Non-Positive Curvature and the Word Problem in Group Theory

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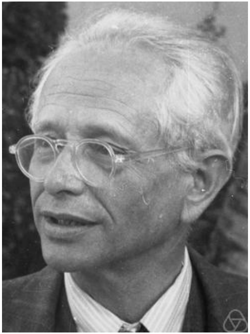
Dehn's Decision Problems

In 1911, German mathematician Max Dehn posed the following problem:

The word problem: An element of the group is given as a product of generators. One is required to give a method whereby it may be decided in a finite number of steps whether this element is the identity or not.

(John Stillwell's translation [7])



Max Dehn, circa 1945 [9]

An Answer to the Word Problem

Theorem (Novikov–Boone)

There exists a finitely presented group for which no algorithm exists to solve the word problem.

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There exists a finitely presented group for which no algorithm exists to solve the word problem.

Not to be deterred, mathematicians still set out on expanding their repository of groups for which the word problem was solvable:

- Automatic groups
- Coxeter groups (Kieran's talk)
- Braid groups
- Polycyclic groups
- Non-positively curved groups (stay tuned)

Bridson's Universe of (Finitely Generated) Groups

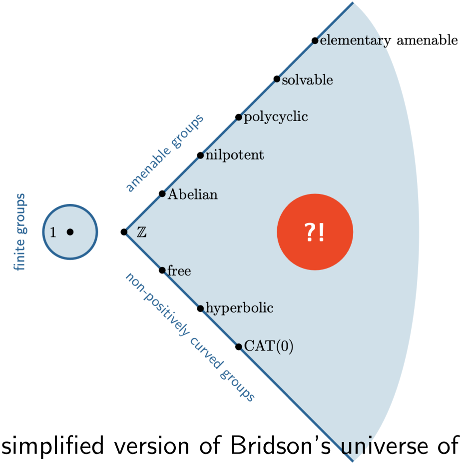

Figure: A simplified version of Bridson's universe of groups [10]

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Free Groups

Write $\mathcal{F}(\mathcal{A})$ to denote the free group on an alphabet $\mathcal{A}$.
A *word* "in $\mathcal{A}$" is a finite sequence of letters in $\mathcal{A}^{\pm 1}$.

The elements of $\mathcal{F}(\mathcal{A})$ are equivalence classes of words under *free reductions*, i.e, deletions of subwords aa^{-1}. The group operation is concatenation.

Example

In $\mathcal{F}(a, b)$, we have: $aa^{-1}b^2b^{-1}a = ba$.

Presentations

A *presentation* for a group Γ is a set of *generators* $\mathcal{A}$ and a set of *relations* $\mathcal{R} \subset \mathcal{F}(\mathcal{A})$ and a surjective homomorphism $\pi\colon \mathcal{F}(\mathcal{A}) \to \Gamma$ whose kernel is the *normal closure* of $\mathcal{R}$.

We usually suppress the mention of π and write $\Gamma = \langle \mathcal{A} \mid \mathcal{R} \rangle$.

Example

$\mathbb{Z} \times \mathbb{Z} = \langle a, b \mid aba^{-1}b^{-1} \rangle$.

First Approaches

We say that a finitely generated group Γ has *solvable* word problem if there exists an algorithm to decide which words in the generators represent the identity, and which do not. The word problem is solvable in:

1. Free groups
2. Finite groups
3. $G \times H$ and $G * H$, whenever G and H both have solvable word problem
4. Finitely generated abelian groups
5. Finitely generated subgroups of groups with solvable word problem

The General Case

Given a relation $r = u_1 u_2 u_3$, I can "apply the relation" to replace any occurrence of u_2 with $u_1^{-1} u_3^{-1}$.

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$$l\, u_2\, r \equiv (x_1 r x_1^{-1}) \;\; l\, (u_1^{-1} u_3^{-1})\, r,$$

where $x_1 := l u_1^{-1}$. Thus applying a relation allows us to write $w = (x_1 r x_1^{-1}) w'$.

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where $x_1 := l u_1^{-1}$. Thus applying a relation allows us to write $w = (x_1 r x_1^{-1}) w'$. Applying another relation allows us to write $w = (x_1 r_1 x_1^{-1})(x_2 r_2 x_2^{-1}) w''$, and so on...

The General Case

If a word w can be reduced to the empty word by applying N relations, then we have

$$w \equiv \prod_{i=1}^{N} x_i r_i x_i^{-1},$$

where $r_i \in \mathcal{R}^{\pm 1}$ and $x_i \in \mathcal{F}(\mathcal{A})$.

What is a Dehn Function?

Definition

Let $\mathcal{P}$ denote a finite presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ defining a group Γ. We say that a word w in $\mathcal{A}^{\pm 1}$ is *null-homotopic* if $w =_\Gamma 1$, or equivalently $w \in \langle\langle \mathcal{R} \rangle\rangle$. If w is null-homotopic, we define the *algebraic area* of w to be

$$\mathrm{Area}(w) := \min\{N \mid w \equiv \prod_{i=1}^{N} x_i r_i x_i^{-1} \text{ with } x_i \in \mathcal{F}(\mathcal{A}), r_i \in \mathcal{R}^{\pm 1}\}.$$

What is a Dehn Function?

Definition

$$\operatorname{Area}(w) := \min\{N \mid w \equiv \prod_{i=1}^{N} x_i r_i x_i^{-1} \text{ with } x_i \in \mathcal{F}(\mathcal{A}), r_i \in \mathcal{R}^{\pm 1}\}.$$

The *Dehn function* of the presentation $\mathcal{P}$ is the function $\delta_P \colon \mathbb{N} \to \mathbb{N}$ defined by

$$\delta_P(n) := \max\{\operatorname{Area}(w) \mid w =_\Gamma 1, |w| \leq n\},$$

where $|w|$ denotes the length of the word w.

How Dehn Functions Help

Theorem

For a finitely presented group, the following are equivalent:

- *The word problem is solvable,*
- *The Dehn function is recursive,*
- *The Dehn function is bounded above by a recursive function.*

Dehn Functions

Example

$\langle a \mid a^m \rangle$ has Dehn function $\delta(n) = \lfloor n/m \rfloor$. Any null-homotpic word is of the form a^{rm}, $r \in \mathbb{Z}$; conversely, a^{rm} cannot be reduced to the identity with fewer than $|r|$ applications of relations.

Example

$\langle a, b \mid aba^{-1}b^{-1} \rangle$ has $(n-3)^2 \leq 16\delta(n) \leq n^2$, so $\delta(n) \simeq n^2$.

A Problem

Then Dehn function of $\langle a \mid \emptyset \rangle$ is $\delta(n) = 0$ and the Dehn function of the presentation $\langle a, b \mid b \rangle$ is $\delta(n) = n$.

Equivalence of Dehn Functions

Definition

Define $\preceq$ to be the relation on functions $[0, \infty) \to [0, \infty)$ defined by $f \preceq g$ if there exists a constant $C > 0$ such that

$$f(x) \leq Cg(Cx + C) + Cx + C$$

for all $x \geq 0$. If $f \preceq g$ and $g \preceq f$, then f and g are said to be $\simeq$*-equivalent*, denoted $f \simeq g$.

Example

$n^2 \simeq 15n^2 + 10n$, and $n^a \simeq n^b \Rightarrow a = b$, and $\forall a, b > 1, a^n \simeq b^n$.

Dehn Functions are Invariants

Proposition

If two finite presentations define isomorphic groups, then the Dehn functions of those presentations are $\simeq$-equivalent.

Therefore, it makes sense to talk about "the Dehn function of a group," albeit up to $\simeq$-equivalence.

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The Isoperimetric Spectrum

A real number α is said to be an *isoperimetric* exponent if there exists a finite presentation P with Dehn function $\delta_P(n) \simeq x^a$. The collection of all isoperimetric exponents is called the *isoperimetric spectrum* and is denoted by $\mathbb{P}$.

We know $\mathbb{P}$ is a countable subset of $[1, \infty)$. We saw that $1 \in \mathbb{P}$ and $2 \in \mathbb{P}$. What more can we know?

What We Know

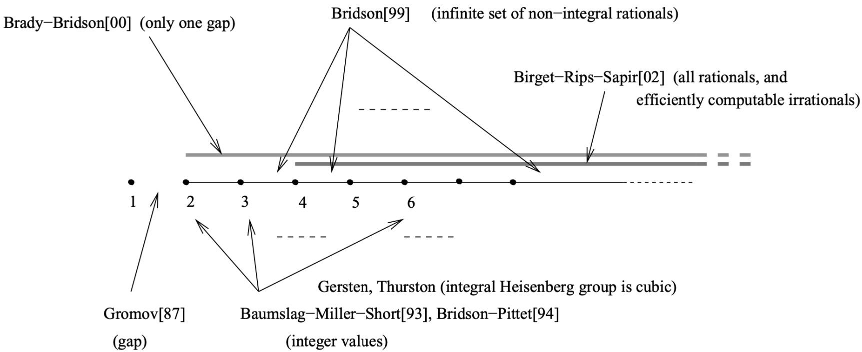

Figure: History of discoveries about the isoperimetric spectrum [2]

Larger Dehn Functions

$\mathcal{P}$	$\delta_{\mathcal{P}}(n)$
$\langle a, b \mid ab = ba^2 \rangle$	$\exp(n)$
$\langle a, b, c \mid ab = ba^2, bc = cb^2 \rangle$	$\exp(\exp(\exp(n)))$
$\langle a, b, c, d \mid ab = ba^2, bc = cb^2, cd = dc^2 \rangle$	$\exp(\exp(n))$
$\vdots$	$\vdots$
$\langle a, b \mid a(b^{-1}ab) = (b^{-1}ab)a^2 \rangle$	$\underbrace{\exp(\cdots(\exp}_{\log n}(n))\cdots)$

But groups with large Dehn functions may have efficient solutions to their word problems. For example, $\langle a, b \mid ab = ba^2 \rangle$ embeds in $\mathrm{Homeo}(\mathbb{R})$ as $a(x) = x - 1,\ b(x) = x/2$.

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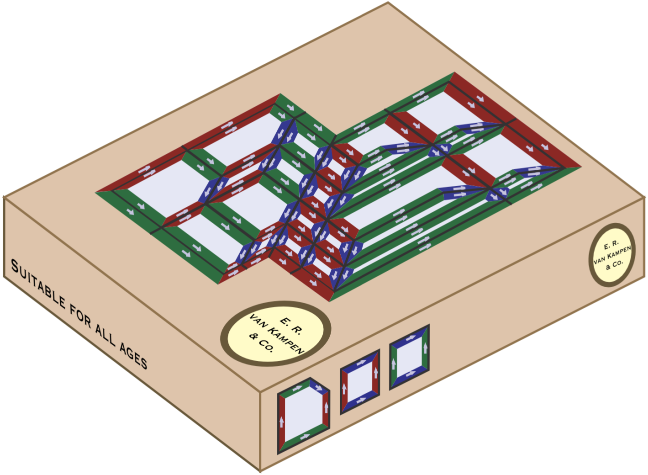
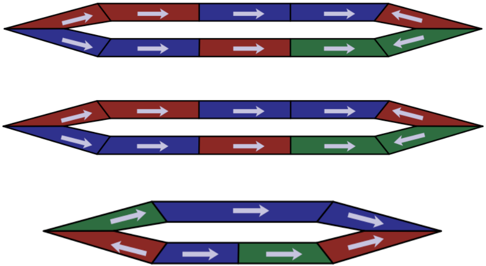
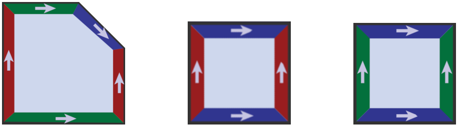
A Puzzle Kit



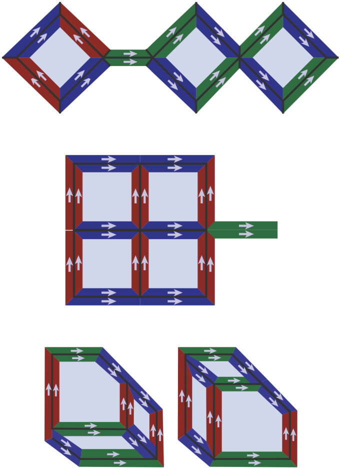
Some Puzzles



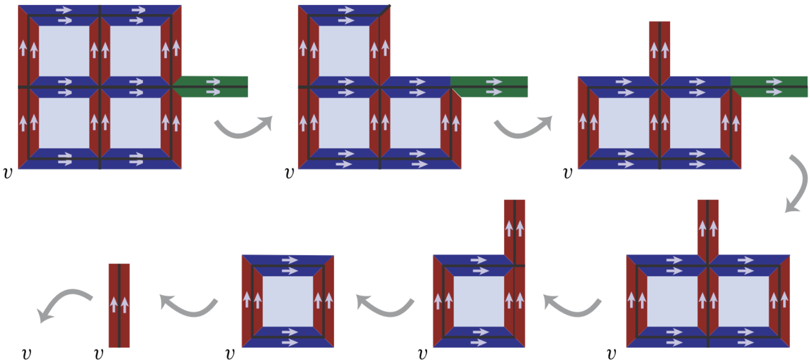
You have:



Solutions



Disassembly



What is Happening?

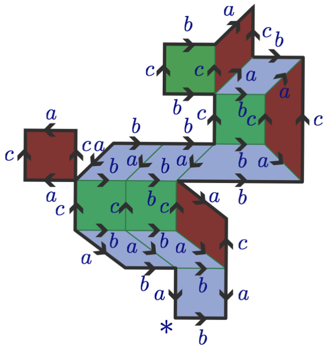
If we assign labels to colors and track what happens to the boundary, we see

$$
\begin{aligned}
c^2baa^{-1}bc^{-2}b^{-2} &\rightsquigarrow c^2baa^{-1}c^{-1}bc^{-1}b^{-2} \rightsquigarrow c^2baa^{-1}c^{-1}bb^{-1}c^{-1}b^{-1} \\
&\rightsquigarrow c^2bc^{-1}bb^{-1}c^{-1}b^{-1} \rightsquigarrow cb^2b^{-1}c^{-1}b^{-1} \rightsquigarrow cbc^{-1}b^{-1} \\
&\rightsquigarrow bb^{-1} \rightsquigarrow \text{empty word.}
\end{aligned}
$$

View in this light, completed puzzle is seen as a *van Kampen diagram* for its boundary label w.

Extending the Analogy

Let $\mathbb{Z}^3 = \langle a, b, c \mid aba^{-1}b^{-1}, bcb^{-1}b^{-1}, aca^{-1}c^{-1} \rangle$.



A van Kampen diagram for
$ba^{-1}ca^{-1}bcb^{-1}ca^{-1}b^{-1}c^{-1}bc^{-1}b^{-2}acac^{-1}a^{-1}c^{-1}aba$

Van Kampen's Lemma

A van Kampen diagram for a word w is a finite planar contractible 2-complex D with edges directed and labelled so that around each 2-cell one reads a defining relation and around ∂D one reads w.

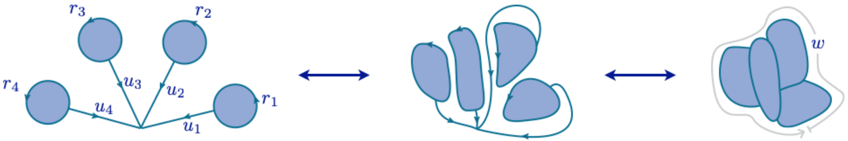
Lemma (Van Kampen)

For a group Γ with a finite presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ and a word w, the following are equivalent:

1. $w = 1$ *in* Γ,
2. w *admits a van Kampen diagram*

Moreover, $\text{Area}(w)$ is the minimum number of faces in a van Kampen diagram for w.

Constructing Van Kampen Diagrams



$$w = u_1 r_1 u_1^{-1} \cdot u_2 r_2 u_2^{-1} \cdot u_3 r_3 u_3^{-1} \cdot u_4 r_4 u_4^{-1}$$

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Dehn's Algorithm

Suppose a group Γ with a finite generating set $\mathcal{A}$ has the following property: there exists a finite list of words $u_1, \ldots, u_n$ such that every word that represents the identity in Γ contains at least one of the u_i, and there exists a corresponding list of words $v_1, \ldots, v_n$ with $|v_i| < |u_i|$ such that $u_i = v_i$ in Γ. Then we can solve the word problem extremely efficiently, in at most $|w|$ steps, hence the group has a linear Dehn function.

Definition

A finite presentation $\langle \mathcal{A} \mid \mathcal{R} \rangle$ is called a *Dehn presentation* if $\mathcal{R} = \{u_1 v_1^{-1}, \ldots, u_n v_n^{-1}\}$, where $u_1, \ldots, u_n$ and $v_1 \ldots, v_n$ are as above.

Small-Cancellation Theory

Consider a van Kampen diagram D over a finite presentation $\mathcal{P}$. If the boundaries of two compact faces have a common arc of intersection, then the corresponding relators (read cyclically) contain a common subword. If these common subwords are always short, then a face that does not meet the boundary meets many other faces.

One proves that if every subword common to at least two relators comprises at most one sixth of any relator, then a group has a Dehn presentation. This is called the $C'(1/6)$ small-cancellation condition. Small-cancellation theory is predicated on the Euler formula for planar graphs: $V + F - E = 2$, since van Kampen diagrams are planar 2-complexes.

Hyperbolicity

Gromov came up with a notion of negative curvature that focuses on the global structure of a space while ignoring its local structure.

Hyperbolicity

Definition

A geodesic triangle $\triangle$ in a metric space, is said to be δ-*slim* if each side of $\triangle$ lies in the δ-neighbourhood of the union of the other two sides. A geodesic metric space X is said to be *hyperbolic* (in the sense of Gromov) if every geodesic triangle in X is δ-slim.

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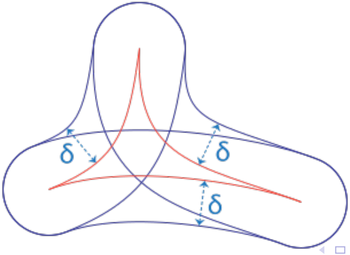


Hyperbolic Groups

Definition

A finitely generated group Γ is said to be *hyperbolic* if its Cayley graph is δ-hyperbolic for some $\delta > 0$.

Recall that the Cayley graph $\mathcal{C}_{\mathcal{A}}(\Gamma)$ of a finitely generated group Γ with respect to a generating set $\mathcal{A}$ is the graph whose vertex set is Γ and has an edge joining γ to γa for every $\gamma \in \Gamma$ and $a \in \mathcal{A}$. Words that represent the identity $\leftrightarrow$ edge loops in $\mathcal{C}_{\mathcal{A}}(\Gamma)$.

Giving the edges length one, the Cayley graph is a geodesic metric space by defining the distance between to points to be the length of the shortest path joining them.

Quasi-Isometry

The Cayley graph of a group depends on the chosen generating set.

Definition

Let (X_1, d_1) and (X_2, d_2) be metric spaces. A (not necesssarily continuous) map $f : X_1 \to X_2$ is called a *quasi-isometric embedding* if there exist constants $\lambda \geq 1$, $\varepsilon \geq 0$ such that

$$\frac{1}{\lambda} d_1(x, y) - \varepsilon \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + \varepsilon$$

for all $x, y \in X_1$. If, in addition, there exists a constant $C \geq 0$ such that every point of X_2 lies in the C-neighbourhood of the image of f, then f is called a *quasi-isometry*. When such a map exists, X_1 and X_2 are said to be *quasi-isometric*.

A Quasi-Isometry

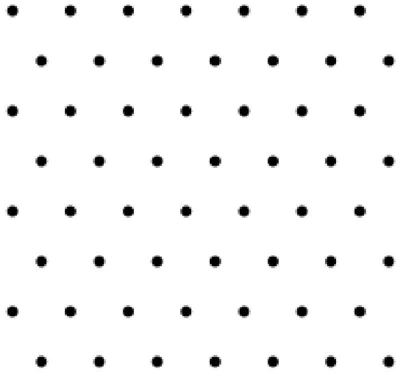


Figure: The lattice is quasi-isometric to the plane. (Try squinting!)

A Quasi-Isometry Invariant

Proposition

1. *If X' is a geodesic space that is quasi-isometric to a δ-hyperbolic space X, then X' is δ'-hyperbolic for some $\delta' > 0$.*
2. *If the Cayley graph of a group Γ with respect to one finite generating set is hyperbolic, then the Cayley graph of Γ with respect to any other generating set is hyperbolic.*

Hyperbolic Groups

Theorem

For a finitely presented group Γ*, the following statements are equivalent:*

1. Γ *is a hyperbolic group.*
2. Γ *has a finite Dehn presentation*
3. Γ *has linear Dehn function*

Ol'Shanksii proved that "almost every" group is hyperbolic [12].

Hyperbolic Groups

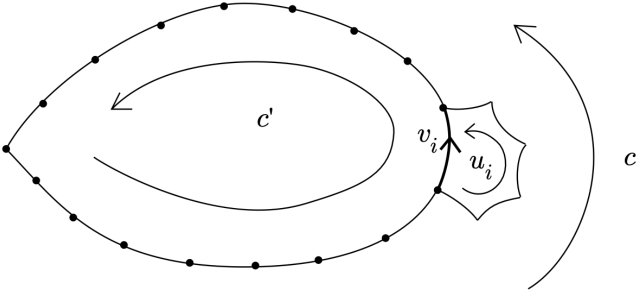


Figure: Applying a relation in Dehn's algorithm

Can We Generalize?

In hyperbolic spaces, we know that triangles in X are uniformly δ-slim, for some $\delta > 0$. What if we instead say that triangles must be "slimmer" than triangles with the same side lengths in some other (hyperbolic, Euclidean, spherical) space? We define *model spaces* M^2_κ:

$$M^2_{-1} : \mathbb{H}^2 \qquad M^2_0 : \mathbb{R}^2 \qquad M^2_1 : \mathbb{S}^2$$

The CAT(κ) Inequality

If $\triangle$ is a geodesic triangle in X for three points $x, y, z \in X$, a *comparison triangle* for $\triangle$ in M_κ^2 is a geodesic triangle $\overline{\triangle}$ in M_κ^2 with vertices $\bar{x}, \bar{y}, \bar{z}$, such that $d(x, y) = d(\bar{x}, \bar{y})$, $d(y, z) = d(\bar{y}, \bar{z})$, and $d(z, x) = d(\bar{z}, \bar{x})$.

For a point $p \in [x, y]$, a point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a *comparison point* in $\overline{\triangle}$ for p if $d(x, p) = d(\bar{x}, \bar{p})$. Comparison points for $[y, z]$ and $[z, x]$ are defined similarly.

A geodesic space X is said to satisfy the CAT(κ) inequality, or *is a* CAT(κ) *space* if, for all geodesic triangles $\triangle$ in X,

$$d(p, q) \leq d(\bar{p}, \bar{q})$$

for all comparison points $\bar{p}, \bar{q} \in \overline{\triangle} \subseteq M_\kappa^2$.

The CAT(κ) Inequality

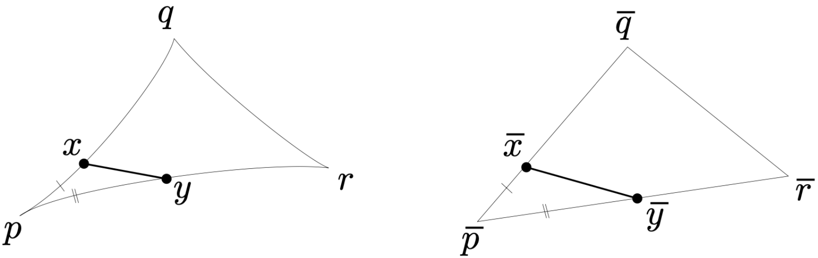


Figure: The CAT(κ) Inequality

Properties of CAT(0) Spaces

- Geodesics are unique
- Contractibility
- Convexity of the metric: given any pair of geodesics $c, c' \colon [0,1] \to X$ parameterized by arc length, we have

$$d(c(t), c'(t)) \leq (1-t)d(c(0), c'(0)) + td(c(1), c'(1))$$

for all $t \in [0,1]$.

CAT(κ) Groups

Definition

A group is said to be a CAT(κ) *group* if it acts properly[1] and cocompactly[2] by isometries on a CAT(κ) space.

This is different from our definition of hyperbolic groups. Unfortunately, we cannot check the CAT(κ) condition on the Cayley graph, and it is no longer invariant under quasi-isometry.

[1] For every compact subset $K \subset X$, the set of elements $\{\gamma \in \Gamma \mid \gamma \cdot K \cap K \neq \emptyset\}$ is finite.

[2] There exists a compact set $K \subseteq X$ such that $X = \Gamma \cdot K$.

Properties of CAT(κ) Groups

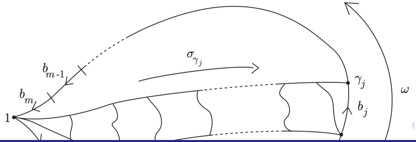
Theorem (CAT(-1) implies hyperbolic)

If G is a CAT(-1) *group, then G is hyperbolic.*

This follows from the definition of the CAT(-1) inequality, and the fact that $M^2_{-1} = \mathbb{H}^2$ is δ-hyperbolic.

Theorem

If Γ is a CAT(0) *group, then its Dehn function is either linear or quadratic.*



Examples of CAT(κ) Groups

The following are CAT(-1) groups:

1. Finite groups
2. Free groups of finite rank
3. Fundamental groups of closed surfaces of genus at least 2
4. Hyperbolic groups

Examples of CAT(κ) Groups

The following are CAT(0) groups:

1. All CAT(-1) groups
2. Trees
3. $\mathbb{Z}^n$ for all n (since it acts on $\mathbb{R}^n$)
4. More generally, $\pi_1 M$ for any closed manifold M of non-positive curvature
5. Right-angled Artin groups
6. Coxeter groups
7. Small-cancellation groups (including $C'(1/6)$ and $C(4)$–$T(4)$ groups)
8. Aut(F_2) and Aut(B_4) (as shown by Piggott, Ruane, and Walsh [13])
9. Products of CAT(0) groups (with the ℓ_2 metric)

M_κ-Polyhedral Complexes

The usefulness of the CAT(0) condition is that they can be readily checked for suitable spaces (certain complexes) via Gromov's *link condition.*

The most general version of these complexes, called *M_κ-complexes*, are constructed from convex cells in some model space. Informally, a *piecewise Euclidean (resp. piecewise hyperbolic, piecewise spherical)* cell complex K is a complex obtained from a collection of convex cells in Euclidean space (resp. hyperbolic space, the sphere) by identifying their faces via isometries.

The formal definition resembles that of charts for a smooth structure on a manifold.

CAT(0) Cubical Complexes

A piecewise Euclidean complex where the convex cells are unit cubes is called a *cubical complex*. There is an especially easy way to tell if a cubical complex admits a CAT(0) metric by looking at their local structure.

If K is a cubical complex and v is a vertex of K, define the link of v, denoted $\mathrm{Lk}(v)$, is the simplicial complex obtained by looking at the ε-sphere around v in X. It consists of:

- a vertex for each edge incident to v
- an edge for each corner of a square at v (connecting the vertices associated to the edges spanning the square)
- a 2-simplex for each corner of a cube at v
- $\vdots$

The Link Condition

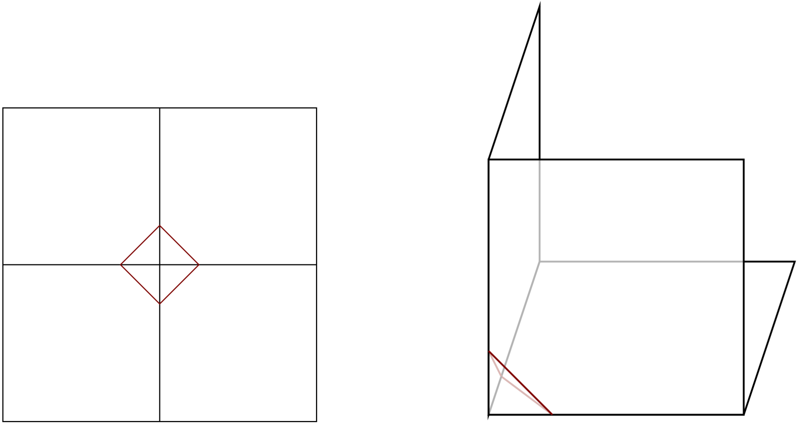


Figure: The links of two vertices in cubical complexes, shown in red [5]

Dehn Functions Subgroups of CAT(0) Groups

It is known that Dehn functions are virtually unchanged when passing to *finite-index* subgroups. However,

Theorem

There exist subgroups of CAT(0) *groups with Dehn functions satisfying* $\delta(x) \simeq x^m$ *for each* $m \geq 3$.

I cover a 3-step method of constructing such subgroups, using the Bieri doubling trick, originating in a paper of Bieri [1]. It is currently unknown if x^α for $\alpha \notin \mathbb{Z}$ occurs as a Dehn function of a subgroup of a CAT(0) group, or if e^x occurs.

The Steps (Simplified)

Step 1: Find a subgroup $H \subset G$ where G is CAT(0) such that

$$1 \to H \to G \to \mathbb{Z} \to 1$$

is a short exact sequence, and H is distorted with $\mathrm{disto}_H^G(n) \succeq n^2$. Show that G is CAT(0) using the *Link Condition*.

Step 2: Form the *double* $\triangle(G, H)$ *of* G *along* H, by amalgamating two copies of G along H. Hence $\triangle = \triangle(G, H) := G *_H G$.

Step 3: The double $\triangle(G, H)$ embeds in $G \times F_2$. Since G and F are CAT(0), so is $G \times F_2$.

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Implications

The undecidability of the word problem implies the undecidability of many other problems in pure mathematics. For example, the homeomorphism problem for closed manifolds in dimension at least 4 is undecidable [11], and there are far-reaching consequences for the global shape of moduli spaces (Adrian's talk).

One can also design a public-key cryptosystem based on the undecidability of the word problem [14].

A Parting Reflection

So, my interest in symmetry has not been misplaced.

H. S. M. Coxeter,
upon learning that his brain displayed
a high degree of bilateral symmetry

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