_1)^{-1}$. This corresponds to rewriting:

$$w = w_1 u_2 w_2 \mapsto w' = w_1 (u_3 u_1)^{-1} w_2,$$

and amounts to recording the move as:

$$w \equiv (x_1 r x_1^{-1}) w',$$

where $x_1 = w_1 u_1^{-1}$. Repeating this process N times corresponds to the product in (3.1). Thus, reducing a word to the identity corresponds to expressing it as a product of conjugates of relators in the free group.

3.3 Dehn Functions

A direct approach to the word problem is to attempt to bound the number of relators required to reduce a word representing the identity to the empty word. This leads us to the following definition.

Definition 3.5. Let w be a word that represents the identity element. The minimal number N of conjugates of relators required to express w in the form of Equation 3.1 is called the *area* of w, denoted $\mathrm{Area}(w)$

The *Dehn function* of a presentation P, denoted $\delta_P \colon \mathbb{N} \to \mathbb{N}$, is defined by

$$\delta_P(n) = \max\{\mathrm{Area}(w) \mid w \in \langle\langle R \rangle\rangle,\ |w| \leq n\}.$$

This function measures the worst-case complexity of proving that a word of length at most n represents the identity.

An *isoperimetric inequality* for the presentation is an upper bound for the Dehn function. That is, a function $f \colon \mathbb{N} \to \mathbb{R}$ is called an isoperimetric function for the presentation if

$$\delta_P(n) \leq f(n) \quad \text{for all } n \in \mathbb{N}.$$

Example 3.6. The Dehn function of the presentation $\langle a \mid a^m \rangle$ for the cyclic group $\mathbb{Z}/m\mathbb{Z}$ is $\delta(n) = \left\lfloor \frac{n}{m} \right\rfloor$.

Proof. Any word w representing the identity must reduce (through free reductions) to a^{rm} for some integer r. Since these free cancellations do not affect the area, we may assume $w = a^{rm}$.

For the upper bound, reducing a^{rm} to the empty word requires exactly $|r|$ applications of the relation $a^m = 1$. Because $|a^{rm}| = |r|m \leq n$, we have $|r| \leq \frac{n}{m}$, giving the bound $\delta(n) \leq \left\lfloor \frac{n}{m} \right\rfloor$.

For the lower bound, note that each application of $a^m = 1$ changes the exponent sum by at most m. Since the exponent sum of a^{rm} is rm, we need at least $|r|$ such applications to reach the identity. The maximum value occurs when $|r| = \left\lfloor \frac{n}{m} \right\rfloor$, completing the proof. □

We can efficiently solve the word problem for certain presentations. Suppose that we are given a finite presentation such that every word in the normal closure $\langle\langle R\rangle\rangle$ contains more than half of some relator (or its inverse), considered up to cyclic permutation. That is, for any word w representing the identity, we can write $w = ur'v$, where there exists a relator $r \in R^{\pm 1}$ (or a cyclic conjugate of it) such that $r = r'r''$ with $|r'| > |r''|$. Since $r = 1$ in the group, we know $r' = (r'')^{-1}$ in the group, so we can replace r' with the shorter word $(r'')^{-1}$, reducing the length of w while preserving its equivalence class.

By iterating this process, we reach the empty word if and only if w represents the identity. This method is known as *Dehn's algorithm*, and it only requires detecting subwords among a finite list of relators and their cyclic permutations. Moreover, it reduces any word w representing the identity in at most $|w|$ steps, yielding a linear upper bound on the Dehn function. We will encounter explicit examples of such presentations.

So far, we have defined the Dehn function for a specific presentation, but we would like to speak meaningfully of the Dehn function of a group itself. A complication arises: different finite presentations of the same group may yield different Dehn functions.

Example 3.7. The Dehn function of the presentation $\langle a \mid \emptyset\rangle$ is $\delta(n) = 0$, while for the presentation $\langle a, b \mid b\rangle$, it is $\delta(n) = n$. However, both define the group $\mathbb{Z}$.

Proof. In the first presentation, there are no relations, so any word representing the identity must be freely trivial. Hence, no area is required: $\delta(n) = 0$.

In the second presentation, the generator b is trivialized by the relation $b = 1$, so all instances of b^k must be reduced using this relation. A word of length n could contain up to n such instances, so the upper and lower bounds are n, giving $\delta(n) = n$. $\square$

To address this, we introduce a notion of asymptotic equivalence.

Definition 3.8. Let $f, g\colon \mathbb{N} \to \mathbb{R}$. We write $f \preceq g$ if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$,

$$f(n) \leq Cg(Cn + C) + Cn + C.$$

We say that f and g are *equivalent*, written $f \simeq g$, if both $f \preceq g$ and $g \preceq f$ hold.

This defines an equivalence relation. Some consequences of this definition include the following:

- All constant functions are equivalent to linear functions, including the zero function.
- All polynomial functions of the same degree > 1 are equivalent.
- All exponential functions are equivalent.

Thus, the Dehn functions in Example 3.7 are equivalent.

The following result is due to Steve Gersten [Ger92], and the argument we present is adapted from Bridson [Bri02].

Proposition 3.9. *If two finite presentations define isomorphic groups, then their Dehn functions are equivalent.*

Proof. By Proposition 3.1, there exists a finite sequence of Tietze transformations between any two such presentations. Therefore, it suffices to show that each such transformation preserves the equivalence class of the Dehn function.

For Type I, adding a redundant relation $r \in \langle\langle R \rangle\rangle$ to $P = \langle A \mid R \rangle$, the new presentation $P' = \langle A \mid R \cup \{r\} \rangle$ satisfies $\delta_{P'}(n) \leq \delta_P(n)$, since additional relations can only decrease the area. Conversely, $\delta_P(n) \leq M\delta_{P'}(n)$, where M is the maximal area of r in P, because any occurrence of r in a word in P' can be replaced by its expression as a product of conjugates from R, multiplying its area by at most M. Thus, $\delta_P \simeq \delta_{P'}$. Type II (removing redundant relations) is the inverse operation and similarly preserves equivalence.

For Type III, adding a redundant generator a with relation $a = w_a$ to $P = \langle A \mid R \rangle$, yielding $P'' = \langle A \cup \{a\} \mid R \cup \{a^{-1}w_a\} \rangle$, we observe that $\delta_{P''}(n) \leq \delta_P(Mn)$, where $M = |w_a|$, since occurrences of a in P can be replaced by w_a, multiplying the word length by at most M. The reverse inequality $\delta_P(n) \leq \delta_{P''}(n)$ holds because the relation $a^{-1}w_a$ allows eliminating a without increasing area. Type IV (removing redundant generators) similarly preserves equivalence as the inverse operation. □

It is thus meaningful to speak of "the" *Dehn function* of a finitely presented group Γ, denoted δ_Γ, well defined up to equivalence. A more general result appears in Chapter 5 (see Proposition 5.12).

We now examine the behavior of Dehn functions in standard group constructions. A subgroup H of G is called a *retract* if there exists a homomorphism $\rho\colon G \to H$ such that $\rho|_H = \mathrm{id}_H$.

Proposition 3.10. *If H is a retract of a finitely presented group G, then H is finitely presented and $\delta_H(n) \leq \delta_G(n)$.*

Proof. Let $\rho\colon G \to H$ be a retraction. Since G is finitely presented, let $G = \langle S_G \mid R_G\rangle$. Set $S_H = S_G \cap H$. As ρ restricts to the identity in H, S_H generates H, and the relations for H must come from R_G, so H is finitely presented.

For the Dehn function, let w be a word over S_H representing the identity in H. Then w also represents the identity in G, so there exists an expression

$$w \equiv \prod_{i=1}^{m} x_i r_i^{\pm 1} x_i^{-1},$$

with $r_i \in R_G$, and $x_i \in F(S_G)$. Applying ρ yields

$$w = \prod_{i=1}^{m} \rho(x_i)\rho(r_i)^{\pm 1}\rho(x_i)^{-1}.$$

Since $\rho(r_i) = 1$ in H for $r_i \in R_H$, we can omit those terms, and the number of remaining conjugates is at most m. Thus, $\delta_H(n) \le \delta_G(n)$. □

Proposition 3.11. *Let G_1 and G_2 be infinite, finitely presented groups. Then:*

$$\delta_{G_1\times G_2}(n) \simeq \max\{n^2, \delta_{G_1}(n), \delta_{G_2}(n)\}, \quad \delta_{G_1 * G_2}(n) \simeq \max\{\delta_{G_1}(n), \delta_{G_2}(n)\}.$$

Proof. For the direct product $G_1 \times G_2$, a standard presentation is obtained by taking the generators and the relations of G_1 and G_2 separately, and adding the commutativity relations $g_1g_2 = g_2g_1$ for each generator $g_1 \in G_1$ and $g_2 \in G_2$.

Given a word of length n that represents the identity in $G_1 \times G_2$, it can be decomposed as a product of words in G_1 and G_2, each of length at most n. At most $\delta_{G_1}(n)$ and $\delta_{G_2}(n)$ relators are needed to reduce them.

However, to bring the word into a form where these relators can be applied, the generators must often be reordered using commutativity relations. Each commutator relation allows for the swapping of adjacent generators from G_1 and G_2. In the worst case, every occurrence of a generator from G_1 must be moved past every occurrence of a generator from G_2, requiring up to n^2 applications of commutativity relations. This gives the upper bound $\max\{n^2, \delta_{G_1}(n), \delta_{G_2}(n)\}$.

A matching lower bound is provided by the word $a^n b^n a^{-n} b^{-n}$ for $a \in G_1$, $b \in G_2$, which requires n^2 commutator relations to reduce to the identity. This proves the first equivalence.

For the free product $G_1 * G_2$, any word representing the identity can be expressed as an alternating product of elements from G_1 and G_2, without two consecutive elements from the same group (see Theorem A.3). Reducing such a word to the identity involves only reducing the individual subwords using the relators of G_1 and G_2. Therefore, the number of relators required is bounded above and below by the maximum of the Dehn functions of the factors, proving the second equivalence. □

3.4 The Isoperimetric Spectrum

Having defined Dehn functions of finitely presented groups, a natural question arises: which functions actually occur as Dehn functions? This leads to the study of the isoperimetric spectrum, denoted $\mathbb{P}$, which captures the growth rates of Dehn functions up to equivalence. We highlight selected results from the survey of the isoperimetric spectrum in [BRS07].

Definition 3.12. A real number α is called an *isoperimetric exponent* if there exists a finite presentation of a group whose Dehn function satisfies $\delta(n) \simeq n^\alpha$. The set of all such exponents is called the *isoperimetric spectrum* and is denoted by $\mathbb{P}$.

By convention, we restrict attention to $\alpha \geq 1$, since any Dehn function must grow at least linearly. Because there are only countably many finite presentations, the isoperimetric spectrum is a countable subset of $[1, \infty)$.

A landmark result of Gromov shows that the gap between exponents 1 and 2 corresponds precisely to hyperbolic groups, which we study in Chapter 6.

Theorem 3.13 (Gromov [Gro87])**.** *Let G be a finitely presented group. The following are equivalent:*

1. *G has a sub-quadratic Dehn function.*
2. *G has a linear Dehn function.*
3. *G is hyperbolic.*

This result implies that there are no Dehn functions equivalent to n^α for $1 < \alpha < 2$. In other words, there is a gap in $\mathbb{P}$ between 1 and 2, which has been a central theme in geometric group theory.

Beyond this gap, researchers have constructed groups with a wide range of polynomial Dehn functions. For example, Gersten and Thurston showed that the Dehn function of the integral Heisenberg group satisfies $\delta(n) \simeq n^3$ [Ger92, ECH$^+$92], and later constructions by Baumslag, Miller, Short [BMS93] and Bridson [Bri99] produced examples with various integer and rational exponents.

Brady and Bridson [BB00] went further and constructed groups whose Dehn functions fill in a dense subset of transcendental exponents in $[2, \infty)$. Thus, the only known gap in $\mathbb{P}$ lies between 1 and 2.

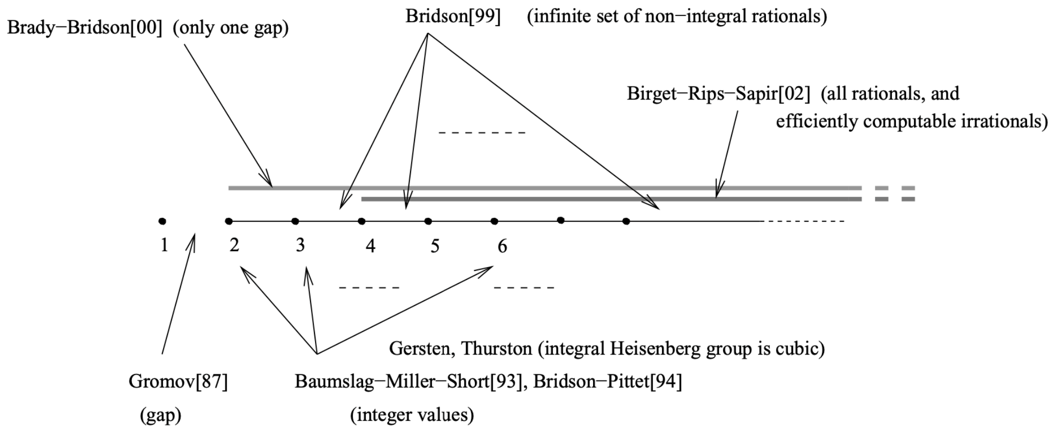


Figure 3.1: Key results concerning the structure of the isoperimetric spectrum $\mathbb{P}$.

A classical example of a group with a non-polynomial Dehn function is the *Baumslag–Solitar group** $\mathrm{BS}(1, 2) = \langle a, b \mid ab = b^2 a\rangle$, which satisfies $\delta(n) \simeq 2^n$. A recursive version of this construction yields a family B_k of groups with Dehn functions $\delta_k(n) = 2^{\delta_{k-1}(n)}$ for $k \geq 1$, where $\delta_1(n) = 2^n$ [Ger92].

Even faster-growing examples exist. For instance, the group

$$B = \langle x, y \mid (yxy^{-1})x = x^2(yxy^{-1})\rangle$$

has Dehn function growing faster than any iterated exponential: $\delta(n) \simeq \delta_n(n)$ [Ger92].

Some results link Dehn function growth to algebraic properties. Gromov showed that nilpotent groups of class c have Dehn functions bounded above by n^{c+1} [Gro92], later clarified by Gersten, Holt, and Riley [GHR02]. Bridson and Vogtmann [BV95] studied $\mathrm{Out}(F_n)$, showing that its Dehn function is linear for $n = 2$, exponential for $n = 3$, and at most exponential for all n.

*The name "Baumslag-Solitar groups" endures from their 1962 paper [BS62] despite Baumslag's well-documented but ultimately unsuccessful objections to the terminology.

Chapter 4

Van Kampen Diagrams

We now introduce our main geometric tool for studying the word problem. These diagrams arose as regions in the hyperbolic plane in Dehn's original study of the word problem [Deh12], but they were later studied in their full generality for arbitrary finitely presented groups by E. van Kampen [Kam33] in 1933.

A van Kampen diagram for a word w in a finitely presented group Γ is a planar diagram that represents an expression for w as a product of conjugates of relators. The 2-cells in the diagram correspond to the relators used in this expression, and the number of 2-cells is at least as large as the area of w. It is possible to construct a diagram that has exactly this minimal number of 2-cells. This will explain the reason for using the term "area."

4.1 Van Kampen's Lemma

Definition 4.1. Let $P = \langle A \mid R \rangle$ be a group presentation. We may assume that all $r \in R$ are cyclically reduced, since cyclic reduction does not change the presented group. Denote by R^C the cyclic closure of R, comprising all cyclic permutations of relators and their inverses.

A van Kampen diagram D for a word $w \in F(A)$ is a finite, connected, planar, oriented graph with:

- Each edge labeled by a generator $a \in A$.
- Each bounded region (or *face*) F of $\mathbb{R}^2 \setminus D$ with boundary labeled by a word in R^C, traversing edges with exponents ± 1 depending on orientation.
- A distinguished vertex on the boundary of the unbounded region of $\mathbb{R}^2 \setminus D$, from which the boundary word w is read.

The *boundary word* w of the diagram D is the word read starting from the base-point along the boundary of the unbounded region of $\mathbb{R}^2 \setminus D$. In this case, we say that D is a van Kampen diagram for w over the presentation P.

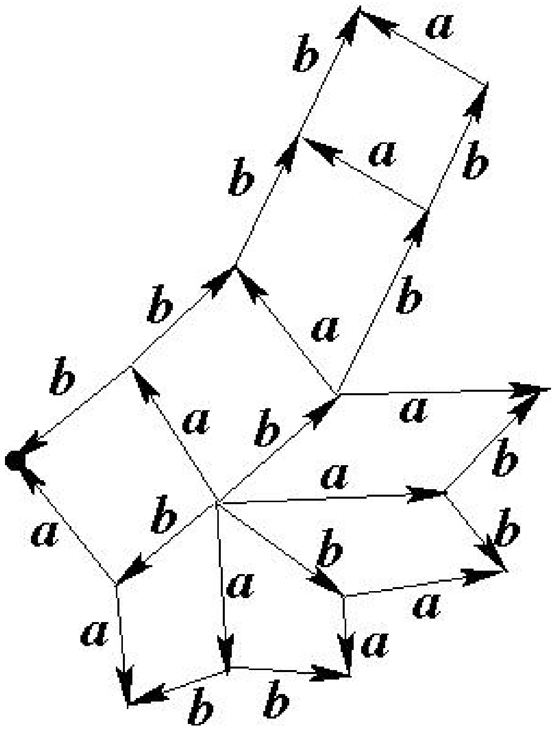


Figure 4.1: A van Kampen diagram over the presentation $\mathbb{Z}\times\mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1}\rangle$ with boundary word $w = a^{-1}ab^{-1}ba^{-1}ab^{-1}ba^{-1}b^2ab^{-3}b$ [Nsk92].

A van Kampen diagram can be viewed as a 2-complex, where the 1-dimensional skeleton of the complex corresponds to the graph, and a 2-cell is attached to each bounded region (or face). Additionally, the entire diagram can be retracted onto a tree by collapsing each 2-cell to a single point.

Van Kampen demonstrated the applicability of his diagrams to the word problem by providing the following result:

Theorem 4.2 (van Kampen's Lemma). *Let $P = \langle A \mid R\rangle$ be a presentation of a group Γ. Then for any word $w \in F(A)$, the following are equivalent:*

1. *The word w represents the identity in Γ.*

2. *There exists a van Kampen diagram for w over P.*

Proof. This proof is adapted from Lyndon and Schupp [LS01].

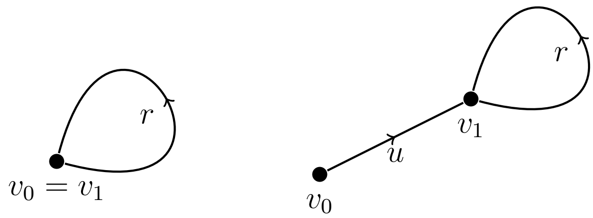
(1) ⇒ (2): Suppose that w is given in reduced form and represents the identity in Γ. Then there exists an expression

$$w \equiv u_1 r_1 u_1^{-1} \cdots u_n r_n u_n^{-1}$$

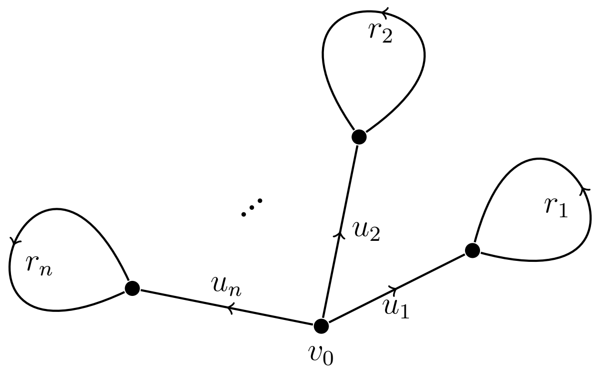
with each $r_i \in R^C$ cyclically reduced. We proceed by induction on the number n of such factors.

Base Case: If w is the empty word, we take D to be a single vertex.

If $w = uru^{-1}$ for some cyclically reduced $r \in R^C$, we construct a vertex v_1 with a loop labeled r. If $u = 1$, we set the basepoint $v_0 = v_1$. Otherwise, we add an edge labeled u from a new vertex v_0 to v_1, as illustrated below. The resulting diagram satisfies the conditions of a van Kampen diagram.



Inductive Step: Suppose that van Kampen diagrams $D_1, \ldots, D_n$ have been constructed for each $u_i r_i u_i^{-1}$. Arrange these around a common basepoint v_0 to form a diagram D'.



If the boundary label of D' is already reduced, we are done. Otherwise, there must be a pair of adjacent edges in the boundary labeled by inverse generators. we perform a folding process to simplify the boundary. By folding one edge over the other, we identify these edges and remove any resulting loops. By iterating this process, we gradually reduce the boundary label of D', as illustrated in Figure 4.2. After finitely many steps, the process terminates, resulting in a van Kampen diagram D whose boundary word is w.

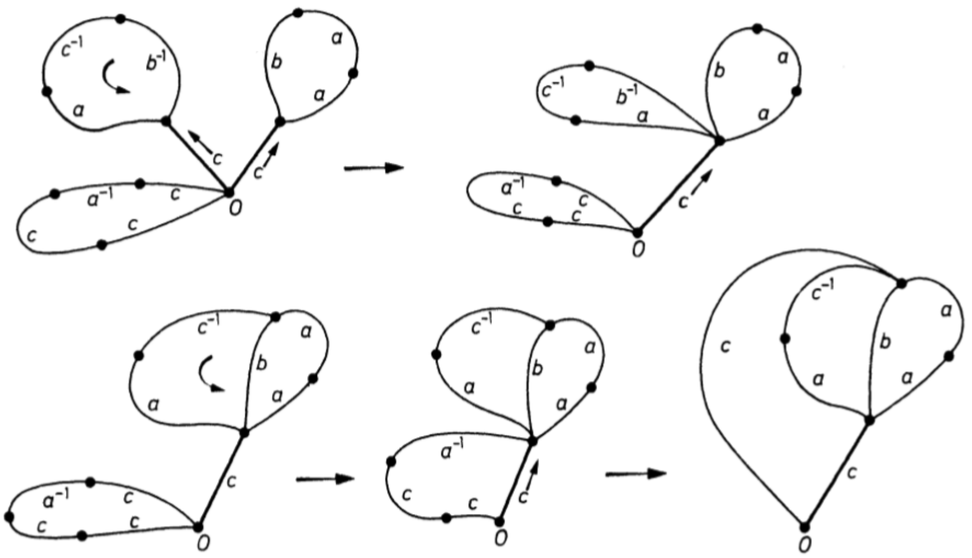


Figure 4.2: Folding adjacent inverse-labeled edges in the boundary during the construction of a van Kampen diagram [LS01].

(2) ⇒ (1): Suppose D is a van Kampen diagram for w. We argue by induction on the number of faces.

Base Case: If D has no faces, it must be a tree, and so the boundary word w is trivial in $F(A)$. Hence, w represents the identity in Γ.

Inductive Step: Let D have $m+1$ faces, and assume that the result holds for all diagrams with at most m faces. There exists a face $F \subset D$ that shares an edge with the outer boundary. Let e be such an edge. Define D' as the diagram obtained from D by deleting e, leaving intact the endpoints of e.

Since w is identically the boundary label of D, we write $w = \beta\epsilon\gamma$, where β and γ are segments of the boundary of D, and ϵ is the label of e. Let η be the remainder of the boundary label of the face F, after starting with the edge e. Then the boundary of D' becomes $\beta\eta^{-1}\gamma$.

Since D' has m faces, the induction hypothesis guarantees that there exist relators $r_1, \ldots, r_m \in R^C$ and elements $u_1, \ldots, u_m \in F(A)$ such that $\beta\eta^{-1}\gamma \equiv (u_1 r_1 u_1^{-1}) \cdots (u_m r_m u_m^{-1})$. Moreover, we can express w as

$$w = \beta\epsilon\gamma \equiv (\beta\eta^{-1}\gamma)(\gamma^{-1}\eta\epsilon\gamma).$$

Since $\eta\epsilon$ is a boundary label for the face F, we define $u_{m+1} := \gamma^{-1}$ and $r_{m+1} := \eta\epsilon$. This allows us to write

$$w \equiv (u_1 r_1 u_1^{-1}) \cdots (u_m r_m u_m^{-1})(u_{m+1} r_{m+1} u_{m+1}^{-1}).$$

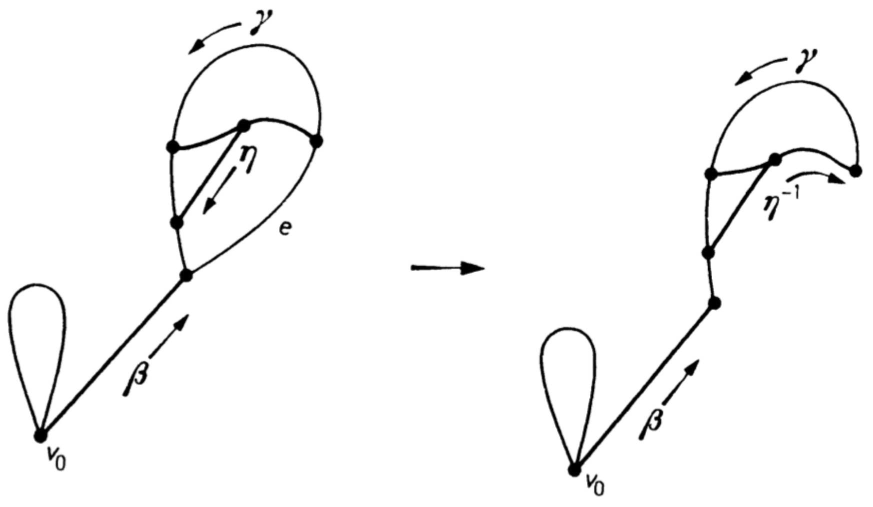


Figure 4.3: Eliminating a face along the boundary of a van Kampen diagram [LS01].

Therefore, w represents the identity in Γ, completing the proof. □

Remark 4.3. Notice that when deconstructing a van Kampen diagram in the second part of Theorem 4.2, the conjugating elements u_i we define always correspond to edge paths in D. The total number of edges in D is at most the sum of the boundary length and the contributions from the faces, which gives the bound:

$$|u_i| \leq |w| + mR, \quad \text{where } R = \max_{r \in R} |r|.$$

We have seen that if w represents the identity element in Γ, then there exists a van Kampen diagram for w over P. Among all such diagrams, there is one with the smallest possible number of faces. Therefore, there exists a van Kampen diagram D for w such that $\text{Area}(D) = \text{Area}(w)$.

4.2 Applications of Diagrams

Diagrammatic methods can be used to deduce important results quickly and concretely.

Theorem 4.4. *A finite presentation has solvable word problem if and only if it satisfies a recursive (i.e., computable) isoperimetric inequality (in which case, the Dehn function itself is recursive).*

Proof. Let $\Gamma = \langle S \mid R \rangle$ be a finite presentation.

Suppose first that the word problem in Γ is solvable. Then, given any word $w \in F(S)$ of length n, we can decide whether w represents the identity in Γ. If w is indeed equivalent to the identity in the group, then the procedure described in Section 3.2 to enumerate $\langle\langle R\rangle\rangle$ will eventually find an expression for w as a product of conjugates and relators. The number of such conjugates is at least $\mathrm{Area}(w)$. Hence, by examining all words of length up to n, we can establish an upper bound for the Dehn function as required.

Conversely, suppose the Dehn function is bounded by a recursive function $f\colon \mathbb{N} \to \mathbb{N}$. Given any word $w \in F(S)$ of length n, we first compute $f(n)$. We then systematically enumerate all possible products of at most $f(n)$ conjugates of relators, testing equivalence with w in the free group. By Remark 4.3, it suffices to check conjugating elements of length at most $|w| + f(n)R$, where $R = \max_{r\in R} |r|$, ensuring that the number of combinations to check is finite. If an expression is found, then w represents the identity in Γ; otherwise, it does not. Thus, the word problem is solvable. $\square$

Thus, to solve the word problem, it suffices to bound the Dehn function by a recursive function. This result was originally proved by Gersten [Ger93]. Since the word problem is not solvable in general, it follows that the Dehn function of a finitely presented group is not recursive in general.

Recall that a finitely presented group G can be made into a metric space by the word metric (see Section 2.3). However, this metric renders G as a discrete set of points. As Gromov observes in the introduction to his text [Gro92]

> This space may appear boring and uneventful to a geometer's eye since it is discrete and the traditional local machinery (e.g., topological and infinitesimal) machinery does not run in G.

This motivates a more geometric model for a group:

Definition 4.5. Given a presentation $P = \langle A \mid R\rangle$, the *presentation complex* K is a 2-dimensional CW-complex with a single vertex, an oriented edge for each generator in A, and a 2-cell for each relator in R, attached along the corresponding loop in the 1-skeleton. The fundamental group of K is the group defined by P.

The *Cayley complex* $\widetilde{K}$ of a group Γ with respect to the presentation P is the universal cover of K. It is obtained by attaching 2-cells to the Cayley graph $\mathrm{Cay}(\Gamma, A)$ along every loop corresponding to a relator in R.

Example 4.6. The following are examples of Cayley complexes:

- $F_2 = \langle a, b \rangle$: The presentation complex has one vertex and two loops labeled a and b; since there are no relators, no 2-cells are attached. The Cayley complex is just the Cayley graph, a regular infinite tree.
- $\mathbb{Z}^2 = \langle x, y \mid xyx^{-1}y^{-1} \rangle$: The presentation complex has one vertex, two loops, and one 2-cell attached along the commutator loop. The Cayley graph is a grid, and attaching 2-cells along each square yields a planar square tiling, the universal cover of the torus.

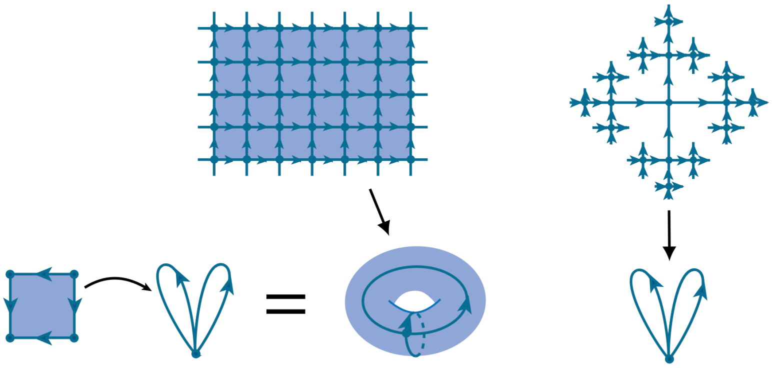


Figure 4.4: The presentation complex and Cayley complex for $\mathbb{Z}^2$ (left) and F_2 (right) [Ril17].

The Cayley complex $\widetilde{K}$ provides a rich geometric stage for studying the word problem. For example, a van Kampen diagram D for w can be seen as a disk in the Cayley complex that fills the loop in $\widetilde{K}$ corresponding to w. That is, there is a unique map from D to $\widetilde{K}$ sending the basepoint of D to the identity vertex, preserving labels and orientations, and mapping the boundary of D to the loop labeled by w. Indeed, if two paths in D start at v_0 and reach the same edge in $\widetilde{K}$, they must differ by a loop bounding a smaller van Kampen diagram. Since this loop represents the identity in Γ, both paths must end at the same point in $\widetilde{K}$, ensuring that the map is well defined. The map extends over the faces of D by mapping each 2-cell to the corresponding 2-cell in $\widetilde{K}$, respecting the relators.

This establishes a precise correspondence between the algebraic and geometric notions of area. The minimal number of conjugates needed to reduce w to the identity corresponds to the minimal number of 2-cells in a van Kampen diagram – the "area" of w. The Dehn function thus plays the role of a combinatorial isoperimetric function: given a loop of length n, how many 2-cells are needed to fill it? This echoes the classical isoperimetric problem in geometry, in which one seeks to maximize the area bounded by a curve of fixed perimeter.

Proposition 4.7. *The Dehn function $\delta(n)$ of a finite presentation of a finite group satisfies*

$$\delta(n) \leq Cn$$

for some constant C.

Proof. Let $G = \langle S \mid R \rangle$ be a finite presentation of a finite group. Then its Cayley complex $\widetilde{K}$ is finite, meaning it has finitely many vertices, edges, and faces. Let C be the number of faces in $\widetilde{K}$.

Take any word w that represents the identity in G. If the corresponding loop in $\widetilde{K}$ is simple (non-self-intersecting), then the area enclosed by it is at most C, since there are only C faces in total. Otherwise, if the loop is not simple, then after freely reducing, it can be decomposed into at most n simple subloops (where $n = |w|$). Each of these bounds at most C face. So, the total area is at most Cn, yielding $\delta(n) \leq Cn$. □

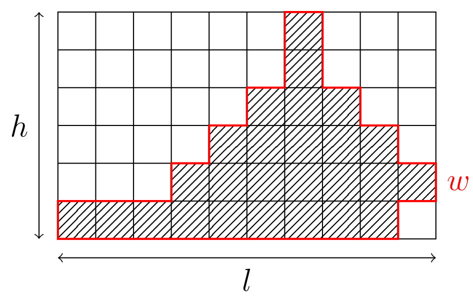
Proposition 4.8. *The Dehn function $\delta(n)$ of $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1} \rangle$ satisfies*

$$\frac{(n-3)^2}{16} \leq \delta(n) \leq \frac{n^2}{16}.$$

Proof. We follow [Zha23].

The Cayley complex of $\mathbb{Z}^2$ is the standard square tiling of the Euclidean plane. Each relator corresponds to a unit square.

Let w be a word of length n that represents the identity. Then w traces a loop in the grid, and we can bound its area by considering the smallest rectangle that encloses it in the grid. Suppose that this rectangle has side lengths l and h (measured in edges). The perimeter of the loop is at least $2(l+h)$, so $2(l+h) \leq n$. The maximum value of the area $l \times h$ subject to the constraint $(l + h) \leq n/2$ occurs when $l = h = n/4$, resulting in $\delta(n) \leq n^2/16$.



For the lower bound, consider the word $w_k = a^k b^k a^{-k} b^{-k}$. This word has length $4k$ and traces out a square with side length k, requiring k^2 squares to be filled. Taking $n = 4k$, we get $\delta(n) = k^2 = n^2/16$. For n not necessarily a multiple of 4, choose k such that $4k \leq n \leq 4k + 3$. Then $k \geq (n-3)/4$, and

$$\delta(n) \geq k^2 \geq \left(\frac{n-3}{4}\right)^2 = \frac{(n-3)^2}{16}. \qquad \square$$

4.3 Small Cancellation Theory

The algebraic structure of a group can be strongly constrained by the combinatorial properties of its presentation. This principle is central to *small cancellation theory*, developed in the 1960s by Lyndon, Greendlinger, and others. Small cancellation theory offers powerful tools for deriving isoperimetric inequalities, and its full development is presented in Chapter V of Lyndon and Schupp's seminal work *Combinatorial Group Theory* [LS01].

Small cancellation conditions are especially valuable because they are both mathematically tractable and general enough to describe random group presentations [Oll04]. At their core, these conditions are combinatorial and planar in nature, drawing on ideas from graph theory, most notably, Euler's formula $V - E + F = 2$ for finite connected planar graphs, where V, E, and F are the numbers of vertices, edges, and complementary regions (faces), respectively.

In the setting of a van Kampen diagram over a finite presentation, whenever two faces share a boundary segment, the corresponding relators must contain matching subwords. If all such overlaps are sufficiently short, then each interior face must neighbor many others. This is the intuition behind small cancellation: overlaps (called pieces) are short relative to the relators they appear in, forcing diagrams to be combinatorially dense.

Definition 4.9. Let $P = \langle A \mid R \rangle$ be a group presentation. We may assume that every relator in R is cyclically reduced, since cyclic reduction does not affect the group presented. Let R^C denote the cyclic closure of R, i.e., the set of all cyclic permutations of the relators and their inverses.

A *piece* is a nontrivial word $p \in F(A)$ that appears as a subword of at least two distinct elements of R^C. That is, there exist $r_1, r_2 \in R^C$ with $r_1 = ps_1$ and $r_2 = ps_2$, where $s_1 \neq s_2$. Every nontrivial subword of a piece is itself a piece.

We define the following small cancellation conditions:

- $C'\left(\frac{1}{\lambda}\right)$ for $\lambda > 0$: For any piece p appearing in a relator $r \in R^C$, the ratio $|p|/|r| < 1/\lambda$.

- $C(k)$: No element of R^C can be written as a product of fewer than k pieces.
- $T(q)$: For any $3 \leq k < q$, and for any sequence of relators $r_1, \ldots, r_k \in R^C$ and pieces $p_1, \ldots, p_k$ satisfying

  $$r_1 = p_1 r_1' p_2^{-1}, \quad r_2 = p_2 r_2' p_3^{-1}, \quad \ldots, \quad r_k = p_k r_k' p_1^{-1},$$

  then for some $i \mod k$, we must have $r_i = r_{i+1}^{-1}$. The intuitive meaning of this condition will be clarified soon.

These conditions become stronger as their parameters increase: $C'\left(\frac{1}{\lambda}\right)$ implies $C'\left(\frac{1}{\lambda'}\right)$ for $\lambda \geq \lambda'$, and $C(k)$ implies $C(k')$ for $k' \leq k$. Moreover, the metric condition $C'\left(\frac{1}{k}\right)$ implies the combinatorial condition $C(k)$.

Example 4.10. The following are small cancellation groups:

- The standard presentation of the genus-two surface group:

  $$\langle a, b, c, d \mid aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

  satisfies the conditions $C'(1/7)$ and $C(8)$. Each generator is a piece, but no sword of length 2 is a piece.
- More generally, the fundamental group of a closed orientable surface of genus g has the following presentation:

  $$\langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$

  This presentation satisfies $C'\left(\frac{1}{4g-1}\right)$ and $C(4g)$.

To understand how these conditions control the geometry of van Kampen diagrams, we introduce a standard simplification process.

Definition 4.11. A van Kampen diagram is called *unreduced* if it contains two distinct faces sharing an edge e, such that the boundary labels of the two faces are equal when read from the initial vertex of e in the direction of orientation. If no such pair exists, the diagram is *reduced*.

To reduce such a diagram, we remove the shared edge e, then progressively fold together the remaining boundaries of the two regions. This process merges the faces, decreasing the total number of regions. See Figure 4.5 for an example of this reduction step.

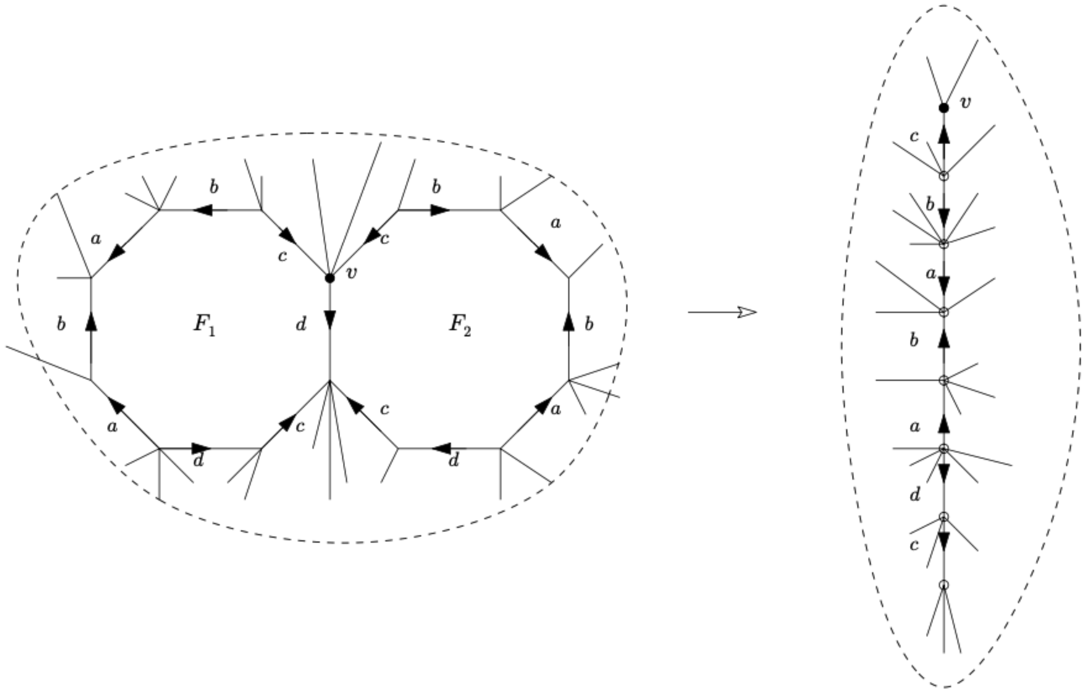


Figure 4.5: Reduction of a van Kampen diagram [BRS07]. The shared edge d is removed and the adjacent edges are folded together.

A minimal-area van Kampen diagram for a given word must be reduced. Moreover, we can always eliminate degree-2 vertices by merging the two incident edges and their labels. Henceforth, we will assume all diagrams are reduced and that all vertices have degree at least 3.

Now, let us see how the small cancellation conditions restrict the shape of possible van Kampen diagrams.

If a presentation satisfies $C(k)$, then in any reduced van Kampen diagram, every internal face (not touching the boundary) has at least k sides. This is because each side must be a piece, and $C(k)$ guarantees that no relator is a product of fewer than k such pieces.

The condition $T(q)$ similarly restricts the valence of internal vertices in a reduced van Kampen diagram: it ensures that every internal vertex has degree at least q. This is because the assertion that $r_i = r_{i+1}^{-1}$ for some $i \mod k$ implies that two faces meeting at p_{i+1} share the same boundary label, which is not possible in a reduced diagram (see Figure 4.6).

Together, these two conditions—$C(k)$ and $T(q)$—ensure that van Kampen diagrams over the presentation must be locally expansive: interior regions must be

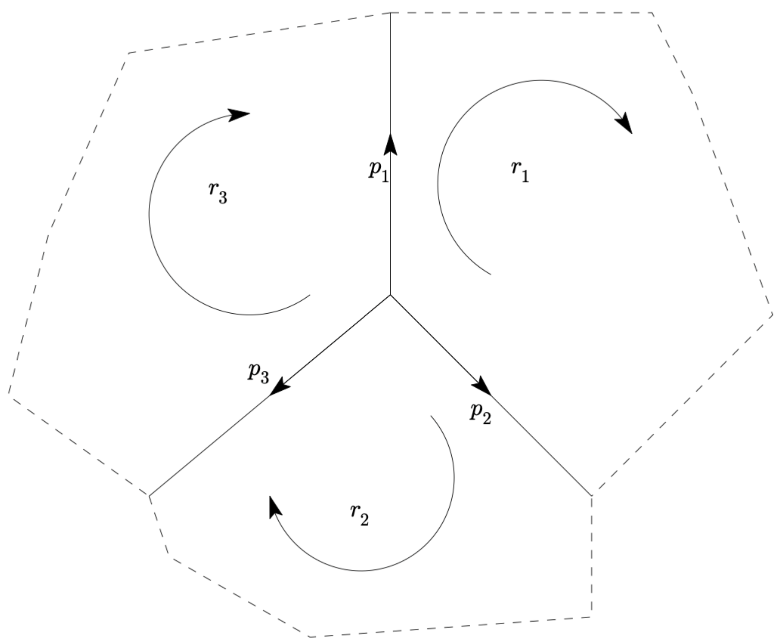


Figure 4.6: The $T(q)$ condition for $q = 3$ [BRS07]

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polygonal with many sides, and internal vertices must be shared by many regions. This combinatorial sparsity leads to strong bounds on the area of diagrams.

The following lemma plays a crucial role in deriving isoperimetric inequalities for groups satisfying certain small cancellation conditions. The results in this section follow the treatment provided by Hamish Short [BRS07].

Lemma 4.12 (Greendlinger's Lemma)**.** *Let D be a van Kampen diagram that is a topological disk with at least two regions. Suppose that every internal region in D has at least 6 sides, and every vertex has degree at least 3. For $i = 1, 2, 3, 4, 5$, let b_i denote the number of regions that meet the boundary of D in exactly one connected segment (which may also include some boundary vertices) and have exactly i internal edges. Then the inequality*

$$3b_1 + 2b_2 + b_3 \geq 6$$

holds.

Proof. We follow the approach of L.I. Greendlinger and M.D. Greendlinger [GG84].

Let D be a reduced van Kampen diagram over a presentation that satisfies the $C(6)$ condition. By assumption, D is a topological disk.

We now form a new diagram G on the 2-sphere S^2 by doubling D across its boundary with a slight twist (less than half the length of the shortest boundary edge). This means that we glue two copies of D along their boundaries and introduce a new vertex in the middle of each boundary edge, creating a planar graph embedded on the sphere.

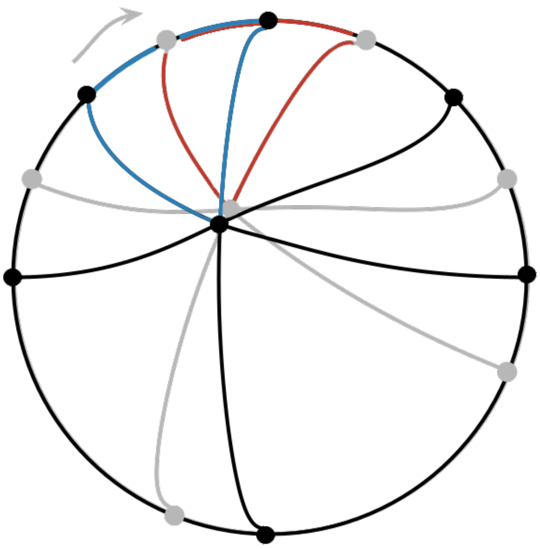


Figure 4.7: The doubling process: a 3-sided region in D becomes two 4-sided regions in G (highlighted in blue and red).

Each region of D contributing to b_i has $i+1$ sides in total. Doubling this contributes two regions to G—one for each half of the doubled diagram. These new regions will have $i+2$ sides, with the additional side coming from the newly created vertex on the boundary edge. For $i = 1, 2, 3$, these regions have 3, 4, or 5 sides, respectively. All other regions of D, including those that meet the boundary in more than one segment or do not meet it at all, double to form regions in G with at least 6 sides.

Let V, E, and F denote the number of vertices, edges, and faces (regions) in G, respectively. We now estimate E in terms of F. Every face has at least 6 sides, except those arising from b_1, b_2, and b_3, which contribute 3, 4, and 5 sides, respectively. Thus, we have:

$$2E \geq 6\left(F - 2b_1 - 2b_2 - 2b_3\right) + 3 \cdot 2b_1 + 4 \cdot 2b_2 + 5 \cdot 2b_3 = 6F - 6b_1 - 4b_2 - 2b_3.$$

Rewriting this gives:

$$F \leq \frac{E}{3} + \frac{3b_1 + 2b_2 + b_3}{3}.$$

Since every vertex has degree at least 3, counting the edges at each vertex yields:

$$2E \geq 3V \quad \Rightarrow \quad V \leq \frac{2E}{3}.$$

Applying Euler's formula on the sphere and substituting the bounds on V and F, we obtain:

$$2 = V - E + F \leq \frac{2E}{3} - E + \frac{E}{3} + \frac{3b_1 + 2b_2 + b_3}{3}.$$

Simplifying, we get:

$$3b_1 + 2b_2 + b_3 \geq 6.$$

This completes the proof. □

Theorem 4.13. *If a finite presentation P satisfies the small cancellation condition $C'(1/6)$, then Dehn's algorithm solves the word problem for P.*

Proof. The condition $C'(1/6)$ provides two crucial facts:

- The combined length of any 3 pieces of a relator is less than half the length of that relator.
- The presentation satisfies $C(7)$ and therefore the $C(6)$ condition required to apply Lemma 4.12.

Let w be a word that represents the identity element in the group defined by P, and let D be a van Kampen diagram for w. We assume that w is cyclically reduced, since any word is conjugate to its cyclically reduced form. Our goal is to show that w contains a subword that matches more than half of some relator (or its inverse), which allows the application of Dehn's algorithm.

The structure of D can be thought of as a collection of topological discs (each made up of regions glued along edges) connected together like a tree. If we treat each disc as a "fat vertex," then the entire diagram forms a tree. Since w is cyclically reduced, we know that there are no "dangling" edges or vertices of degree 1 on the boundary.

Now, we consider two cases:

Case 1: The diagram D is a single disc. In this case, Lemma 4.12 applies, and since at least one of b_1, b_2, or b_3 must be nonzero for the inequality to hold, there must exist a face in the diagram whose boundary intersects the outer boundary ∂D in a single segment and which has at most 3 internal edges. From the first observation, we know that this boundary segment must be longer than half the length of the relator labeling that face. Thus, w must contain a subword u that is longer than half of some relator r, say $r = uv$ with $|u| > |v|$. Hence, Dehn's algorithm can be applied.

Case 2: The diagram D contains multiple discs. In this case, the tree-like structure of the diagram ensures that there are at least two "extremal discs," discs connected to the rest of the diagram at a single vertex. In each of these extremal discs, Lemma 4.12 applies and gives us a face whose boundary overlaps with the outer boundary in a segment longer than half a relator. Again, this implies that the word w contains a subword longer than half of some relator, allowing the application of Dehn's algorithm. □

Consequently, presentations that satisfy the $C'(1/6)$ condition satisfy a linear isoperimetric inequality. Groups having such presentations are often referred to as sixth groups.

We can apply a similar method to Lemma 4.12 to prove the following:

Lemma 4.14. *Let P be a finite presentation satisfying the conditions $C(4)$ and $T(4)$, and let D be a reduced van Kampen diagram over P homeomorphic to a disk, with more than one face. Then D contains two bounded faces F_1 and F_2 such that for each $i = 1, 2$, the intersection $\partial F_i \cap \partial D$ consists of a connected segment containing all but at most two pieces of ∂F_i.*

Proof. We follow the approach of L.I. Greendlinger and M.D. Greendlinger [GG84].

Let D be a reduced van Kampen diagram satisfying the conditions $C(4)$ and $T(4)$, and assume that D has more than one face. We glue two copies of D along their boundaries ∂D, but this time we omit the twist so that the boundary vertices of both copies coincide. This results in a cell complex G embedded in the 2-sphere S^2.

By this gluing process, the degree of each boundary vertex increases by one, so all boundary vertices now have degree at least 4. Furthermore, the condition $T(4)$ ensures that all internal vertices of D have degree at least 4. Hence, every vertex in G has degree at least 4.

Let b_1 and b_2 represent the number of regions in D that intersect the boundary of D along a single connected segment and have 1 or 2 internal edges, respectively. Each region contributing to b_i has $i + 1$ sides in total. Doubling these regions results in two regions in G with the same number of sides, which, for $i = 1, 2$, have 2 or 3 sides, respectively. The $C(4)$ condition ensures that all internal regions of D have at least 4 sides, so all other regions of D, including those that meet the boundary in more than one segment or do not meet it at all, double to form regions in G with at least 4 sides.

Let V, E, F represent the number of vertices, edges, and faces in G. We count the total edge and face degrees:

$$2E \geq 4(F - 2b_1 - 2b_2) + 2 \cdot 2b_1 + 3 \cdot 2b_2 = 4F - 4b_1 - 2b_2$$

This simplifies to:

$$F \leq \frac{E}{2} + \frac{2b_1 + b_2}{2}$$

Since every vertex has degree at least 4, we count the edges at each vertex:

$$2E \geq 4V \quad \Rightarrow \quad V \leq \frac{E}{2}.$$

Applying Euler's formula and substituting the bounds on V and F gives:

$$2 = V - E + F \leq \frac{E}{2} - E + \frac{E}{2} + \frac{2b_1 + b_2}{2}.$$

Simplifying yields:

$$2b_1 + b_2 \geq 4,$$

which implies $b_1 + b_2 \geq 2$. This completes the proof. □

Using Lemma 4.14, we can derive the following result:

Theorem 4.15. *If a group presentation satisfies either the $C(6)$ condition or both the $C(4)$ and the $T(4)$ conditions, then it satisfies a quadratic isoperimetric inequality.*

Proof. Consider a van Kampen diagram D satisfying the $C(6)$ condition, where all degree-2 vertices have been removed. First, assume that D is topologically a disk.

Case 1: The diagram can be split along an edge. If there exists a non-boundary edge in D with endpoints v_1 and v_2 both on the boundary, cutting along this edge divides D into two smaller diagrams D_1 and D_2. Let δ_i denote the number of boundary vertices in D_i after suppressing degree-2 vertices. If v_1 or v_2 has degree 3 in D, then it has degree 2 and becomes suppressed in D_i. Thus, $\delta_1 + \delta_2 + 2 \geq \delta$. Also, since $\delta_1 + \delta_2$ cannot exceed δ by more than 2, we have $\delta_1 + \delta_2 - 2 \leq \delta$. By induction, if the number of faces in such a diagram is bounded by the square of the boundary length, we have

$$F = F_1 + F_2 \leq \delta_1^2 + \delta_2^2 \leq (\delta_1 + \delta_2 - 2)^2 \leq \delta^2$$

if $\delta_1, \delta_2 \geq 4$.

Case 2: No splitting edge exists. If no edge in D has both endpoints on the boundary, then any region that meets the boundary in two or more segments must have at least 6 sides (see Figure 4.8).

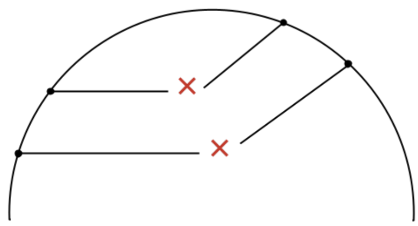


Figure 4.8: Any attempt to produce such a region with less than 6 sides would create a splitting edge. Intermediate vertices at the crosses are required.

We use the notation b_i and G (with a small twist) from Lemma 4.12, and define b_5' as the number of regions meeting the boundary in one segment with 5 or more internal edges or meeting the boundary in two or more segments (all having at least 6 sides, as above). Let V' denote the number of vertices of degree at least 4 in D.

Note that regions contributing to b_4 give rise to regions in G with 6 sides, and those contributing to b_5' give rise to regions with at least 7 sides.

We can count the number of edges in G in two ways.

From faces:

$$\begin{aligned}2E &\geq 6(F - 2(b_1 + b_2 + b_3 + b_4 + b_5')) + 3 \cdot 2b_1 + 4 \cdot 2b_2 + 5 \cdot 2b_3 + 6 \cdot 2b_4 + 7 \cdot 2b_5' \\ &= 6F - 6b_1 - 4b_2 - 2b_3 + 2b_5',\end{aligned}$$

which simplifies to:

$$F \leq \frac{E}{3} + \frac{3b_1 + 2b_2 + b_3 - b_5'}{3}.$$

From vertices:

$$2E \geq 3(V - 2V') + 4 \cdot 2V' \quad \Rightarrow \quad V \leq \frac{2E - 2V'}{3}.$$

Applying Euler's formula for planar graphs and substituting the inequalities gives:

$$2 = V - E + F \leq \frac{2E - 2V'}{3} - E + \frac{E}{3} + \frac{3b_1 + 2b_2 + b_3 - b_5'}{3},$$

which simplifies to:

$$6 \leq (3b_1 + 2b_2 + b_3) - (2V' + b_5').$$

Therefore, the number of boundary regions that contribute to $(3b_1 + 2b_2 + b_3)$ must exceed the number of vertices and regions contributing to $(2V' + b_5')$ by at least 6. We intuitively think of regions contributing to the former as simple, boundary-hugging, low-degree faces, and vertices/regions contributing to the lattter as complex, high-degree objects.

If, between every pair of regions contributing to $3b_1 + 2b_2 + b_3$, there is a vertex or region contributing to $2V' + b_5'$, then we would have at least as many high-degree objects as low-degree boundary-hugging faces, i.e.,

$$2V' + b_5' \geq V' \geq 3b_1 + 2b_2 + b_3,$$

which is a contradiction. Thus, between some pair of regions that contribute to $3b_1 + 2b_2 + b_3$, there must be no objects contributing to $2V' + b_5'$. In other words, since we do not have enough high-degree vertices or complex regions to "break up" all the small boundary-hugging faces, there must exist an uninterrupted chain of adjacent small faces.

More specifically, we can find a segment s on the boundary that touches a chain of regions $F_1, F_2, \ldots, F_k$, where:

- F_1 and F_k each contribute at most 3 internal edges (as they contribute to $3b_1 + 2b_2 + b_3$),
- The intermediate regions F_i contribute to b_4 (since they do not contribute to either $3b_1 + 2b_2 + b_3$ or $2V' + b_5'$),
- The endpoints v_0 and v_k and the vertices v_i through which the regions are chained have degree 3 (since they do not contribute to $2V' + b_5'$).

This chain provides a region that we can safely cut out, after which we can apply the induction hypothesis to the remaining diagram. This allows us to bound the number of faces in D by the square of δ, the number of vertices on its boundary.

Suppose that one of the internal vertices u_i (see Figure 4.9) that connects two regions in the chain has an edge e (other than f_i) with an endpoint on the boundary of D. Then u_i must have degree at least 4 (since it is already part of at least 3 edges: one from F_i, one from f_i, and one from F_{i+1}). Cutting D along

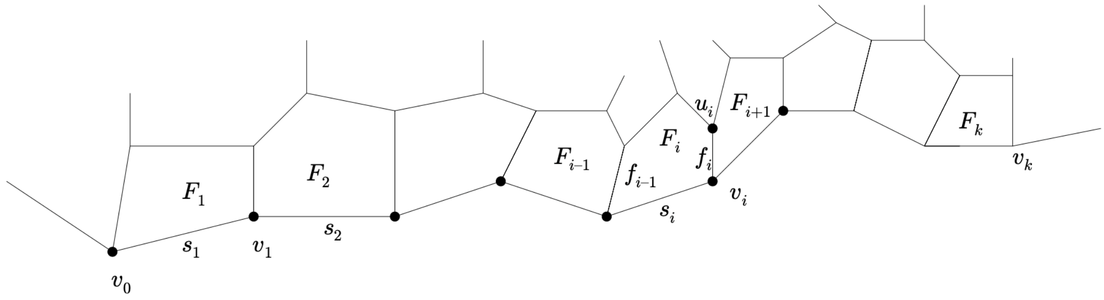


Figure 4.9: The chain of adjacent regions [BRS07].

the edges f_i and e would split D into two diagrams, each with at least 4 vertices on its boundary. Hence, as in Case 1, the induction hypothesis applies and shows that $F \leq \delta^2$.

Now, we assume that none of the vertices u_i has other edges with an endpoint on the boundary. If, for every such chain, some u_i had degree at least 4, then we would have $V' \geq 3b_1 + 2b_2 + b_3$, which is a contradiction as before. Therefore, we can assume that every u_i in the chain has degree 3.

Finally, we remove the entire chain of faces $F_1, \ldots, F_k$, along with the boundary segments $s_1, \ldots, s_k$ and the interior edges $f_1, \ldots, f_k$, resulting in a smaller diagram D'. The resulting diagram is a topological disk with k fewer faces. After suppressing the vertices v_0 and v_k and the u_i, which all have degree 2, the new boundary length δ' satisfies $\delta' \leq \delta - 2$. According to the induction hypothesis, the number of faces in D' is at most δ'^2. Hence, we have:

$$F = F' + k \leq (\delta - 2)^2 + k = \delta^2 - 4\delta + 4 + k.$$

Since we are removing at most δ boundary adjacent faces, we have $k \leq \delta$. Therefore:

$$F \leq \delta^2 - 4\delta + 4 + \delta = \delta^2 - 3\delta + 4.$$

For $\delta > 1$, this implies that $F \leq \delta^2$.

We have proven that the number of faces in a disk-shaped van Kampen diagram without degree 2 vertices is at most the square of the number of boundary vertices. A general van Kampen diagram can be decomposed into such disk components, and the total area is the sum of their areas, which satisfies the same inequality.

To handle diagrams with degree 2 vertices, observe that each edge label in a face for the relator r is a placeholder for at most $C = \max_{r \in R} |r|$ vertices of degree 2. Therefore, we obtain

$$\text{Area}(w) \leq C \cdot |w|^2,$$

where C is a constant depending on the presentation.

A directly analogous argument applies to presentations satisfying $C(4)$ and $T(4)$, involving chains between the regions provided by Lemma 4.14. $\square$

Chapter 5

Large-Scale Geometry of Groups

One of the central concepts in geometric group theory is that of a quasi-isometry, which formalizes the notion of two spaces having the same large-scale geometry. A property of a group is called geometric if it is invariant under quasi-isometry. Consequently, geometric group theory is primarily concerned with those properties of groups that depend only on their quasi-isometry class.

For instance, while the Cayley graph of a group depends on the choice of generating set, any two Cayley graphs of the same finitely generated group are quasi-isometric. Thus, the large-scale geometric properties of the Cayley graph are intrinsic to the group itself.

5.1 Quasi-Isometries

We now define quasi-isometries more precisely.

Definition 5.1. Let X and Y be metric spaces. A (not necessarily continuous) map $f\colon X \to Y$ is a (λ, ε)-*quasi-isometric embedding* if there exist constants $\lambda \geq 1$ and $\varepsilon \geq 0$ such that for all $x, x' \in X$,

$$\frac{1}{\lambda} d_X(x, x') - \varepsilon \leq d_Y(f(x), f(x')) \leq \lambda d_X(x, x') + \varepsilon.$$

If, in addition, there exists a constant $C \geq 0$ such that every point in Y lies within distance C of the image of f, then f is called a (λ, ε)-*quasi-isometry*. In this case, we say that X and Y are *quasi-isometric*.

The intuition is that a quasi-isometry distorts distances only by a controlled amount and covers the target space up to a bounded error. Quasi-isometric spaces therefore share the same large-scale structure, though they may differ significantly in local geometry.

Example 5.2. The following are examples of quasi-isometries:

- Any bounded metric space is quasi-isometric to a single point.
- The strip $\mathbb{R} \times [0, 1]$ is quasi-isometric to $\mathbb{R}$; the projection map $\pi_1(x, y) = x$ is a quasi-isometry.
- The Cayley graph of $\mathbb{Z}$ with respect to the generating set $\{2, 3\}$ (see Example 2.9) is quasi-isometric to $\mathbb{R}$, just as $\mathbb{R}$ can be viewed as the Cayley graph of $\mathbb{Z}$ with respect to $\{1\}$.
- The inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$ is a quasi-isometry. The integer-part function serves as a quasi-inverse.
- The inclusion $\mathbb{Z} \times \mathbb{Z} \hookrightarrow \mathbb{R} \times \mathbb{R}$ is a quasi-isometric embedding of the Cayley graph of $\mathbb{Z} \times \mathbb{Z}$ in the Euclidean plane.

Intuitively, quasi-isometric spaces share the same large-scale geometric properties, even if they may differ in finer details. For example, if one squints at the Cayley graph of $\mathbb{Z}$ with respect to the set $\{2, 3\}$, it resembles the real line $\mathbb{R}$.

Proposition 5.3. *Let $f \colon X \to Y$ be a (λ, ε)-quasi-isometry between metric spaces. Then there exists a (λ', ε')-quasi-isometry $g \colon Y \to X$ and a constant $K \geq 0$ such that*

$$d_X(g(f(x)), x) \leq K \quad \textit{and} \quad d_Y(f(g(y)), y) \leq K$$

for all $x \in X$, $y \in Y$. The map g is called a quasi-inverse to f.

Proof. Since f is a quasi-isometry, there exists a constant $C \geq 0$ such that for every $y \in Y$, some $x \in X$ satisfies $d_Y(f(x), y) \leq C$. Using the axiom of choice, define $g \colon Y \to X$ by selecting such a point x_y for each y, and set $g(y) := x_y$.

For all $x \in X$, we estimate:

$$d_X(g(f(x)), x) = d_X(x_{f(x)}, x) \leq \lambda d_Y(f(x_{f(x)}), f(x)) + \lambda\varepsilon \leq \lambda(C + \varepsilon).$$

Similarly, for all $y \in Y$, we have:

$$d_Y(f(g(y)), y) = d_Y(f(x_y), y) \leq C.$$

Hence, $K := \lambda(C + \varepsilon)$ suffices.

To show g is a quasi-isometric embedding, observe that for any $y, y' \in Y$,

$$\begin{aligned} d_X(g(y), g(y')) &= d_X(x_y, x_{y'}) \\ &\leq \lambda d_Y(f(x_y), f(x_{y'})) + \lambda\varepsilon \\ &\leq \lambda(d_Y(f(x_y), y') + d_Y(y, y') + d_Y(y, f(x_{y'}))) + \lambda\varepsilon \\ &\leq \lambda(C + d_Y(y, y') + C) + \lambda\varepsilon \\ &= \lambda d_Y(y, y') + \lambda(2C + \varepsilon), \end{aligned}$$

and similarly for the lower bound:

$$\begin{aligned} d_X(g(y), g(y')) &\geq \frac{1}{\lambda} d_Y(f(x_y), f(x_{y'})) - \frac{\varepsilon}{\lambda} \\ &\geq \frac{1}{\lambda} d_Y(y, y') - \frac{2C + \varepsilon}{\lambda}. \end{aligned}$$

Thus, g is a $(\lambda, \lambda(2C+\varepsilon))$-quasi-isometric embedding, hence a quasi-isometry. □

Proposition 5.4. *Let $f\colon X \to Y$ be a (λ, ε)-quasi-isometric embedding and $g\colon Y \to Z$ a (λ', ε')-quasi-isometric embedding. Then the composition $g \circ f\colon X \to Z$ is a $(\lambda\lambda', \lambda'\varepsilon + \varepsilon')$-quasi-isometric embedding. If f and g are quasi-isometries, then so is $g \circ f$.*

Proof. Let $x, x' \in X$. Then:

$$\begin{aligned} d_Z(g(f(x)), g(f(x'))) &\leq \lambda' d_Y(f(x), f(x')) + \varepsilon' \\ &\leq \lambda'(\lambda d_X(x, x') + \varepsilon) + \varepsilon' \\ &= \lambda\lambda' d_X(x, x') + \lambda'\varepsilon + \varepsilon'. \end{aligned}$$

For the lower bound:

$$d_Z(g(f(x)), g(f(x'))) \geq \frac{1}{\lambda'} d_Y(f(x), f(x')) - \varepsilon' \geq \frac{1}{\lambda\lambda'} d_X(x, x') - \left(\frac{\varepsilon}{\lambda'} + \varepsilon'\right).$$

Since $\lambda' \geq 1$, this gives a lower bound of the form $\frac{1}{\lambda\lambda'} d_X(x, x') - (\lambda'\varepsilon + \varepsilon')$. The composition is therefore a quasi-isometric embedding with parameters $(\lambda\lambda', \lambda'\varepsilon + \varepsilon')$.

If both f and g are quasi-isometries, then for each $z \in Z$, choose $y \in Y$ with $d_Z(g(y), z) \leq C'$, and then $x \in X$ with $d_Y(f(x), y) \leq C$. Then:

$$\begin{aligned} d_Z(g(f(x)), z) &\leq d_Z(g(f(x)), g(y)) + d_Z(g(y), z) \\ &\leq \lambda' d_Y(f(x), y) + \varepsilon' + C' \\ &\leq \lambda' C + \varepsilon' + C', \end{aligned}$$

so every point in Z is within bounded distance of the image of $g \circ f$. Hence, it is a quasi-isometry. □

Corollary 5.5. *Quasi-isometry defines an equivalence relation on the class of metric spaces.*

Let Γ be a group with a finite generating set A. As introduced in Chapter 2, the Cayley graph $\mathrm{Cay}(\Gamma, A)$ can be considered a metric space by assigning each edge length 1 and measuring the distance between vertices by the shortest path in the graph. The resulting metric on the vertex set of the Cayley graph coincides with the *word metric* d_A on Γ, where the distance between two group elements $g, h \in \Gamma$ is given by the minimal length of a word in the free group $F(A)$ representing the element gh^{-1} in Γ. In fact, the inclusion of (Γ, d_A) in $(\mathrm{Cay}(\Gamma, A), d_A)$ is a $(1, 0)$-quasi-isometry.

This viewpoint permits us to interpret Cayley graphs—and, by extension, finitely generated groups—as metric spaces. A foundational result that underpins much of geometric group theory is due independently to Švarc [Sch55] and Milnor [Mil68]. Their theorem establishes that under suitable conditions a group acting on a metric space is quasi-isometric to that space. Before stating the result precisely, we introduce some terminology concerning metric spaces and group actions.

Definition 5.6. Let (X, d) be a metric space.

- X is called *proper* if every closed ball $\overline{B}_r(x) = \{y \in X \mid d(x, y) \leq r\}$ is compact.

- X is a *length space* if for any $x, y \in X$, the distance $d(x, y)$ is equal to the infimum of the lengths of the rectifiable curves joining x to y. A curve $\gamma : [0, 1] \to X$ is called *rectifiable* if its length

$$|\gamma| = \sup \left\{ \sum_{i=1}^{n} d\left(\gamma(t_{i-1}), \gamma(t_i)\right) \,\middle|\, 0 = t_0 < t_1 < \cdots < t_n = 1 \right\}$$

  is finite.

Definition 5.7. Let Γ be a group that acts on a metric space X. The action is said to be:

- *proper* if for every compact subset $K \subset X$, the set $\{g \in \Gamma \mid g \cdot K \cap K \neq \emptyset\}$ is finite;

- *cocompact* if there exists a compact subset $K \subset X$ such that $\Gamma \cdot K = X$;

- *by isometries* if for each $g \in \Gamma$, the map $x \mapsto g \cdot x$ is an isometry of X, i.e., $d_X(g \cdot x, g \cdot y) = d_X(x, y)$ for all $x, y \in X$.

The geometric nature of a group action often reflects deep properties of the group itself. For example, Serre [Ser02] showed that every finite group acting on a tree must fix a point. The following fundamental result by Švarc and Milnor provides a powerful tool for understanding finitely generated groups via their actions on metric spaces.

Theorem 5.8 (Švarc–Milnor Lemma)**.** *Let (X, d_X) be a proper length space. Suppose that Γ acts properly and cocompactly by isometries on X. Then:*

- *Γ is finitely generated;*
- *For any basepoint $x_0 \in X$, there exists a finite generating set A of Γ such that the orbit map*
$$\Gamma \to X, \quad g \mapsto g \cdot x_0,$$
is a quasi-isometry.

Proof. We follow Bridson and Haefliger [BH99].

Since the action is cocompact, there exists a compact set $K \subset X$ with $\Gamma \cdot K = X$. For any $x_0 \in X$, choose $r \geq 1$ such that $K \subset B_r(x_0)$. By properness of both X and the group action, the set

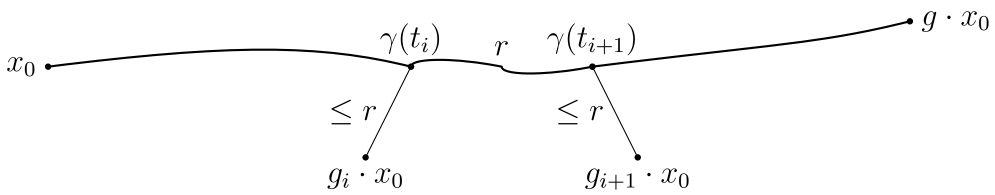
$$A := \{g \in \Gamma \mid g \cdot \overline{B}_{3r}(x_0) \cap \overline{B}_{3r}(x_0) \neq \emptyset\}$$

is finite.

We will show that A generates Γ, and that the map $g \mapsto g \cdot x_0$ defines a quasi-isometry between (Γ, d_A) and (X, d_X). Since the action is by isometries, we have $d_X(g \cdot x_0, g' \cdot x_0) = d_X(x_0, g^{-1}g' \cdot x_0)$, and by left-invariance of the word metric, we have $d_A(g, g') = d_A(1, g^{-1}g')$. Thus, it suffices to compare $d_X(x_0, g \cdot x_0)$ and $d_A(1, g)$ for arbitrary $g \in \Gamma$.

Fix any $g \in \Gamma$. Since X is a length space, there exists a rectifiable path $\gamma : [0, 1] \to X$ from x_0 to $g \cdot x_0$ with length $|\gamma| \leq d_X(x_0, g \cdot x_0) + 1$. Partition the path into subsegments of length approximately r, obtaining times $0 = t_0 < t_1 < \cdots < t_N = 1$, such that each segment has length r (except possibly the last, which is $\leq r$).

Using the cocompactness of the action, we can find group elements $1 = g_0, g_1, \ldots, g_N = g$ such that $d_X(\gamma(t_i), g_i \cdot x_0) \leq r$.



The triangle inequality gives $d_X(g_i \cdot x_0, g_{i+1} \cdot x_0) \leq 3r$, hence $g_i^{-1} g_{i+1} \in A$. Setting $a_i := g_{i-1}^{-1} g_i$, we have:

$$g = g_0(g_0^{-1} g_1) \cdots (g_{N-1}^{-1} g_N) = a_1 a_2 \cdots a_N,$$

so A generates Γ.

Since $(N-1)r \leq |\gamma| \leq Nr$, it follows that

$$N \leq \frac{d_X(x_0, g \cdot x_0) + 1}{r} + 1.$$

Since g can be written as a word of length N, we have

$$d_A(1, g) \leq \frac{d_X(x_0, g \cdot x_0) + 1}{r} + 1,$$

and rearranging gives the lower bound:

$$r d_A(1, g) - (r+1) \leq d_X(x_0, g \cdot x_0).$$

For the other bound, suppose $d_A(1, g) = n$, and write $g = a_1 \cdots a_n$ with each $a_i \in A$. Define partial products $g_i = a_1 \cdots a_i$, so $g_0 = 1$ and $g_n = g$. Then the triangle inequality gives:

$$d_X(x_0, g \cdot x_0) \leq \sum_{i=1}^{n} d_X(g_{i-1} \cdot x_0, g_i \cdot x_0).$$

Each term is at most $\mu := \max_{a \in A} d_X(x_0, a \cdot x_0)$, so:

$$d_X(x_0, g \cdot x_0) \leq \mu n = \mu \cdot d_A(1, g).$$

Thus, the orbit map satisfies:

$$r \cdot d_A(1, g) - (r+1) \leq d_X(x_0, g \cdot x_0) \leq \mu \cdot d_A(1, g),$$

which shows that it is a (λ, ε)-quasi-isometry, with $\lambda = \max\{1/r, \mu\}$ and $\varepsilon = r + 1$. □

An immediate consequence of the Švarc–Milnor Lemma is the following:

Corollary 5.9. *If a finitely generated group* Γ *acts properly and cocompactly by isometries on two proper length spaces* X_1 *and* X_2*, then* X_1 *and* X_2 *are quasi-isometric.*

The quasi-isometry $\Gamma \to X$ in the second part of Theorem 5.8 can be viewed as a quasi-isometry on the vertex set of $\mathrm{Cay}(\Gamma, A)$. By arbitrarily assigning images to the edges incident to the identity and extending equivariantly under the action of Γ, we obtain a quasi-isometry $\mathrm{Cay}(\Gamma, A) \to X$. The following proposition shows that the existence of such a quasi-isometry is independent of the choice of finite generating set for Γ.

Proposition 5.10. *Let* A *and* B *be two finite generating sets for the group* Γ*. Then the Cayley graphs* $\mathrm{Cay}(\Gamma, A)$ *and* $\mathrm{Cay}(\Gamma, B)$ *are quasi-isometric via a map that is the identity on the vertices.*

Proof. We follow [BRS07].

Let $A = \{a_1, \dots, a_p\}$ and $B = \{b_1, \dots, b_q\}$. Since both sets generate Γ, each generator $a_i \in A$ can be written as a word $u_i(B) \in F(B)$ representing the same element in Γ, and similarly each $b_j \in B$ corresponds to a word $v_j(A) \in F(A)$.

Suppose $g, g' \in \Gamma$ and let $k = d_A(g, g')$. Then there is a word $w(A) = a_{\epsilon_1} a_{\epsilon_2} \dots a_{\epsilon_k} \in F(A)$ such that $gw(A) = g'$. Replacing each a_{ϵ_i} by the corresponding word $u_{\epsilon_i}(B)$, we obtain a word over B that connects g to g' in $\mathrm{Cay}(\Gamma, B)$. Let λ_1 denote the maximum length of the words $u_i(B)$. Then:

$$d_B(g, g') \leq \lambda_1 \cdot d_A(g, g').$$

Similarly, expressing each $b_j \in B$ as a word $v_j(A) \in F(A)$ and letting λ_2 be the maximum of these lengths, we get:

$$d_A(g, g') \leq \lambda_2 \cdot d_B(g, g').$$

Combining both inequalities:

$$\frac{1}{\lambda_2} d_A(g, g') \leq d_B(g, g') \leq \lambda_1 d_A(g, g'),$$

so the identity map on Γ defines a $(\lambda, 0)$-quasi-isometry between (Γ, d_A) and (Γ, d_B), where $\lambda = \max\{\lambda_1, \lambda_2\}$.

To extend this to a quasi-isometry between the entire graphs, define maps $\varphi\colon \mathrm{Cay}(\Gamma, A) \to \mathrm{Cay}(\Gamma, B)$ and $\psi\colon \mathrm{Cay}(\Gamma, B) \to \mathrm{Cay}(\Gamma, A)$ that send each interior point of an edge to one of its endpoints and act as the identity on vertices. For any t, t' on edges of $\mathrm{Cay}(\Gamma, A)$, we estimate:

$$d_B(\varphi(t), \varphi(t')) \leq \lambda_1 d_A(\varphi(t), \varphi(t')) \leq \lambda_1 (d_A(t, t') + 2) \leq \lambda_1 d_A(t, t') + 2\lambda_1,$$

and similarly in the reverse direction.

Since the maps are identity on vertices, the 1-neighbourhoods of their images cover the Cayley graphs. Furthermore, for any point t in $\mathrm{Cay}(\Gamma, A)$, we have

$$d_A(\psi \circ \varphi(t), t) \leq 1 \quad \text{and} \quad d_B(\varphi \circ \psi(t), t) \leq 1,$$

so φ and ψ are quasi-inverses.

Hence, $\mathrm{Cay}(\Gamma, A)$ and $\mathrm{Cay}(\Gamma, B)$ are quasi-isometric. □

The first part of this proof shows that, for any two finite generating sets A and B of a group Γ, the identity map $(\Gamma, d_A) \to (\Gamma, d_B)$ is a quasi-isometry.

We are now able to define the notion of quasi-isometry for groups.

Definition 5.11. Two finitely generated groups Γ and Γ' are said to be *quasi-isometric* if there exist finite generating sets A and A' such that the Cayley graphs $\mathrm{Cay}(\Gamma, A)$ and $\mathrm{Cay}(\Gamma', A')$ are quasi-isometric metric spaces. By Proposition 5.10, this definition does not depend on the choice of finite generating sets and is therefore well defined.

A group-theoretic property is called *geometric* if it is preserved under quasi-isometries of groups.

The following result, due to Alonso [Alo90], shows that finite presentability and solvability of the word problem are geometric properties.

Theorem 5.12 ([Alo90])**.** *Let Γ and Γ' be quasi-isometric finitely generated groups. Suppose Γ admits a finite presentation $P = \langle A \mid R \rangle$. Then Γ' also admits a finite presentation $Q = \langle B \mid S \rangle$, and the Dehn functions of P and Q are equivalent.*

Proof. We adapt the approach of Short in [BRS07].

Let $\phi : \mathrm{Cay}(\Gamma, A) \to \mathrm{Cay}(\Gamma', B)$ be a quasi-isometry with quasi-inverse $\psi : \mathrm{Cay}(\Gamma', B) \to \mathrm{Cay}(\Gamma, A)$. Up to adjusting constants, we may assume that both maps send vertices to vertices, and preserve the identity elements, that is, $\phi(1_\Gamma) = 1_{\Gamma'}$ and $\psi(1_{\Gamma'}) = 1_\Gamma$. Suppose that the quasi-isometry constants are (λ, ε), and that for all $g \in \Gamma'$, we have

$$d_B(\phi \circ \psi(g), g) \leq C.$$

Let $w_B = y_{j_1} \cdots y_{j_n} \in F(B)$ be a word over B representing a loop in $\mathrm{Cay}(\Gamma', B)$ based at the identity. Let $v_i \in \Gamma'$ denote the vertex reached by the prefix $y_{j_1} \cdots y_{j_i}$, with $v_0 = v_n = 1$.

Apply ψ to lift the path into Γ: set $u_i = \psi(v_i) \in \Gamma$. Since each $d_B(v_i, v_{i+1}) = 1$, the quasi-isometry property implies $d_A(u_i, u_{i+1}) \leq \lambda + \varepsilon$. For each i, choose a word $\alpha_i \in F(A)$ of length at most $\lambda + \varepsilon$ that labels a path in $\mathrm{Cay}(\Gamma, A)$ from u_i to u_{i+1}.

The concatenation $w_A = \alpha_0 \cdots \alpha_n$ labels a loop in $\mathrm{Cay}(\Gamma, A)$ based at the identity. Since Γ has presentation $P = \langle A \mid R \rangle$, there exists a van Kampen diagram D over P with boundary word w_A.

We now move this diagram to $\mathrm{Cay}(\Gamma', B)$. Apply ϕ to the 1-skeleton of D, replacing each edge labeled with a generator from A with a path in $\mathrm{Cay}(\Gamma', B)$ of length at most $\lambda + \varepsilon$. Each relator $r \in R$ becomes a word over B of length at most $\lambda \cdot |r| + \varepsilon$. This process produces a new, relabeled diagram D' over (Γ', B). The boundary path of the resulting diagram is $\phi(w_A) = \phi(\alpha_0) \cdots \phi(\alpha_n)$, which starts and ends at the identity of Γ'.

We now compare $\phi(w_A) = \phi(\alpha_0) \cdots \phi(\alpha_n)$ and $w_B = y_{j_1} \cdots y_{j_n}$. The path $\phi(w_A)$ starts and ends at the identity, since $\phi \circ \psi(v_n) = \phi \circ \psi(1) = 1$, since the maps preserve identities. For each prefix $\phi(\alpha_0) \cdots \phi(\alpha_i)$, the endpoint is $\phi(\psi(v_i))$, which, as previously observed, lies within distance C of v_i, the endpoint of each prefix $y_{j_1} \cdots y_{j_i}$. Thus, there exists a word h_i of length at most C that connects $\phi(\psi(v_i))$ and v_i.

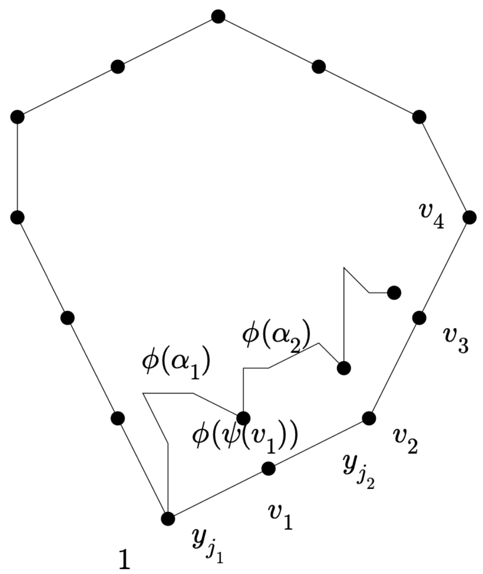


Figure 5.1: The paths w_B and $\phi \circ \psi(w_B)$ in $\mathrm{Cay}(\Gamma', B)$ [BRS07].

We can now construct a van Kampen diagram for the original word w_B from D', the relabeled version of D, by filling in the difference between $\phi(w_A)$ and w_B.

More precisely, for each i, we add a small van Kampen diagram filling each loop labeled by $\phi(\alpha_i)h_i y_{j_i}^{-1} h_{i-1}^{-1}$, which has length at most $(\lambda+\varepsilon)+C+1+C$. The relators required for this construction have length at most

$$\max\left\{2C+1+\lambda+\varepsilon,\ (\lambda+\varepsilon)\cdot\max_{r\in R}|r|\right\}.$$

Thus, we can define a finite presentation $S \subset F(B)$ consisting of these relators.

The area of D' is equal to the area of D (since relabeling does not change the number of faces), which is $\mathrm{Area}_P(w_A)$. Moreover, the word w_A has length at most $(\lambda+\varepsilon)n = K\cdot|w_B|$, for some constant K. Finally, the filled correction loops contribute at most $(2C+1+\lambda+\varepsilon)n = K'\cdot|w_B|$ to the area.

$$\mathrm{Area}_Q(w_B) \leq K''\cdot\delta_P(K\cdot|w_B|)+K'\cdot|w_B|,$$

for some constants K, K', K''. This implies $\delta_Q \preceq \delta_P$. By the symmetry of the quasi-isometry relation, the same inequality holds with P and Q interchanged. Therefore, the Dehn functions of Γ and Γ' are equivalent. □

Observe that Theorem 3.9 is a corollary of this result.

5.2 Commensurability

Proposition 5.13. *Let G be a finitely generated group, and let H be a finite-index subgroup. Then H is finitely generated, and H is quasi-isometric to G.*

Proof. Let A be a finite generating set for G. Consider the Cayley graph $\mathrm{Cay}(G, A)$, on which G acts properly and cocompactly by isometries. Restricting this action to the subgroup H, we obtain an isometric action of H on $\mathrm{Cay}(G, A)$.

The action remains proper since properness is preserved under restriction. Because H has finite index in G, there exist finitely many left cosets $g_1H, \ldots, g_nH$ that partition G. Let $K = \{g_1, \ldots, g_n\}$; then $H \cdot K = G$, so the action is also cocompact.

By the Švarc–Milnor lemma (Theorem 5.8), it follows that H is finitely generated and that the orbit map $H \to \mathrm{Cay}(G, A)$, defined by $h \mapsto h\cdot 1$, is a quasi-isometry. Since $\mathrm{Cay}(G, A)$ is quasi-isometric to G, it follows that H and G are quasi-isometric. □

The Švarc–Milnor lemma provides a powerful method for proving that two groups are quasi-isometric: it suffices to construct proper, cocompact actions by isometries on the same metric space. In Proposition 5.13, the key point is that cocompactness is preserved under restriction to finite-index subgroups. Indeed, if G acts cocompactly on a metric space X, then there exists a compact subset $K \subset X$ such that $G \cdot K = X$. For any finite-index subgroup $H \leq G$, choose coset representatives $g_1, \ldots, g_n$ for G/H, and define $K' = \bigcup_{i=1}^n g_i \cdot K$. Then K' is compact and $H \cdot K' = X$, verifying the cocompactness of the H-action.

It follows from Proposition 5.13 that quasi-isometry invariants, such as being finitely presentable or having a solvable word problem, are preserved when passing to finite-index subgroups. Moreover, by Theorem 5.12, the Dehn function is also preserved up to equivalence.

Definition 5.14. Two groups G_1 and G_2 are said to be *commensurable* if they have isomorphic finite-index subgroups. That is, there exist finite index subgroups H_1 of G_1 and H_2 of G_2 such that H_1 is isomorphic to H_2. Commensurability is an equivalence relation.

A group is said to be *virtually* P if it contains a finite-index subgroup with property P.

Common examples of such terminology include "virtually free," "virtually abelian," and "virtually torsion-free." Note that every finite group is virtually trivial, since its identity subgroup is finite index.

Applying Proposition 5.13, we obtain the following consequence:

Proposition 5.15. *If G and G' are finitely generated commensurable groups, then G and G' are quasi-isometric.*

This naturally leads to the question of whether the converse holds. Are quasi-isometric groups necessarily commensurable? In general, the answer is negative, but there are special cases where it holds. For instance, quasi-isometric groups are commensurable when:

- one of the groups is finite (hence both are finite),
- one of the groups is virtually $\mathbb{Z}$, or
- both groups are virtually abelian.

See Bowditch [Bow95] for further discussion of these results.

Chapter 6

Non-Positive Curvature

6.1 Hyperbolic Groups

In this section, we show that the class of hyperbolic groups satisfies a linear isoperimetric inequality. We begin by formalizing the notion of geodesic spaces.

Definition 6.1. Let (X, d) be a metric space. A *geodesic* in X is an isometric embedding $\gamma\colon [0, L] \to X$, where $[0, L]$ is equipped with the standard Euclidean metric. That is, for all $t, t' \in [0, L]$, we have

$$d(\gamma(t), \gamma(t')) = |t - t'|.$$

In particular, $d(\gamma(0), \gamma(L)) = L$. If $\gamma(0) = x$ and $\gamma(L) = y$, the image of γ is called a *geodesic segment* from x to y, denoted $[x, y]$.

We say that X is a *geodesic space* if every pair of points in X is connected by a geodesic.

An equivalent definition is that a geodesic space is a metric space in which, for all $x, y \in X$, there exists a rectifiable path from x to y whose length equals $d(x, y)$. More generally, in a length space, the distance between two points is the infimum of the lengths of all rectifiable paths connecting them. In a geodesic space, this infimum is always *attained*. Most of the metric spaces we have considered so far are geodesic.

Definition 6.2. Let X be a geodesic space. For any three points $x, y, z \in X$, a *geodesic triangle* $\Delta(x, y, z)$ consists of geodesic segments connecting each pair of points: $[x, y]$, $[y, z]$, and $[x, z]$.

Given $\delta > 0$, we say the triangle is *δ-slim* if each side is contained in the δ-neighbourhood of the union of the other two sides. The space X is called *δ-hyperbolic* if every geodesic triangle in X is δ-slim. In this case, we also say that X is *hyperbolic*.

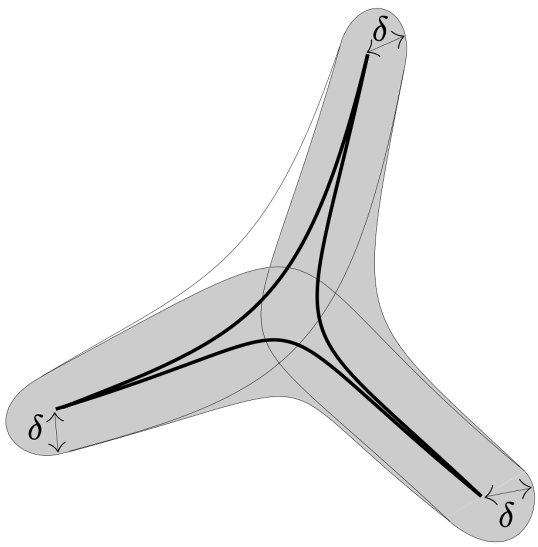


Figure 6.1: A δ-slim triangle [BH99]

Intuitively, δ-slim triangles resemble trees more than they resemble thick Euclidean triangles. In fact, trees are 0-hyperbolic.

One can show that the slim triangle condition carries over to quasi-geodesic triangles (i.e. images of geodesic triangles under a quasi-isometry): if X is δ-hyperbolic, then quasi-geodesic triangles in X are uniformly slim. More precisely, for every $\lambda \geq 1$ and $\varepsilon \geq 0$, there exists a constant $M = M(\lambda, \varepsilon)$ such that every (λ, ε)-quasi-geodesic triangle in X is M-slim [BH99]. This leads to the following result.

Proposition 6.3. *Let X and X' be geodesic metric spaces. If X is hyperbolic and $f \colon X' \to X$ is a quasi-isometric embedding, then X' is also hyperbolic.*

Proof. Suppose X is δ-hyperbolic and $f \colon X' \to X$ is a (λ, ε)-quasi-isometric embedding. Let Δ' be a geodesic triangle in X' with sides $\gamma_1, \gamma_2, \gamma_3$. The images $f(\gamma_i)$ form a quasi-geodesic triangle in X. Since X is δ-hyperbolic, there exists a constant $M = M(\delta, \lambda, \varepsilon)$ such that this quasi-geodesic triangle is M-slim.

Fix a point $x \in \gamma_1$. Then there exists $y \in \gamma_2 \cup \gamma_3$ such that $d_X(f(x), f(y)) \leq M$. Using the quasi-isometric inequality, we estimate:

$$d_{X'}(x, y) \leq \lambda \cdot d_X(f(x), f(y)) + \lambda\varepsilon \leq \lambda M + \lambda\varepsilon.$$

Thus, each point of γ_1 lies within distance $\lambda M + \lambda\varepsilon$ of the union $\gamma_2 \cup \gamma_3$, and similarly for the other sides. Therefore, Δ' is δ'-slim in X' with $\delta' = \lambda M + \lambda\varepsilon$, and so X' is hyperbolic. $\square$

Combining Proposition 6.3 with the fact that Cayley graphs associated with different finite generating sets are quasi-isometric (Proposition 5.10), we may define the following:

Definition 6.4. A finitely generated group G is called *hyperbolic* if its Cayley graph (with respect to any finite generating set) is a δ-hyperbolic metric space for some $\delta > 0$. This property is independent of the choice of finite generating set.

Example 6.5. Examples of hyperbolic groups include:

- Finite groups: their Cayley graphs are bounded, and thus trivially hyperbolic.

- Finitely generated free groups: their Cayley graphs are trees which are 0-hyperbolic.

- Random groups: a remarkable result of Gromov is that "almost every" finitely presented group is hyperbolic, in a suitable probabilistic sense. Consider presentations with $m \geq 2$ generators and $n \geq 1$ relators, where each relator is a cyclically reduced word of fixed length ℓ. By choosing relators uniformly at random, one obtains a random group Γ_ℓ. Gromov showed that as $\ell \to \infty$, the probability that Γ_ℓ is infinite and hyperbolic tends to 1, for all values of m and n [Gro87].

Our goal in this section is to prove that hyperbolic groups satisfy a linear isoperimetric inequality. To do so, we require the notion of a local geodesic.

Definition 6.6. Let (X, d) be a metric space, and let $k > 0$. A path $c\colon [a, b] \to X$ is called a *k-local geodesic* if the restriction of c to every subinterval of length at most k is an isometric embedding. That is, for all $t, t' \in [a, b]$ with $|t - t'| \leq k$, we have

$$d(c(t), c(t')) = |t - t'|.$$

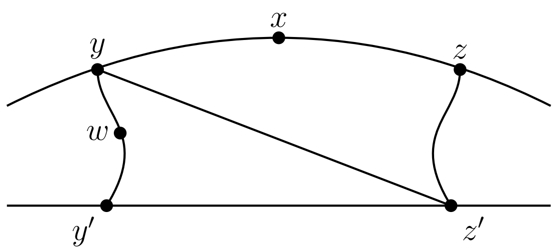
We first show that local geodesics remain close to global geodesics in δ-hyperbolic spaces.

Lemma 6.7. *Let X be a δ-hyperbolic space, and fix $k > 8\delta$. If $c\colon [a, b] \to X$ is a k-local geodesic, then the image of c lies within the 2δ-neighbourhood of any geodesic segment joining $c(a)$ and $c(b)$.*

Proof. We follow the argument in [BH99].

Let $x = c(t)$ be a point on the image of c that maximizes the distance to the geodesic segment $[c(a), c(b)]$. First, assume that x is at least 4δ away from both endpoints, i.e., $t - a > 4\delta$ and $b - t > 4\delta$. Then we can choose a subarc of c, centered at x, of length strictly between 8δ and k. Let y, z be its endpoints and let y', z' be their respective closest-point projections to $[c(a), c(b)]$.

We now consider the geodesic quadrilateral with vertices y, z, z', y', divided into two geodesic triangles along the diagonal $[y, z']$.



By δ-hyperbolicity, there is a point on either $[y, z']$ or $[z, z']$ within δ of x. In the former case, by the second triangle, there is a point either on $[y, y']$ $[y', z']$ within 2δ of x. In the latter case, similarly we get some point on either $[z, z']$ or $[y', z']$ within 2δ of x. In either case, the point w within 2δ of x cannot lie on the original local geodesic $[y, z]$ and must lie on one of the other three sides of the quadrilateral.

Suppose first that $w \in [y, y']$. Then we can estimate:

$$\begin{aligned} d(x, y') - d(y, y') &\le d(x, w) + d(w, y') - [d(y, w) + d(w, y')] \\ &= d(x, w) - d(y, w) \\ &\le 2d(x, w) - d(y, x) \\ &< 4\delta - 4\delta = 0. \end{aligned}$$

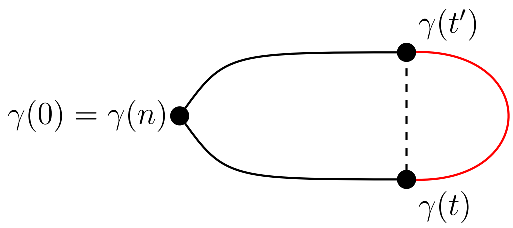
This implies $d(x, y') < d(y, y')$, contradicting the fact that y' is the closest-point projection of y to $[c(a), c(b)]$. A similar contradiction arises if $w \in [z, z']$. Therefore, w must lie on $[y', z']$, and we conclude that $d(x, [c(a), c(b)]) \le 2\delta$.

In the remaining case, where x is within 4δ of either endpoint, the conclusion follows even more directly: the geodesic segment $[c(a), c(b)]$ must pass within δ of x, as the endpoints are already within 4δ.

Hence, in all cases, the image of c remains within a 2δ-neighbourhood of $[c(a), c(b)]$. □

Next, we show that cycles in Cayley graphs of hyperbolic groups always contain short sub-paths that fail to be geodesic. This will be crucial when constructing relators that satisfy Dehn's algorithm later.

Lemma 6.8. *Let G be a hyperbolic group with finite generating set A, and suppose that the Cayley graph $\mathrm{Cay}(G, A)$ is δ-hyperbolic. Let $\gamma\colon [0, n] \to \mathrm{Cay}(G, A)$ be the piecewise linear realization of a graph-theoretic cycle of length $n > 0$. Then there exist $t, t' \in [0, n]$ such that the sub-path $\gamma|_{[t,t']}$ has length at most 8δ and is not geodesic.*



Proof. We follow Löh [Lö17].

Suppose for contradiction that γ is a k-local geodesic for some $k > 8\delta$. Since γ is a cycle, we must have $\gamma(0) = \gamma(n)$, and thus $n > 8\delta$ (otherwise, the entire path would be a geodesic and therefore constant).

By Lemma 6.7, the image of any such k-local geodesic lies within the 2δ-neighbourhood of a geodesic connecting its endpoints. But in this case, the geodesic is constant at $\gamma(0)$, so the image of γ is entirely contained in a ball of radius 2δ centered at $\gamma(0)$. Therefore, the diameter of γ is at most 4δ.

This contradicts the existence of an intermediate point $\gamma(5\delta)$, which must satisfy

$$d_A(\gamma(0), \gamma(5\delta)) = 5\delta > 4\delta.$$

Therefore, γ cannot be a k-local geodesic for any $k > 8\delta$.

It follows that γ must not be an 8δ-local geodesic. That is, there exist $t, t' \in [0, n]$ with $|t - t'| \leq 8\delta$ such that the subpath $\gamma|_{[t,t']}$ is not geodesic. Since γ is a cycle, the length of $\gamma|_{[t,t']}$ is exactly $|t - t'| \leq 8\delta$. □

This result shows that long cycles in hyperbolic groups cannot uniformly behave like geodesics: there must exist short sub-paths of bounded length (at most 8δ) where the path fails to take the shortest possible route.

Finally, we show that hyperbolic groups satisfy a linear isoperimetric inequality.

Theorem 6.9. *Let G be a group with a finite generating set A. If G is hyperbolic, then G has a finite presentation for which Dehn's algorithm solves the word problem.*

Proof. Since G is hyperbolic, its Cayley graph $\mathrm{Cay}(G, A)$ is δ-hyperbolic for some $\delta > 0$. Define $D := \lceil 8\delta \rceil + 2$. Let $\pi\colon F(A) \to G$ be the canonical projection, which is the homomorphism that sends each word in $F(A)$ to the corresponding element in G that the word represents.

The key idea is as follows: for every word u of length $\leq D$ that is not a geodesic (i.e. $|u| > d_A(1, u)$), we add a relator uv^{-1}, where v is a shorter word representing the same group element. We define R as the set of all such words. Note that R is finite and since $D > 2$, R contains all words of length 2 representing the identity (taking v as the empty word).

We now verify the property required for Dehn's algorithm. Specifically, we must show that any nontrivial word w representing the identity in G contains a subword that is more than half of a relator in R. We proceed by induction on $|w|$:

- *Base Case*: If $|w| = 0$, then w is the empty word (trivial).
- *Inductive Step*: Assume that for any word of length less than or equal to k, the property holds, and consider a word w of length greater than k. We now examine two cases:
  - *Case 1*: If w contains a subword ab that represents the identity in G (that is, ab labels a cycle of length 2 in the Cayley graph), then w contains more than half (in this case, the entirety) of the relator in $ab \in R$.
  - *Case 2*: If no such ab exists, then w (or a subword of w) forms a cycle in $\mathrm{Cay}(G, A)$. By Lemma 6.8, there exists a subword u of w such that $d_A(1, u) < |u| \leq D$. This means that there exists a shorter word v representing the same element as u in G. Therefore, w contains more than half of the relator $uv^{-1} \in R$.

Since we have shown that any non-trivial word w can be reduced to the identity in this way, and S already generates G, it follows that $\langle S \mid R \rangle$ is the desired finite presentation of G. $\square$

We see that hyperbolicity ensures that every sufficiently long word w representing the identity must contain a "detour" (a subword that can be shortened).

Theorem 6.9 implies that hyperbolic groups satisfy a linear isoperimetric inequality. The converse is less direct, and a proof using filling functions can be found in Section III.H.2 of [BH99]. Gromov showed that a group satisfying a subquadratic isoperimetric inequality is hyperbolic, hence the "gap" in the isoperimetric spectrum. Bowditch gives a short proof that a subquadratic isoperimetric inequality implies a linear one [Bow95].

6.2 CAT(0) Groups

In the last chapter, we defined a hyperbolic metric space as a geodesic space where geodesic triangles are "slim," meaning that each side stays close to the union of the other two. Intuitively, this means that triangles in such spaces are not much fatter than those in trees, which are extremely thin. To develop a notion of non-positive curvature for general metric spaces, we shift the comparison: instead of comparing triangles to those in trees, we use triangles from the Euclidean plane as our reference.

Definition 6.10. Let X be a geodesic metric space. A *comparison triangle* for a geodesic triangle $\Delta(p, q, r)$ is a triangle $\Delta(\overline{p}, \overline{q}, \overline{r})$ in the Euclidean plane $\mathbb{E}^2$ with the same side lengths (such triangles are unique up to Euclidean isometry).

For any point x on a side of $\Delta(p, q, r)$, say on the side $[p, q]$, a *comparison point* for p is the point $\overline{x}$ on the corresponding edge $[\overline{p}, \overline{q}]$ in the comparison triangle such that $d_X(p, x) = d_{\mathbb{E}^2}(\overline{p}, \overline{x})$.

A geodesic triangle Δ is said to satisfy the CAT(0) *inequality* if for all $x, y \in \Delta$ and corresponding comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$

$$d_X(x, y) \leq d_{\mathbb{E}^2}(\overline{x}, \overline{y}).$$

We call X a CAT(0) *space** if every geodesic triangle in X satisfies the CAT(0) inequality.

Remark 6.11. The "0" in CAT(0) refers to the space having curvature less than or equal to zero; they are commonly called *non-positively curved* spaces. If we

*The name CAT is an acronym derived from the initials of mathematicians Élie Cartan, Aleksandr Aleksandrov, and Victor Toponogov, who made foundational contributions to the study of spaces with curvature bounds.

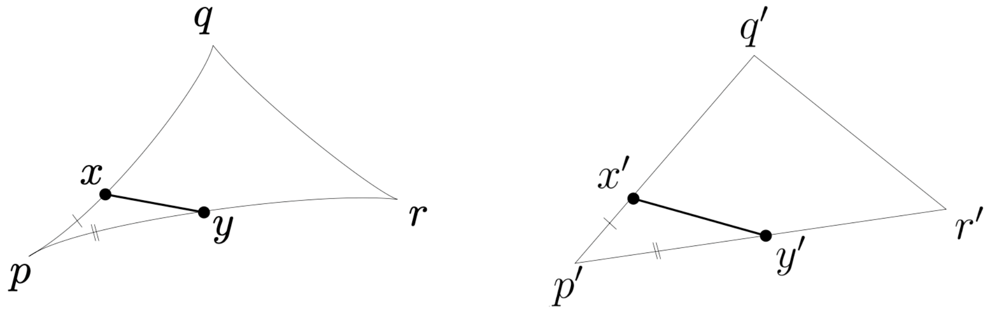


Figure 6.2: Tne CAT(0) inequality

use other two-dimensional spaces with constant curvature, like the sphere or the hyperbolic plane, as our models for comparison, we get a more general class of spaces called CAT(κ) spaces. For example, CAT(-1) spaces are those where triangles are no fatter than the corresponding triangles in the hyperbolic plane, which has constant negative curvature.

We can also consider *locally* CAT(0) spaces – spaces that satisfy the CAT(0) condition only in small neighborhoods. However, it turns out that if a space is simply connected and locally CAT(0), then it is actually CAT(0) everywhere (see the Cartan–Hadamard theorem in Chapter II.4 of [BH99]).

Example 6.12. The following are CAT(0) spaces:

- The standard Euclidean spaces $\mathbb{R}^n$ are CAT(0) for all $n \in \mathbb{N}$. Every geodesic triangle in $\mathbb{R}^n$ lies in a 2-dimensional subspace and satisfies the CAT(0) inequality.
- The hyperbolic plane $\mathbb{H}^2$ is a CAT(0) space, as its geodesic triangles are thinner than their Euclidean comparison triangles. This extends to all hyperbolic spaces $\mathbb{H}^n$ for $n \geq 2$.
- Any (possibly infinite-dimensional) Hilbert space is CAT(0), generalizing the Euclidean case to infinite dimensions.
- Any simply connected Riemannian manifold with non-positive sectional curvature is a CAT(0) space, by the Cartan–Hadamard theorem [BH99].
- Any metric tree is CAT(0), as geodesic triangles in a tree are always tripods and hence satisfy the CAT(0) inequality.
- If X and Y are CAT(0), then the product space $X \times Y$, equipped with the ℓ^2-metric, is also CAT(0). In particular, a finite product of trees is CAT(0).

The following is *not* a CAT(0) space:

- The round 2-sphere S^2 is not CAT(0), since its geodesic triangles are fatter than their Euclidean comparison triangles.

Also, the 2-sphere S^2 is quasi-isometric to a single-point space, which *is* a CAT(0) space. Therefore, unlike hyperbolicity, whether a space is CAT(0) is *not* preserved under quasi-isometries.

We describe a few important properties of CAT(0) spaces.

Proposition 6.13. *If X is a* CAT(0) *space, then the distance function $d\colon X \times X \to \mathbb{R}$ is convex in the following sense: If $c, c'\colon [0,1] \to X$ are reparameterized geodesics, then for all $t \in [0,1]$, we have*

$$d(c(t), c'(t)) \leq (1-t)d(c(0), c'(0)) + td(c(1), c'(1)).$$

Proof. We follow [BH99].

First, assume $c(0) = c'(0)$. Consider a comparison triangle Δ for points $c(0)$, $c(1)$, $c'(1)$ and the corresponding straight-line paths $\overline{c}$, $\overline{c}'$ in Δ. By Euclidean geometry:

$$d(\overline{c}(t), \overline{c}'(t)) = t \cdot d(\overline{c}(1), \overline{c}'(1)) = t \cdot d(c(1), c'(1))$$

The CAT(0) condition implies:

$$d(c(t), c'(t)) \leq d(\overline{c}(t), \overline{c}'(t)) = t \cdot d(c(1), c'(1))$$

Now we turn to the general case. For arbitrary geodesics c, c', construct an intermediate geodesic c'' from $c(0)$ to $c'(1)$. We apply the special case twice. Between c and c'' we get

$$d(c(t), c''(t)) \leq t \cdot d(c(1), c''(1)) = t \cdot d(c(1), c'(1))$$

Between c'' and c' (reversed) we get:

$$d(c''(t), c'(t)) \leq (1-t) \cdot d(c''(0), c'(0)) = (1-t) \cdot d(c(0), c'(0))$$

The triangle inequality then gives

$$d(c(t), c'(t)) \leq d(c(t), c''(t)) + d(c''(t), c'(t)) \leq (1-t)d(c(0), c'(0)) + td(c(1), c'(1))$$

□

Proposition 6.14. *If X is a* CAT(0) *space, then any two points in X are connected by a unique geodesic segment.*

Proof. Suppose γ and δ are two geodesics connecting the same pair of points, that is, $\gamma(0) = \delta(0)$ and $\gamma(1) = \delta(1)$. By the convexity of the distance function (Proposition 6.13), the distance between corresponding points on γ and δ satisfies

$$d(\gamma(t), \delta(t)) \leq (1-t) \cdot \underbrace{d(\gamma(0), \delta(0))}_{=0} + t \cdot \underbrace{d(\gamma(1), \delta(1))}_{=0} = 0.$$

Thus, $d(\gamma(t), \delta(t)) = 0$ for all $t \in [0,1]$ meaning $\gamma(t) = \delta(t)$ for every point along the paths. $\square$

Proposition 6.15. *If X is a* CAT(0) *space, then X is contractible.*

Proof. Choose a basepoint $x_0 \in X$. We construct an explicit contraction as follows. For each point $x \in X$, let $\gamma_x \colon [0,1] \to X$ be the unique geodesic from x_0 to x (guaranteed by Proposition 6.14). The map $x \mapsto \gamma_x$ is continuous because geodesics in CAT(0) spaces vary continuously with their endpoints (a consequence of the convexity of the distance function). The map

$$H \colon X \times [0,1] \to X, \quad (x,t) \mapsto \gamma_x(t)$$

gives a homotopy between, at $t = 0$, the constant map $H(x, 0) = x_0$ for all x, and, at $t = 1$, the identity map $H(x, 1) = x$ for all x. $\square$

Since the CAT(0) property is not preserved under quasi-isometries, we cannot define CAT(0) groups in terms of their Cayley graphs, unlike the case for hyperbolic groups.

Definition 6.16. A group is a CAT(0) *group* if it acts properly and cocompactly by isometries on a CAT(0) space.

Example 6.17. The following are CAT(0) groups:

- For each $n \in \mathbb{N}$, the group $\mathbb{Z}^n$ acts by translations on $\mathbb{R}^n$, which is a CAT(0) space. Thus, $\mathbb{Z}^n$ is a CAT(0) group.
- Every finite group is a CAT(0) group, since it acts trivially on a one-point space, which is trivially CAT(0).
- Every finitely generated free group is a CAT(0) group, since it acts on its Cayley graph, which is a tree and hence a CAT(0) space.
- Groups satisfying $C'(1/6)$, $C(4)$–$T(4)$ small cancellation conditions are CAT(0) groups [Wis04].

- $\mathrm{Aut}(F_2)$ is a CAT(0) group [PRW10].
- The direct product of two CAT(0) groups is again a CAT(0) group.

The question of whether every hyperbolic group is a CAT(0) group remains open. CAT(0) groups are closed under passage to finite index subgroups: if G is CAT(0) and $H \leq G$ has finite index, then H is also CAT(0). However, the property does not generally pass to finite index supergroups, as cocompactness of the action may be lost. Thus, CAT(0) is not preserved under commensurability: being commensurable to a CAT(0) group implies that one is virtually CAT(0), but not necessarily CAT(0) itself.

Our next goal is to show that CAT(0) groups satisfy a quadratic isoperimetric inequality. To do this, we first need a result that produces a generating set from a group action:

Theorem 6.18. *Let X be a connected topological space and suppose a group Γ acts on X by homeomorphisms. If there is an open set $U \subset X$ such that $X = \Gamma \cdot U$ (i.e., the translates of U cover X), then the set*

$$S = \{\gamma \in \Gamma \mid \gamma \cdot U \cap U \neq \emptyset\}$$

generates Γ.

Proof. Let $H = \langle S \rangle$ be the subgroup generated by S. We will show $H = \Gamma$.

Let $V = H \cdot U$, the union of H-translates of U, and let $V' = (\Gamma \setminus H) \cdot U$, the union of all other translates.

We first show disjointness. If $V \cap V' \neq \emptyset$, there exist $h \in H$ and $h' \in \Gamma \setminus H$ with $h \cdot U \cap h' \cdot U \neq \emptyset$. But then:

$$h^{-1}h' \cdot U \cap U = \emptyset \implies h^{-1}h' \in S \subset H \implies h' \in H,$$

contradicting $h' \in \Gamma \setminus H$. Thus, $V \cap V' = \emptyset$.

Note that V is non-empty (contains U), both V and V' are open (since U is open and actions are by homeomorphisms), and $X = V \sqcup V'$ is a disjoint union of open sets. By connectivity of X, V' must be empty, so $\Gamma \cdot U = H \cdot U$ and hence $\Gamma = H$. $\square$

For the remainder of this section, let Γ be a group acting properly and cocompactly by isometries on a CAT(0) space X. We follow the method in [BH99]. Fix a point $x_0 \in X$ and choose $D > 0$ such that the collection of balls $\gamma \cdot B(x_0, D/3)$ for $\gamma \in \Gamma$ covers X.

Lemma 6.19. *Let* Γ *and* D *be as described above. Define*

$$A = \{a \in \Gamma \mid d(a \cdot x_0, x_0) \leq D + 1\}.$$

Then A *generates* Γ*. Moreover, if* $\gamma \in \Gamma$ *satisfies* $d(x_0, \gamma \cdot x_0) \leq 2D + 1$*, then* γ *can be written as a product of at most four elements of* A*; that is,* $\gamma = a_1 a_2 a_3 a_4$ *for some* $a_i \in A$*.*

Proof. By Theorem 6.18 applied to $U = B(x_0, D/3)$, the set $S = \{\gamma \in \Gamma \mid \gamma \cdot B(x_0, D/3) \cap B(x_0, D/3) \neq \emptyset\}$ generates Γ. Since any such $\gamma \in S$ satisfies $d(x_0, \gamma \cdot x_0) \leq 2D/3 < D + 1$, we have $S \subset A$, and therefore A also generates Γ.

For the second claim, fix $\gamma \in \Gamma$ with $d(x_0, \gamma \cdot x_0) \leq 2D + 1$. If this distance is already at most $D + 1$, then $\gamma \in A$ and we are done. Otherwise, we approximate the geodesic from x_0 to $\gamma \cdot x_0$ by successive steps of length at most $D + 1$, using cocompactness at each stage to find elements of A.

Let x_1 be a point on the geodesic at distance $\frac{2D}{3} + 1$ from x_0. By cocompactness, there exists $a_1 \in \Gamma$ with $x_1 \in a_1 \cdot B(x_0, D/3)$, which implies

$$d(x_0, a_1 \cdot x_0) \leq d(x_0, x_1) + d(x_1, a_1 \cdot x_0) \leq \left(\frac{2D}{3} + 1\right) + \frac{D}{3} = D + 1$$

so $a_1 \in A$. The remaining distance to $\gamma \cdot x_0$ is at most

$$d(a_1 \cdot x_0, \gamma \cdot x_0) \leq d(a_1 \cdot x_0, x_1) + d(x_1, \gamma \cdot x_0) \leq \frac{D}{3} + \left[(2D + 1) - \left(\frac{2D}{3} + 1\right)\right] = \frac{5D}{3}.$$

If this is $\leq D + 1$, then $a_1^{-1}\gamma \in A$, so $\gamma = a_1(a_1^{-1}\gamma)$ is a product of two elements from A.

If not, repeat: pick x_2 on the geodesic from $a_1 \cdot x_0$ to $\gamma \cdot x_0$ at distance $\frac{2D}{3}$ from $a_1 \cdot x_0$, and find $a_2 \in \Gamma$ such that $x_2 \in a_1 a_2 \cdot B(x_0, D/3)$. Then

$$d(a_1 \cdot x_0, a_1 a_2 \cdot x_0) \leq \frac{2D}{3} + \frac{D}{3} = D,$$

so $a_2 \in A$, and the remaining distance to $\gamma \cdot x_0$ is at most

$$d(a_1 a_2 \cdot x_0, \gamma \cdot x_0) \leq d(a_1 a_2 \cdot x_0, x_2) + d(x_2, \gamma \cdot x_0) \leq \frac{D}{3} + \left(\frac{5D}{3} - \frac{2D}{3}\right) = \frac{4D}{3}.$$

If this is $\leq D + 1$, then $\gamma = a_1 a_2 (a_2^{-1} a_1^{-1} \gamma)$ is a product of three elements of A.

If not, proceed in a similar way: choose x_3 on the geodesic from $a_1 a_2 \cdot x_0$ to $\gamma \cdot x_0$ at distance $\frac{2D}{3}$, and find $a_3 \in \Gamma$ with $x_3 \in a_1 a_2 a_3 \cdot B(x_0, D/3)$. Then $a_3 \in A$, and

$$d(a_1 a_2 \cdot x_0, a_1 a_2 a_3 \cdot x_0) \leq \frac{2D}{3} + \frac{D}{3} = D,$$

and the remaining distance to $\gamma \cdot x_0$ is at most

$$d(a_1a_2a_3 \cdot x_0, \gamma \cdot x_0) \leq d(a_1a_2a_3 \cdot x_0, x_3) + d(x_3, \gamma \cdot x_0) \leq \frac{D}{3} + \left(\frac{4D}{3} - \frac{2D}{3}\right) = D.$$

Thus $a_3^{-1}a_2^{-1}a_1^{-1}\gamma \in A$, and $\gamma = a_1a_2a_3(a_3^{-1}a_2^{-1}a_1^{-1}\gamma)$ is a product of four elements in A, as claimed. □

In fact, the set A is finite. To see this, let $K = \overline{B}(x_0, D+1)$ denote the closed ball of radius $D+1$ centered at x_0. By definition of A, for each $\gamma \in A$, we have $\gamma \cdot x_0 \in K$, and hence $\gamma \cdot K \cap K \neq \emptyset$. That is, each $\gamma \in A$ maps the compact set K so that it intersects itself.

Since the action of Γ on X is proper, it follows that the set

$$\{\gamma \in \Gamma \mid \gamma \cdot K \cap K \neq \emptyset\}$$

is finite. Therefore, $A \subset \Gamma$ is finite.

Now we show that CAT(0) groups satisfy a quadratic isoperimetric inequality.

Theorem 6.20. *Let Γ and A be as in the previous lemma. Define $R \subset F(A)$ as the set of all reduced words of length at most 10 in the free group on A, which represent the identity element in Γ. Then a word w in the letters $A^{\pm 1}$ represents the identity in Γ if and only if it can be written in the free group as:*

$$w \equiv \prod_{i=1}^{N} x_i r_i x_i^{-1}$$

where each $r_i \in R$ and $N \leq (D+1)|w|^2$, and each x_i has length at most $(D+1)|w|$. In particular Γ satisfies a quadratic isoperimetric inequality.

Proof. Let $x_0 \in X$ be a fixed basepoint. For each $\gamma \in \Gamma$, we construct a word σ_γ in the generators A that represents γ. Define σ_γ using the following process:

Let c_γ be the geodesic from x_0 to $\gamma \cdot x_0$ in X. For each integer $0 \leq i < d(x_0, \gamma \cdot x_0)$, choose $\sigma_\gamma(i) \in \Gamma$ such that $c_\gamma(i) \in \sigma_\gamma(i) \cdot B(x_0, D/3)$. Set $\sigma_\gamma(0) = 1$, and for $i \geq d(x_0, \gamma \cdot x_0)$, define $\sigma_\gamma(i) = \gamma$. Define the word $\sigma_\gamma = a_1 \cdots a_n$, where each letter $a_i := \sigma_\gamma(i-1)^{-1}\sigma_\gamma(i) \in A$ since

$$\begin{aligned} d(a_i \cdot x_0, x_0) &= d(\sigma_\gamma(i-1) \cdot x_0, \sigma_\gamma(i) \cdot x_0) \\ &\leq d(\sigma_\gamma(i-1) \cdot x_0, c_\gamma(i-1)) + d(c_\gamma(i-1), c_\gamma(i)) + d(c_\gamma(i), \sigma_\gamma(i) \cdot x_0) \\ &\leq \frac{D}{3} + 1 + \frac{D}{3} \leq D + 1, \end{aligned}$$

and n is the smallest integer greater than $d(x_0, \gamma \cdot x_0)$. This word σ_γ represents γ in Γ. Define σ_1 as the empty word.

Now suppose $\gamma' = \gamma b$ for some $b \in A$. We want to compare the words σ_γ and $\sigma_{\gamma'}$. To do this, pad them with identity letters so that both have length $n = n(\gamma, \gamma') = \max\{|\sigma_\gamma|, |\sigma_{\gamma'}|\}$. Since $b \in A$, we have $d(\gamma \cdot x_0, \gamma' \cdot x_0) \leq D + 1$. By the convexity of the metric on X, the points $c_\gamma(i)$ and $c_{\gamma'}(i)$ satisfy $d(c_\gamma(i), c_{\gamma'}(i)) \leq i \cdot d(\gamma \cdot x_0, \gamma' \cdot x_0) \leq D + 1$ for all i. Hence,

$$\begin{aligned} d(\sigma_\gamma(i) \cdot x_0, \sigma_{\gamma'}(i) \cdot x_0) &\leq d(\sigma_\gamma(i) \cdot x_0, c_\gamma(i)) + d(c_\gamma(i), c_{\gamma'}(i)) + d(c_{\gamma'}(i), \sigma_{\gamma'}(i) \cdot x_0) \\ &\leq \frac{D}{3} + (D+1) + \frac{D}{3} < 2D + 1. \end{aligned}$$

By Lemma 6.19, for each i, we can choose a word $\alpha(i) \in A^{\pm 1}$ of length at most 4 such that $\alpha(i) = \sigma_\gamma(i)^{-1}\sigma_{\gamma'}(i)$. Set $\alpha(0) = 1$ and $\alpha(n) = b$, since $\gamma' = \gamma b$.

Let $p_\gamma(i) = a_1 \cdots a_i$ be the prefix of σ_γ, and similarly for $p_{\gamma'}(i)$. Then we have the identity in the free group $F(A)$:

$$\sigma_\gamma b \sigma_{\gamma'}^{-1} \equiv \prod_{i=0}^{n(\gamma,\gamma')-1} p_{\gamma'}(i) \cdot \left(\alpha(i)^{-1} a_{i+1} \alpha(i+1) a'^{-1}_{i+1}\right) \cdot p_{\gamma'}(i)^{-1}. \tag{6.1}$$

Each bracketed expression is a conjugate of a relator from R (by length bounds).

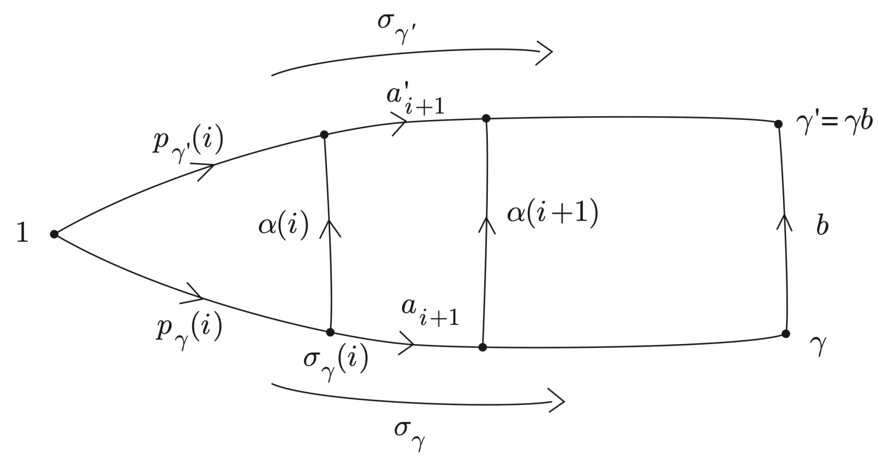


Figure 6.3: The equivalence in Equation 6.1 [BH99].

Now suppose $w = b_1 \cdots b_m \in A^{\pm 1}$ represents the identity in Γ. Define $\gamma_0 = 1$, and for $j = 1, \ldots, m$, let $\gamma_j = b_1 \cdots b_j$. Note that $\gamma_m = 1$, so $\sigma_{\gamma_m} = \sigma_{\gamma_0}$ is the empty word. We can write

$$w \equiv \prod_{j=1}^{m} \sigma_{\gamma_{j-1}} b_j \sigma_{\gamma_j}^{-1}.$$

Apply Equation 6.1 to each factor with $\gamma = \gamma_{j-1}$, $b = b_j$, and $\gamma' = \gamma_j$, expressing each as a product of conjugates of words from R.

We now estimate the number of such conjugates. The path $\gamma_j \cdot x_0$ remains within distance $(D+1)|w|/2$ of x_0, as each generator moves at most $D+1$. Hence $n(\gamma_j, \gamma_{j+1}) \leq 1 + \frac{(D+1)|w|}{2}$. Since there are $m = |w|$ such steps, the total number of conjugates used is less than $(D+1)|w|^2$. Moreover, the lengths of the conjugating prefixes $p_{\gamma_j}(i)$ are bounded by $n(\gamma_j, \gamma_{j+1})$ and hence by $(D+1)|w|$. Thus, any word w representing the identity can be written as a product of at most $(D+1)|w|^2$ conjugates of elements of R, with conjugating words of length at most $(D+1)|w|$. □

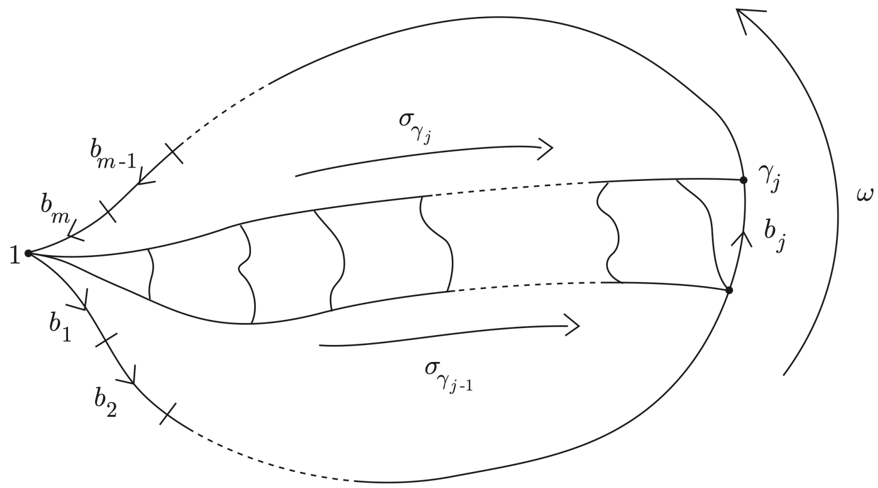


Figure 6.4: Illustration of the van Kampen diagram construction for a word $w = b_1 \cdots b_m$ applied in Theorem 6.20. Each segment between successive elements γ_{j-1} and γ_j is filled with relator faces derived from short relations in R, yielding a diagram with area on the order of $|w|^2$.

In particular, CAT(0) groups are finitely presented.

Gromov [Gro87] showed that a linear isoperimetric inequality characterizes hyperbolic groups (Theorem 3.13). One might therefore wonder whether the converse of Theorem 6.20 holds: Is every group with a quadratic isoperimetric inequality CAT(0)? However, the class of groups with quadratic Dehn functions is much broader. Steve Gersten writes in his notes [Ger98]:

> I call this a zoo, because I am unable to see any pattern in this bestiary of groups. It would be striking if there existed a reasonable characterization of groups with quadratic Dehn functions, which was more enlightening than saying that they have quadratic Dehn functions.

6.3 Subgroup Distortion and Doubles

The notion of distortion captures how the intrinsic geometry of a subgroup compares to the geometry it inherits from the ambient group. Informally, it quantifies how paths in the subgroup may be significantly longer than in the larger group, even when they represent the same group element.

Definition 6.21. Let $H \subset G$ be finitely generated groups, and suppose that d_G and d_H denote the respective word metrics on G and H, each defined using some finite generating set. The *distortion* of H in G is the function

$$\delta_H^G(n) = \max\{d_H(1,h) \mid h \in H,\, d_G(1,h) \leq n\}.$$

That is, $\delta_H^G(n)$ gives the maximum distance from the identity to an element of H, measured in the metric of H, among all elements of H that are within a ball of radius n in G.

This function is well defined up to equivalence: changing the generating sets of G or H alters $\delta_H^G(n)$ only by a multiplicative constant. Thus, the growth type of $\delta_H^G(n)$ is independent of the choice of generators.

Example 6.22. The inclusion $H \hookrightarrow G$ is a quasi-isometric embedding if and only if $\delta_H^G(n) \leq Kn$ for some constant K.

Proof. Suppose $H \hookrightarrow G$ is a quasi-isometric embedding. Then there exist constants $\lambda \geq 1$, $\varepsilon \geq 0$ such that for all $h \in H$,

$$\frac{1}{\lambda} d_H(1,h) - \varepsilon \leq d_G(1,h).$$

Thus, if $d_G(1,h) \leq n$, then $d_H(1,h) \leq \lambda n + \lambda\varepsilon$, so $\delta_H^G(n) \leq \lambda n + \lambda\varepsilon$, i.e., $\delta_H^G(n) \preceq n$.

Conversely, suppose $\delta_H^G(n) \leq Kn$ for some constant $K > 0$. Since generators of H are expressible in terms of those of G, there exists $L > 0$ such that $d_G(1,h) \leq L d_H(1,h)$ for all $h \in H$. Also, for $h \in H$, $d_H(1,h) \leq \delta_H^G(d_G(1,h)) \leq K d_G(1,h)$. Hence,

$$\frac{1}{K} d_H(1,h) \leq d_G(1,h) \leq L d_H(1,h),$$

showing that the inclusion is a quasi-isometric embedding. □

We now state our main theorem. Recall that $D_H(G)$ denotes the double of the group G along the subgroup H, that is, the amalgamated free product $G *_H G$. For further details, see Appendix A.

Theorem 6.23. *Let G be a finitely presented group with Dehn function $\delta_G(n)$, and let $H \subset G$ be a finitely presented subgroup. Let $\delta_H^G(n)$ denote the distortion of H in G, with respect to chosen word metrics. Then the Dehn function $\delta_{D_H(G)}(n)$ of the double of G along H satisfies:*

$$\max\{\delta_G(n), \delta_H^G(n)\} \preceq \delta_{D_H(G)}(n) \preceq n \cdot \delta_G(\delta_H^G(n)).$$

Proof. We follow [BH99].

First, note that since $D_H(G)$ retracts onto G, Proposition 3.10 implies

$$\delta_G(n) \preceq \delta_{D_H(G)}(n).$$

We now establish the other lower bound $\delta_H^G(n) \preceq \delta_{D_H(G)}(n)$.

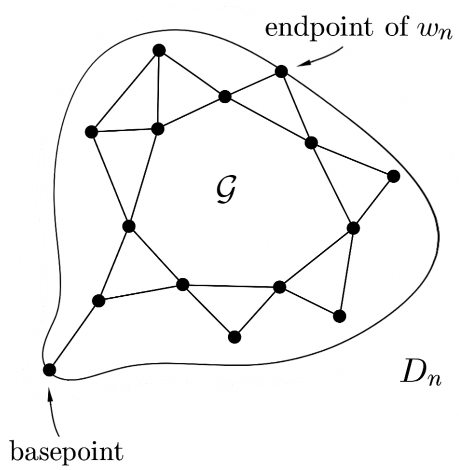
Let $\langle A \mid R \rangle$ be a finite presentation for G, where A contains a generating set B for the subgroup H. Then $D_H(G)$ admits the presentation:

$$\langle A, A' \mid R, R', h^{-1}h' = 1 \text{ for all } h \in B \rangle,$$

where A' and R' are disjoint copies of A and R, and the relations $h^{-1}h' = 1$ identify each $h \in B$ with its primed copy.

Fix a geodesic word $w_n \in H$ over A with $|w_n| \leq n$ and $d_H(1, w_n) = \delta_H^G(n)$. Let w_n' be the corresponding word over A'. Then $w_n^{-1}w_n'$ represents the identity in $D_H(G)$. Let D_n be a minimal van Kampen diagram for this word.

Each 2-cell labeled $h^{-1}h'$ is a digon; we draw an arc across each such 2-cell and label it with the corresponding $h \in B$. These arcs form a graph $\mathcal{G} \subset D_n$.



We focus on the basepoint of D_n and the terminal vertex of the path labeled w_n. Any path between these two vertices must traverse only through the digons, since w_n and w'_n lie in distinct halves of the diagram (associated to R and R', respectively), and the only relations connecting primed and unprimed generators are the digons.

We now focus on two specific vertices in D_n: the basepoint of the diagram and the terminal vertex of the subpath labeled w_n along the boundary. We claim that these two vertices lie in the same connected component of $\mathcal{G}$.

To see this, observe that the word $w_n^{-1}w'_n$ represents the identity in $D_H(G)$, so there must be a path in D_n from the basepoint to the endpoint of w_n, and then on to the endpoint of w'_n, traversing the 2-cells of the diagram. Since the path labeled by w_n lies entirely in the part of the diagram built from relators in R (i.e., the "unprimed" part), and w'_n lies in the part from R' (the "primed" part), any interior path connecting these must cross from unprimed to primed generators.

The only 2-cells in the presentation that connect unprimed and primed generators are digons labeled $h^{-1}h'$. Therefore, any path in D_n from the endpoint of w_n to the endpoint of w'_n must pass through at least one digon. So, to reach the basepoint from the endpoint of w_n, the path must pass through a sequence of these digon arcs — that is, it lies in $\mathcal{G}$. Hence, the basepoint and the endpoint of w_n are connected by a path in $\mathcal{G}$.

Let σ be a shortest such path in $\mathcal{G}$. The label along σ is a word in $B^{\pm 1}$ that represents w_n in G, so by the definition of distortion, $|\sigma| \geq \delta_H^G(n)$. Each edge of σ passes through a distinct digon, so D_n contains at least $\delta_H^G(n)$ such 2-cells.

Since $|w_n^{-1}w'_n| \leq 2n$, this shows:

$$\delta_{D_H(G)}(2n) \geq \delta_H^G(n),$$

completing the proof of the lower bound.

For the upper bound, let $w \in A \cup A'$ be a word of length n representing the identity in $D_H(G)$. Write w in alternating form:

$$w = u_1 v_1 u_2 v_2 \cdots u_\ell v_\ell,$$

where each u_i is a word over A and each v_i over A', with all but possibly u_1 and v_ℓ non-empty. The number ℓ is the alternating length.

Since $w = 1$ in $D_H(G)$, at least one of the subwords u_i or v_i must be in H (see Proposition A.8). Suppose $u_i \in H$, and let $U_i \in B^*$ be a geodesic representative in H, so that $|U_i| \leq \delta_H^G(|u_i|)$. Let $m = |u_i| + |U_i|$, and let U'_i denote the primed version of U_i.

We now relate u_i and u'_i via the identity:

$$u_i = PQP'^{-1}u'_i,$$

where: - P is a product of at most $\delta_G(m)$ conjugates of relators from R, expressing $u_i U_i^{-1}$, - Q is a product of at most $|U_i| \leq m$ relators of the form $h^{-1}h'$, expressing $U_i U_i'^{-1}$, - P' is the primed version of P, expressing $U_i' u_i'^{-1}$.

Substituting this expression for u_i into w gives:

$$w = u_1 v_1 \cdots v_{i-1}(PQP'^{-1})u_i' v_i \cdots v_\ell.$$

We now conjugate the correction term PQP'^{-1} with the suffix s of the word that follows it to push it to the end:

$$w = u_1 v_1 \cdots [u_{i-1} u_i' v_i] \cdots v_\ell (s^{-1} PQP'^{-1} s).$$

The subword in square brackets now lies entirely in A', so the alternating length has decreased by one. Repeating this process at most $N \leq n$ times, we obtain a word W over just A or just A', and a correction term Π composed of at most $2N(\delta_G(m) + m)$ conjugates of relators. Filling W in G requires at most $\delta_G(n)$ relators.

Thus, the total area needed to fill w is:

$$\delta_G(n) + 2N(\delta_G(m) + m).$$

Since $m \leq n + \delta_H^G(n)$, and $N \leq n$, we conclude:

$$\delta_{D_H(G)}(n) \preceq n \cdot \delta_G(\delta_H^G(n)). \qquad \square$$

This result has several important applications, one of which we will explore in the next section.

6.4 The Bieri Doubling Trick

We have established that hyperbolic groups satisfy a linear isoperimetric inequality, while CAT(0) groups satisfy a quadratic isoperimetric inequality. Together with the Gromov gap, this implies that the Dehn function of a CAT(0) group must either be linear or quadratic. Given this precise control over the Dehn functions of hyperbolic and CAT(0) groups, a natural question arises: do the subgroups of these groups exhibit similar growth restrictions on their Dehn functions?

In this section, we explore a construction technique that demonstrates how subgroups of CAT(0) groups can have Dehn functions that grow significantly faster than the quadratic bound of the ambient group. A systematic method for producing such examples is the *Bieri doubling trick*, originally due to Bieri [Bie76] and further developed by Baumslag, Bridson, Miller, and Short [BBMS96].

The key idea is to start with a distorted subgroup of a CAT(0) group, double the ambient group along this subgroup, and embed the resulting group into a larger CAT(0) group. We describe this three-step process below, as presented by Pallavi Dani in a guest lecture and summarized in [ber11].

Step 1: Constructing a Distorted Subgroup

The first step is to construct a CAT(0) group G containing a distorted subgroup H, such that the quotient $G/H \cong \mathbb{Z}$, and the distortion function satisfies

$$\delta_H^G(x) \succeq x^2.$$

Example 6.24. Let $H = F(a, b)$ be the free group on two generators, and define an automorphism $\phi\colon H \to H$ by

$$\phi(a) = a, \qquad \phi(b) = ab.$$

Consider the ascending HNN extension (see Appendix A)

$$G = \langle a, b, t \mid tat^{-1} = a,\ tbt^{-1} = ab \rangle,$$

which corresponds to adjoining a stable letter t implementing the automorphism ϕ. Since ϕ is an automorphism, G is isomorphic to the semidirect product $F_2 \rtimes_\phi \mathbb{Z}$, with $\mathbb{Z}$ generated by t.

We claim that the subgroup $H = F(a, b)$ is quadratically distorted in G. This can be seen by examining the family of elements $t^n b^n t^{-n} \in G$. Conjugating b^n by t^n applies ϕ^n to b^n, resulting in:

$$t^n b^n t^{-n} = \phi^n(b^n) = \phi^{-1}(ab \cdots ab) = \cdots = aaa \cdots aabaaa \cdots aab \cdots ,$$

where the number of a's between each b grows with n, with the total length growing quadratically in n. Therefore, $\delta_H^G(n) \succeq n^2$.

To verify that G is a CAT(0) group, we change variables by setting $\alpha := ta^{-1}$ and $\beta := bt$ to obtain:

$$G = \langle \alpha, \beta, t \mid t\alpha t^{-1} = \alpha,\ \beta t \beta^{-1} = \alpha \rangle.$$

This gives rise to a presentation complex with one vertex, three oriented edges labeled α, β, and t, and two square 2-cells corresponding to the relators $t\alpha t^{-1}\alpha^{-1}$ and $\beta t \beta^{-1}\alpha^{-1}$.

Gromov's link condition [Gro87] provides a criterion for determining whether a square complex is non-positively curved, and hence whether its universal cover is CAT(0). The *link* of a vertex is a graph whose vertices correspond to the edge directions at that vertex and whose edges correspond to corners of 2-cells. The link condition asserts that the link of each vertex must contain no cycles of length less than four.

In this case, the link has six vertices corresponding to the directions $\alpha^{\pm 1}$, $\beta^{\pm 1}$, and $t^{\pm 1}$. Each 2-cell contributes a 4-cycle to the link: one from the relator $t\alpha t^{-1}\alpha^{-1}$, and another from $\beta t\beta^{-1}\alpha^{-1}$. Since there are no cycles of length less than four, the link condition is satisfied.

Consequently, the complex is non-positively curved, and its universal cover is a CAT(0) space. As the fundamental group of the complex, G acts properly and cocompactly by deck transformations on its universal cover. Therefore, G is a CAT(0) group.

Step 2: Doubling Along the Distorted Subgroup

Next, we form the double of G along H, denoted $D_H(G) := G *_H G$, which is constructed by taking two copies of G and identifying their shared subgroup H (see Appendix A). Then, we bound the Dehn function of the double. By Theorem 6.23, the Dehn function of the double $D_H(G)$ is asymptotically bounded below by both the Dehn function of G and the distortion of H in G.

Example 6.25. In the above example, the subgroup $H = \langle a, b\rangle$ is quadratically distorted in G with distortion function $\delta_H^G(n) \succeq n^2$. This already implies a lower bound $\delta_{D_H(G)}(n) \succeq n^2$ for the Dehn function of the double $D_H(G) = G *_H G$.

However, we can obtain a stronger bound by analyzing van Kampen diagrams. The double $D_H(G)$ admits the presentation

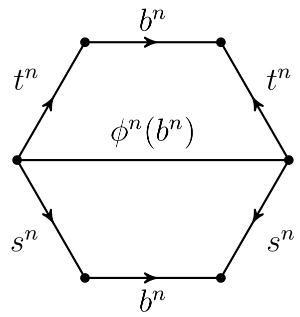
$$\langle a, b, t, s \mid tat^{-1} = sas^{-1} = a, \quad tbt^{-1} = sbs^{-1} = ab\rangle,$$

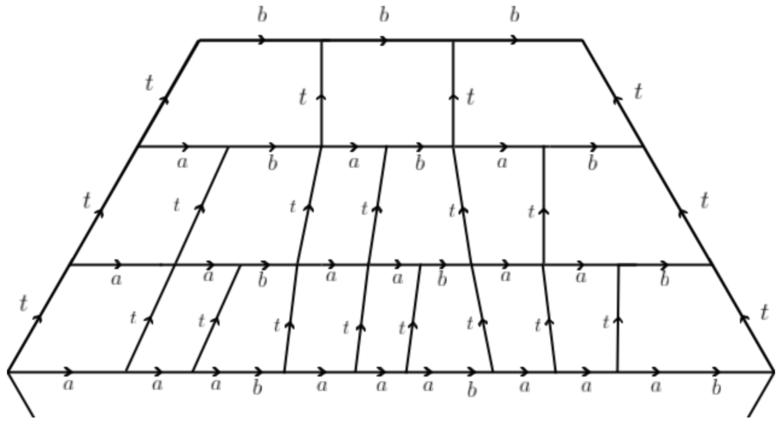
where the stable letters t and s act on H via the same automorphism $\phi\colon a \mapsto a,\ b \mapsto ab$.

In the Cayley complex $\widetilde{K}_{D_H(G)}$, consider the word

$$w_n = t^n b^n t^{-n} s^n b^{-n} s^{-n},$$

which has length linear in n. This loop bounds a van Kampen diagram shaped like a hexagon: two sides labeled b^n, two sides labeled s^n, two sides labeled t^n, and a horizontal edge labeled $\phi^n(b^n)$, which equals both $t^n b^n t^{-n}$ and $s^n b^{-n} s^{-n}$ by the relations in the group.





As we have seen, in $H = F(a, b)$, the word $\phi^n(b^n)$ has length at least n^2, since each application of ϕ replaces b with ab. An example filling for $n = 3$ is shown on the right. We see that the final layer contains n^2 faces and, more generally, the ith layer for $1 \le i \le n$ contains $i \cdot n$ faces. So, the total area of the diagram is $n + 2n + \cdots + n^2 \sim n^3$.

Since $\widetilde{K}_{D_H(G)}$ is 2-dimensional and aspherical, every loop in it has a unique minimal-area filling up to homotopy. Thus, these embedded disks with area $\sim n^3$ are actual realizations of the Dehn function, and we conclude that

$$\delta_{D_H(G)}(n) \succeq n^3.$$

Step 3: Embedding into a CAT(0) Group

In the final step of the construction, we embed the double $D_H(G) = G *_H G$ in a larger CAT(0) group. Specifically, we show that $D_H(G)$ embeds in the direct product $G \times F_2$, which is a CAT(0) group since both G and the free group F_2 are CAT(0) spaces.

Example 6.26. Let $F(u, v)$ denote the free group on two generators. Consider the direct product

$$G \times F(u, v) = (F(a, b) \rtimes \langle t \rangle) \times F(u, v).$$

Define the elements $t_1 = tu$ and $t_2 = tv$, where u and v are the generators of $F(u, v)$. Since the second factor commutes with all elements of G, conjugation by t_1 and t_2 acts on a and b in the same way as conjugation by t:

$$t_1 a t_1^{-1} = tat^{-1}, \quad t_2 a t_2^{-1} = tat^{-1}, \quad t_1 b t_1^{-1} = tbt^{-1}, \quad t_2 b t_2^{-1} = tbt^{-1}.$$

Thus, the subgroups $\langle a, b, t_1\rangle$ and $\langle a, b, t_2\rangle$ are both isomorphic to G, and their intersection is precisely $H = \langle a, b\rangle$. This implies that the subgroup $\langle a, b, t_1, t_2\rangle$ is isomorphic to the double $D_H(G) = G *_H G$.

It remains to check that no further relations are imposed. To do this, we use the fact that a homomorphism $G *_H G \to K \times L$ is injective if the kernels of the projections $\phi\colon G *_H G \to K$ and $\psi\colon G *_H G \to L$ intersect trivially [BBMS96].

Define two homomorphisms on $\langle a, b, t_1, t_2\rangle \subset G \times F(u, v)$ as follows:

- ϕ sends $a \mapsto a$, $b \mapsto b$, $t_1 \mapsto t$, and $t_2 \mapsto t$, collapsing the stable letters and identifying the two copies of G.
- ψ sends $a, b \mapsto 1$, and $t_1 \mapsto tu$, $t_2 \mapsto tv$, projecting to the free group generated by tu and tv, which is isomorphic to $G/H * G/H \cong F_2$.

Since ϕ kills the $F(u, v)$ factor and ψ kills H, their kernels intersect trivially, ensuring that the subgroup $\langle a, b, t_1, t_2\rangle$ embeds into $G \times F_2$ as a copy of $D_H(G)$. That is,

$$D_H(G) = \langle H, (tu), (tv)\rangle \hookrightarrow \langle H, t\rangle \times \langle u, v\rangle = G \times F_2.$$

This construction shows that subgroups of CAT(0) groups can exhibit super-quadratic Dehn functions: in this case, a cubic lower bound on the Dehn function.

Example 6.27. By varying the pair (G, H) in Step 1, we can construct further examples of subgroups of CAT(0) groups with even faster-growing Dehn functions.

- Brady [BRS07] constructed a family of examples $F_k \subset G_k$, where each G_k is a CAT(0) group of the form $F_k \rtimes \mathbb{Z}$, and the subgroup F_k is polynomially distorted in G_k with distortion of degree k. The example we discussed earlier corresponds to the case $k = 2$; the general construction employs Morse theory on CAT(0) cube complexes. Doubling along F_k, one obtains $D_{F_k}(G_k) = G_k *_{F_k} G_k$, which embeds in the CAT(0) group $G_k \times F_2$. By Theorem 6.23, this implies that for each integer $k \geq 3$, the function n^k appears as the Dehn function of a subgroup of a CAT(0) group.
- Another family arises from closed hyperbolic 3-manifolds that fiber over the circle. Let M^3 be such a manifold, so that its fundamental group fits into a short exact sequence

$$1 \to \pi_1(\Sigma) \to \pi_1(M^3) \to \mathbb{Z} \to 1,$$

where Σ is a closed hyperbolic surface. The group $\pi_1(M^3)$ is CAT(0), and $\pi_1(\Sigma)$ is embedded as a normal subgroup of infinite index.

Bridson and Haefliger [BH99] proved that if G is a hyperbolic group and $H \subset G$ is a finitely generated, infinite normal subgroup such that G/H is also infinite, then the distortion of H in G is exponential: there exists $k > 1$ such that $\delta_H^G(n) \simeq k^n$. Applying this result to the inclusion $\pi_1(\Sigma) \subset \pi_1(M^3)$, we find that the surface subgroup is exponentially distorted in the 3-manifold group.

It follows from Theorem 6.23 that doubling $\pi_1(M^3)$ along $\pi_1(\Sigma)$ produces a group with an exponential Dehn function:

$$\delta_{D_{\pi_1(\Sigma)}(\pi_1(M^3))}(n) \simeq e^n.$$

This construction provides a geometric example of a subgroup of a CAT(0) group with exponential Dehn function, derived from the distortion of a surface subgroup in a hyperbolic 3-manifold group.

Brady and Forester developed new examples of subgroups of CAT(0) groups using a "snowflake" construction. Their approach involves building special free-by-cyclic groups and assembling them via a graph of spaces to form a CAT(0) group that contains a distorted "snowflake subgroup." This leads to the following result:

Theorem 6.28 (Brady–Forester [BF17]). *The set*

$$\{\alpha \in [2, \infty) \mid n^\alpha \textit{ is the Dehn function of a subgroup of a CAT(0) group}\}$$

is dense in $[2, \infty)$.

Their construction shows that CAT(0) groups can have subgroups whose Dehn functions grow like n^α for a dense set of exponents α. It remains an open question whether Dehn functions that grow faster than e^n can arise from subgroups of CAT(0) groups.

Appendix A

Group-Theoretic Constructions

A.1 Free Products

The free product of groups is a fundamental construction in combinatorial group theory that provides a way to combine groups while preserving their individual structures as much as possible.

Definition A.1. Let G and H be groups. A *word* in G and H is a sequence of elements taken from either G or H: $s_1, \ldots, s_n$, where each s_i is an element of G or H. Words can be simplified using the following rules:

- Remove occurrences of the identity element from either G or H.
- Replace adjacent elements from the same group with their product (i.e., if $g_1, g_2 \in G$, then replace $g_1 g_2$ with their product in G, and similarly for $h_1, h_2 \in H$).

A word that cannot be simplified further is called a *reduced word.* The *free product* of G and H, denoted $G * H$, is the group whose elements are reduced words in G and H, with multiplication given by concatenation followed by reduction.

Example A.2. Suppose $G = \langle x \rangle$ and $H = \langle y \rangle$ are both infinite cyclic groups. In this case, every element of $G * H$ is a sequence of alternating powers of x and y. This shows that $G * H$ is isomorphic to the free group generated by x and y.

Proposition A.3 (Normal Form Theorem)**.** *Every element of $G * H$ can be written uniquely as a reduced word of the form $g_1 h_1 g_2 h_2 \cdots g_k h_k$, where each g_i is a non-identity element of G and each h_i is a non-identity element of H, except possibly the first or last term.*

The free product can also be described using group presentations:

Proposition A.4. *Suppose G and H are given by presentations $G = \langle S_G \mid R_G \rangle$ and $H = \langle S_H \mid R_H \rangle$. Then the free product has the presentation $G * H = \langle S_G \cup S_H \mid R_G \cup R_H \rangle$. That is, $G * H$ is generated by the union of the generating sets of G and H, with relations coming from both G and H.*

A.2 Amalgamated Free Products

The amalgamated free product generalizes the free product by identifying a common subgroup of two groups. This allows one to glue groups together along shared structure, rather than freely.

Definition A.5. Let G and H be groups, and let A be a group with injective homomorphisms $\phi_G : A \hookrightarrow G$ and $\phi_H : A \hookrightarrow H$. The *amalgamated free product* of G and H over A, denoted $G *_A H$, is the group obtained from the free product $G * H$ by imposing the additional relations $\phi_G(a) = \phi_H(a)$ for all $a \in A$. That is,

$$G *_A H = \langle G, H \mid \phi_G(a) = \phi_H(a) \text{ for all } a \in A\}.$$

Example A.6. Let $G = \langle x \rangle$ and $H = \langle y \rangle$ be infinite cyclic groups, and let $A = \mathbb{Z}$, with $\phi_G(1) = x^2$ and $\phi_H(1) = y^3$. Then $G *_A H$ is given by

$$\langle x, y \mid x^2 = y^3 . \rangle$$

This is the fundamental group of the space obtained by gluing two circles via a map of degree 2 on one and degree 3 on the other.

The double of a group is a special case of the amalgamated free product in which a group is glued to an isomorphic copy of itself along a shared subgroup.

Definition A.7. Let G be a group, and let $A \subset G$ be a subgroup. The *double of G along A*, denoted $D_A(G)$, is the amalgamated free product

$D_A(G) = G *_A G'$, where G' is an isomorphic copy of G, and the amalgamation identifies $A \subset G$ with the corresponding subgroup $A' \subset G'$ under the isomorphism.

In other words, $D_A(G)$ is the group generated by two copies of G, with the relation that corresponding elements of A and A' are identified.

Proposition A.8 (Normal Form Theorem for Amalgamated Free Products)**.** *Let $G *_A H$ be an amalgamated free product, with A embedded in both G and H. Then every element of $G *_A H$ can be represented by a reduced word*

$$s_1 s_2 \cdots s_n,$$

where each s_i lies in $G \setminus A$ or $H \setminus A$, adjacent terms come from alternating factors, and no term is the identity.

Two such reduced words represent the same element if and only if they are related by a finite sequence of applications of the relations from G, H, and the identifications of elements in A.

A.3 Semidirect Products

A related construction is the semidirect product, which generalizes the direct product by allowing one group to act on the other by automorphisms.

Definition A.9. Let N and H be groups, and let $\phi : H \to \mathrm{Aut}(N)$ be a homomorphism. The *semidirect product* of N and H with respect to ϕ, denoted $N \rtimes_\phi H$, is the set $N \times H$ with the group operation

$$(n_1, h_1)(n_2, h_2) = (n_1 \cdot \phi(h_1)(n_2), h_1 h_2).$$

This defines a group where N is a normal subgroup, H is isomorphic to a subgroup, and the conjugation action of H on N is given by ϕ.

Example A.10. Consider the groups $N = \mathbb{Z}$ and $H = \mathbb{Z}/2\mathbb{Z}$, and let ϕ be the homomorphism that sends $1 \in H$ to the automorphism of $\mathbb{Z}$ that negates elements, i.e., $\phi(1)(n) = -n$. The semidirect product $\mathbb{Z} \rtimes_\phi \mathbb{Z}/2\mathbb{Z}$ is then the group consisting of pairs (n, h) with $n \in \mathbb{Z}$ and $h \in \mathbb{Z}/2\mathbb{Z}$, where the group operation is defined as

$$(n_1, h_1)(n_2, h_2) = (n_1 + \phi(h_1)(n_2), h_1 + h_2).$$

For example, we have

$$(2, 1)(3, 0) = (2 + \phi(1)(3), 1 + 0) = (-1, 1).$$

Thus, this semidirect product provides a non-trivial interaction between the two groups.

A.4 HNN Extensions

The HNN extension is a construction in combinatorial group theory that allows one to enlarge a group by adding a new element that conjugates one subgroup isomorphically onto another.

Definition A.11. Let G be a group, and let A and B be subgroups of G. Suppose $\phi : A \to B$ is an isomorphism. The *HNN extension* of G with respect to ϕ is the group

$$G_\phi = \langle G, t \mid tat^{-1} = \phi(a) \text{ for all } a \in A \rangle.$$

This group is obtained by adjoining a new generator t to G, and imposing the relations that conjugation by t sends each element of A to its image under ϕ. The new generator t is called the *stable letter*.

The original group G embeds into G_ϕ, and both A and B are isomorphic to their images in the larger group.

Example A.12. Let $G = \langle a, b \rangle$ be the free group on two generators, and define an automorphism ϕ of G by $\phi(a) = a$ and $\phi(b) = ab$. Then the HNN extension

$$G_\phi = \langle a, b, t \mid tat^{-1} = a,\ tbt^{-1} = ab \rangle$$

is an ascending HNN extension, since ϕ maps G injectively into itself. This group is isomorphic to the semidirect product $F_2 \rtimes_\phi \mathbb{Z}$, where the generator t corresponds to the generator of $\mathbb{Z}$ acting on F_2 via ϕ.

Proposition A.13 (Normal Form Theorem for HNN Extensions)**.** *Let $G_\phi = \langle G, t \mid tat^{-1} = \phi(a) \text{ for all } a \in A \rangle$ be the HNN extension of G. Then every element of G_ϕ can be represented uniquely (up to relations in G) by a reduced word of the form*

$$g_0 t^{\epsilon_1} g_1 t^{\epsilon_2} \cdots t^{\epsilon_n} g_n,$$

where $g_i \in G$, each $\epsilon_i \in \{\pm 1\}$, and no subword tgt^{-1} (resp. $t^{-1}gt$) appears with $g \in A$ (resp. $g \in B$).

That is, the word is reduced if no cancellation of the defining relations is possible.

These constructions play a key role in many results concerning solvability of the word problem, as referenced in Proposition 3.3 and in Chapter 6.

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