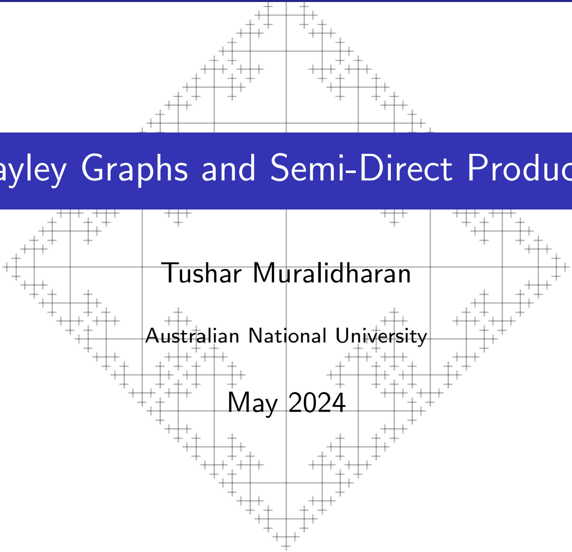

Cayley Graphs and Semi-Direct Products

Tushar Muralidharan

Australian National University

May 2024

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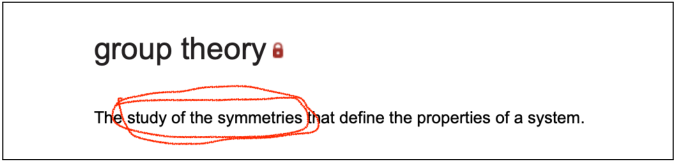
What is Group Theory?



group theory

The study of the symmetries that define the properties of a system.

Figure: Oxford definition of group theory [7]

Groups

A group is a set G together with a binary operation "$\cdot$" on G that satisfies the following three properties:

- *Associativity.* For all $a, b, c \in G$ one has $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- *Identity.* There exists an element $e \in G$ such that for all $a \in G$ one has $e \cdot a = a$ and $a \cdot e = a$.
- *Inverses.* For each $a \in G$, there exists an element $b \in G$ such that $a \cdot b = e$ and $b \cdot a = e$, where e is the identity element.

The Symmetric Group

Example

The *symmetric group* S_n is defined as the set of permutations of the set $\{1, 2, 3, \ldots, n\}$.
Alternatively, it is the set of symmetries of n points.

$$\bullet \qquad \bullet \qquad \bullet \qquad \bullet \qquad \bullet \qquad \cdots$$

The Dihedral Group

Example

The *dihedral group* D_n is **defined as** the set of symmetries of the regular n-gon.
For example, the group D_8 consists of the 8 rotations and 8 reflections of the regular octagon.

Reverse-Engineering

Question: For what object is $\mathbb{Z}/n\mathbb{Z}$ the group of symmetries?

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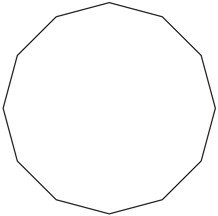

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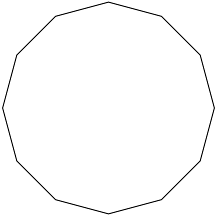

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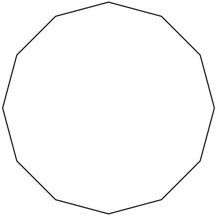

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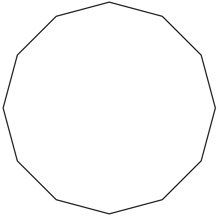

If we choose an n-gon, we get D_n. But, we can identify $\mathbb{Z}/n\mathbb{Z}$ as the subgroup of rotational symmetries of D_n.

Idea: Draw a little arrow in the middle of each edge.

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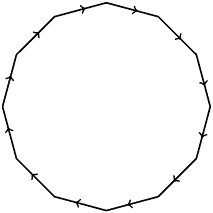

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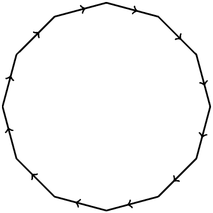

Symmetries must now preserve the arrows.

More Reverse-Engineering

Question: For what object is $\mathbb{Z}$ the group of symmetries?

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But...

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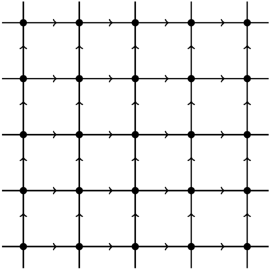

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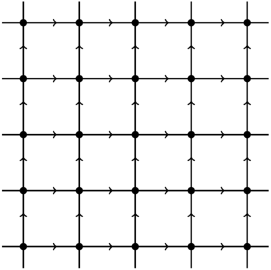

But...this doesn't work. Although every $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ corresponds to a translation (left or right) by a units and (up and down) by b units, we also have reflectional symmetries.

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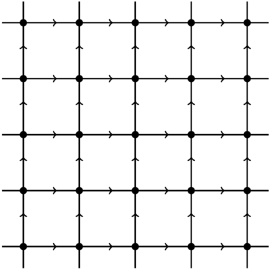

But...this doesn't work. Although every $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ corresponds to a translation (left or right) by a units and (up and down) by b units, we also have reflectional symmetries. Idea: Colors!

A Different Technique

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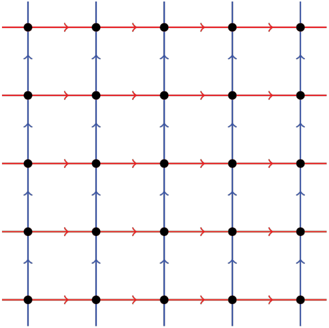

Symmetries must now preserve the arrows *and* the colors.

Where From Here?

I hope to convince you that every finitely presented group is exactly the symmetries of some geometric object, namely, its Cayley graph.

This should make some sense of the definition of group theory as the study of symmetries.

Let's modify our earlier techniques. (Foreshadowing.)

Drawing arrows $\rightarrow$ Adding directions to edges of a graph

Adding colors $\rightarrow$ Adding labels to edges of a graph

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What is a Cayley Graph?

Definition

The *Cayley graph* for a group G with respect to a generating set S is a directed, labeled graph whose *vertices correspond to the elements* of G, and for each $g \in G$ and $s \in S$ contains an edge from g to gs labeled by s.

$$g \xrightarrow{s} gs$$

Note: The definition of the Cayley graph depends on the generating set S chosen for G. More on this later.

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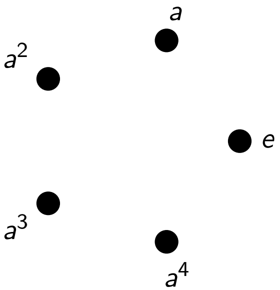
Sometimes, we color-code (or vary the patterns of) the edges by their generator instead of labeling them. Also, if there is an edge with arrows on both sides, we sometimes omit the arrowheads.

A First Example

The Cayley graph for $C_5 = \langle a \rangle = \{e, a, a^2, a^3, a^4\}$ with respect to the generating set $\{a\}$ is



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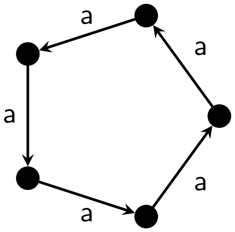

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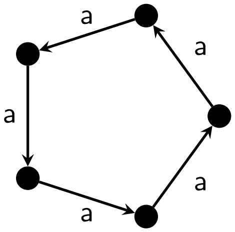
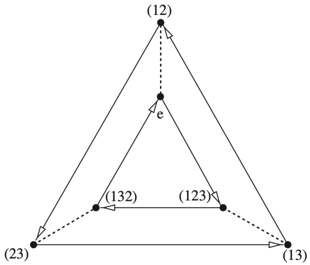


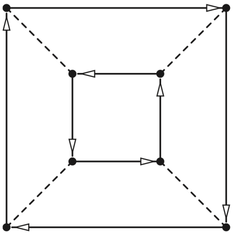
After removing the vertex labels, what are the symmetries (preserving labels and directions) of this graph?

More Cayley Graphs

Can you tell which generators correspond to which edge patterns?



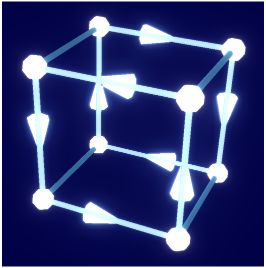
S_3 with respect to $\{(1\,2), (1\,2\,3)\}$



D4 with respect to a rotation and a reflection

What happens if you look at the graph of the dihedral group in 3D?

Dihedral Groups in 3D



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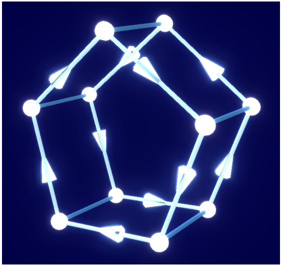
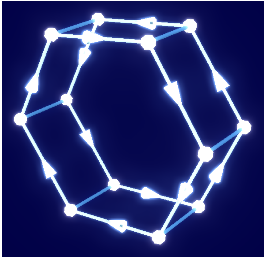


D_5 with respect to a rotation and a reflection



D_6 with respect to a rotation and a reflection

Can you see why these groups might be called “dihedral?”

A More Complicated Cayley Graph

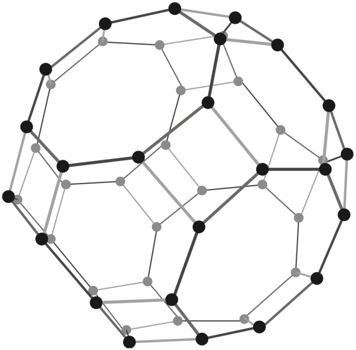


Figure: The Cayley graph of the symmetry group of a cube (generated by three reflections).

Cayley's Theorem

Let $Cay(G, S)$ denote the Cayley graph of a group G with respect to the generating set S.

Cayley's Theorem

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Theorem

Every group G with finite generating set S is isomorphic to the (direction, label)-preserving symmetries of $Cay(G, S)$.

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Proof Sketch

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Every group G with finite generating set S is isomorphic to the (direction, label)-preserving symmetries of $Cay(G, S)$.

Proof Sketch

Every element $g \in G$ determines a symmetry γ_g of $Cay(G, S)$,

which sends $\overset{h}{\bullet} \rightsquigarrow \overset{gh}{\bullet}$ and

$$\begin{tikzcd} \overset{hs}{\bullet} & & \overset{ghs}{\bullet} \\ \underset{h}{\bullet} \arrow[u, "s"] \arrow[rr, phantom, "\rightsquigarrow"] & & \underset{gh}{\bullet} \arrow[u, "s"'] \end{tikzcd}$$

.

Cayley's Theorem

Theorem

Every group G with finite generating set S is isomorphic to the (direction, label)-preserving symmetries of $Cay(G, S)$.

Proof Sketch

Now we show that every symmetry of $Cay(G, S)$ arises in this way.
Claim: If ρ is a symmetry of $Cay(G, S)$, then $\rho = \gamma_g$ for the element g such that $\rho(\bullet_e) = \bullet_g$.
Consider the symmetry $\rho \cdot \gamma_g^{-1}$. This takes $\bullet_e$ to $\bullet_e$ and edges incident to e (since it preserves directions and labels). Hence $\rho \cdot \gamma_g^{-1}$ fixes all vertices adjacent to $\bullet_e$. But then it will fix edges incident to these, and so on. Thus $\rho \cdot \gamma_g^{-1} = \mathrm{id}$, i.e., $\rho = \gamma_g$. □

Uniqueness

Question: Are Cayley graphs unique?

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Answer: No, $Cay(G, S)$ depends on the generating set S.

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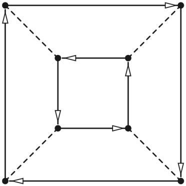
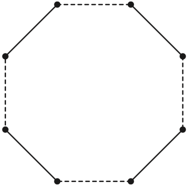


D_4 with respect to a rotation and reflection



D_4 with respect to two adjacent reflections

But...

Uniqueness

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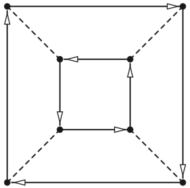
Answer: No, $Cay(G, S)$ depends on the generating set S.



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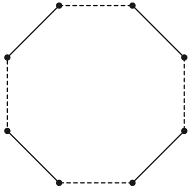

D_4 with respect to two adjacent reflections

But... Cayley graphs *are* unique up to quasi-isometry(!)

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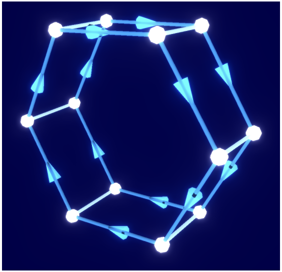
Direct Products

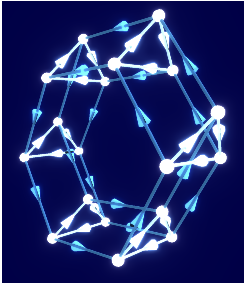
Given groups $(G, \star)$ and $(H, \triangle)$, recall the notion of the direct product $G \times H$, whose binary operation is defined component-wise: $(g_1, h_1) \cdot (g_2, h_2) = (g_1 \star g_2, h_1 \triangle h_2)$.

What does the Cayley graph of $G \times H$ look like? Let's look at the case where G and H are cyclic.

Direct Products



Cayley graph of $C_2 \times C_6$



Cayley graph of $C_3 \times C_6$

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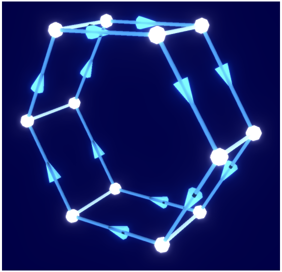

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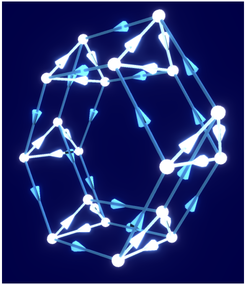

Cayley graph of $C_3 \times C_6$

Let $C_2 = \langle a \rangle$ and $C_6 = \langle b \rangle$. Then multiplying by the generator (a, e) takes us along the white lines and multiplying by (e, b) takes us around the hexagon.

Generalizing the Direct Product

Let's compare the Cayley graph of $C_2 \times C_6$ with that of D_6.

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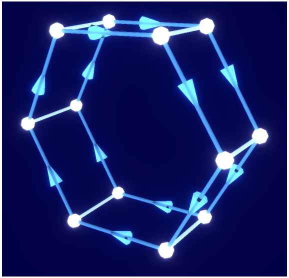
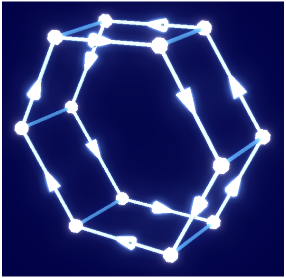
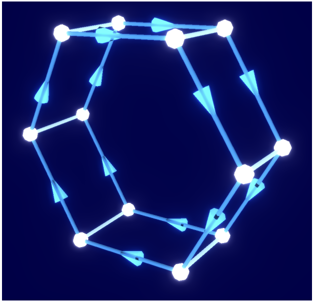


$C_2 \times C_6$



D_6

Generalizing the Direct Product



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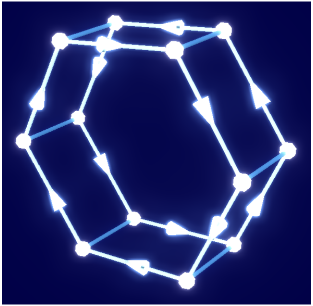
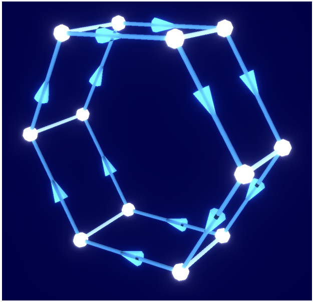


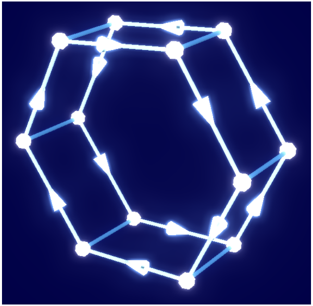
D_6

They are the same, except that we traverse the back hexagon in the opposite direction.

Generalizing the Direct Product



$C_2 \times C_6$



D_6

Recall that in $C_2 \times C_6$, moving around the hexagon corresponds to multiplying by (e, h) for some h.

Generalizing the Direct Product

That is, we have

$$(e, h) \cdot (a, b) = (a, hb),$$

so the first component is fixed and the second is multiplied by h.

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What if we relax the condition that we must stay at the same first component?

Generalizing the Direct Product

That is, we have

$$(e, h) \cdot (a, b) = (a, hb),$$

so the first component is fixed and the second is multiplied by h. What if we relax the condition that we must stay at the same first component? In the general case for two groups G and H, we write

$$(e, h_1) \cdot (g_2, h_2) = (f_{h_1}(g_2), h_1\, h_2),$$

where $f_{h_1}(g_2)$ need not be equal to g_2.

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$$(e, h_1) \cdot (g_2, h_2) = (f_{h_1}(g_2), h_1\, h_2),$$

where $f_{h_1}(g_2)$ need not be equal to g_2. As we vary g_2, we might expect $f_{h_1}(g_2)$ to vary across all elements of G, so we enforce that f_{h_1} is a bijection for all $h_1 \in H$.

Generalizing the Direct Product

Let's denote our new group by $G \rtimes H$. Since we want our multiplication to be well-defined, we must have

$$\begin{aligned}
(f_{h_1}(g_1) \cdot f_{h_1}(g_2), h_1\, h_2) &= (f_{h_1}(g_1), e) \cdot (f_{h_1}(g_2), h_1\, h_2) \\
&= (e, h_1) \cdot (g_1, h_1^{-1}) \cdot (e, h_1) \cdot (g_2, h_2) \\
&= (e, h_1) \cdot (g_1, e) \cdot (g_2, h_2) \\
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so $f_{h_1}(g_1) \cdot f_{h_1}(g_2) = f_{h_1}(g_1 g_2)$, i.e. f_{h_1} must be a homomorphism.

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\end{aligned}$$

so $f_{h_1}(g_1) \cdot f_{h_1}(g_2) = f_{h_1}(g_1 g_2)$, i.e. f_{h_1} must be a homomorphism. Since it is also a bijection, f_{h_1} must be an automorphism for all $h_1 \in H$.

Generalizing the Direct Product

Hence we have a map $\varphi\colon H \to Aut(G)$ defined by $h \mapsto f_h$.
Moreover,

Generalizing the Direct Product

Hence we have a map $\varphi\colon H \to Aut(G)$ defined by $h \mapsto f_h$. Moreover,

$$\begin{aligned}
(f_{h_1\,h_2}(g_1), h_1\,h_2) &= (e, h_1\,h_2)\cdot(g_1, e)\\
&= (e, h_1)\cdot(e, h_2)\cdot(g_1, e)\\
&= (e, h_1)\cdot(f_{h_2}(g_1), h_2)\\
&= (f_{h_1}(f_{h_2}(g_1)), h_1\,h_2)\\
&= ((f_{h_1}\circ f_{h_2})(g_1), h_1\,h_2)
\end{aligned}$$

so $f_{h_1\,h_2}(g_1) = (f_{h_1}\circ f_{h_2})(g_1)$.

Generalizing the Direct Product

Hence we have a map $\varphi\colon H \to Aut(G)$ defined by $h \mapsto f_h$. Moreover,

$$\begin{aligned}(f_{h_1\,h_2}(g_1), h_1\,h_2) &= (e, h_1\,h_2)\cdot(g_1, e)\\ &= (e, h_1)\cdot(e, h_2)\cdot(g_1, e)\\ &= (e, h_1)\cdot(f_{h_2}(g_1), h_2)\\ &= (f_{h_1}(f_{h_2}(g_1)), h_1\,h_2)\\ &= ((f_{h_1}\circ f_{h_2})(g_1), h_1\,h_2)\end{aligned}$$

so $f_{h_1\,h_2}(g_1) = (f_{h_1}\circ f_{h_2})(g_1)$. In other words, the map φ is a homomorphism.

Generalizing the Direct Product

Finally, we derive the group operation for $G \rtimes H$.

$$\begin{aligned}
(g_1, h_1) \cdot (g_2, h_2) &= (g_1, e) \cdot (e, h_1) \cdot (g_2, h_2) \\
&= (g_1, e) \cdot (f_{h_1}(g_2), h_1\, h_2) \\
&= (g_1 f_{h_1}(g_2), h_1\, h_2) \\
&= (g_1 \varphi(h_1)(g_2), h_1\, h_2).
\end{aligned}$$

Therefore, we arrive at the definition of a semi-direct product of groups.

The Definition

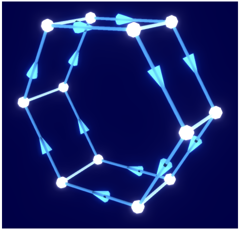
Definition

The *semi-direct product* of two groups G and H *with respect to a homomorphism* $\varphi\colon H \to Aut(G)$, denoted $G \rtimes_\varphi H$, is the set $G \times H$ with the binary operation defined by

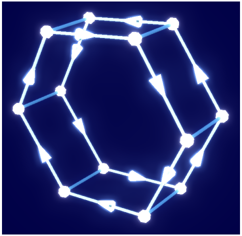
$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \varphi(h_1)(g_2), h_1\, h_2).$$

If we choose φ to be the trivial homomorphism, then the semi-direct product reduces to the direct product.

Our Original Example

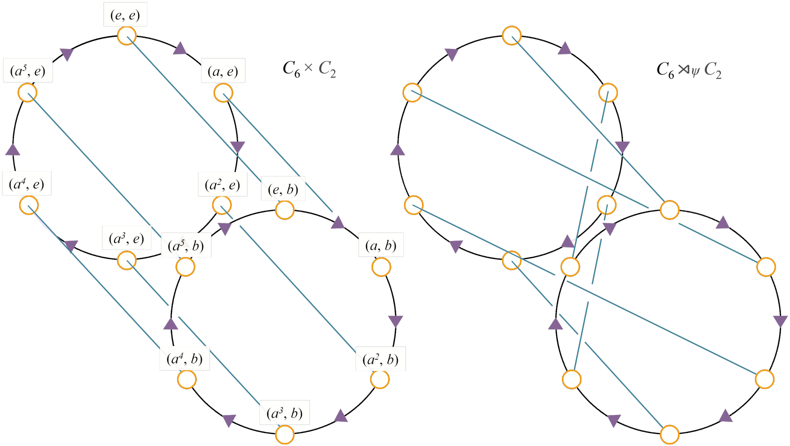


$C_2 \times C_6 \cong C_6 \times C_2$

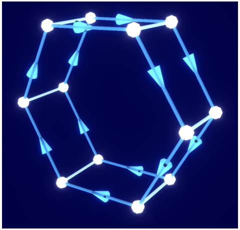


D_6

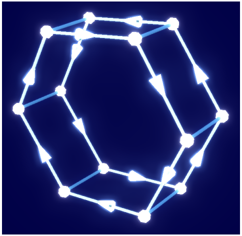
A Twist


(e, e)
(a^5, e)
(a, e)
$C_6 \times C_2$
(a^4, e)
(a^2, e)
(e, b)
(a^3, e)
(a^5, b)
(a, b)
(a^4, b)
(a^2, b)
(a^3, b)
$C_6 \rtimes_\psi C_2$

Our Original Example



$C_2 \times C_6 \cong C_6 \times C_2$



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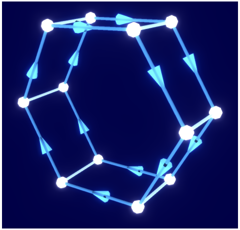

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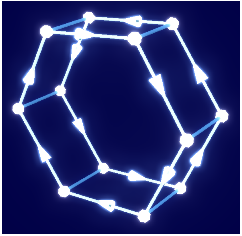
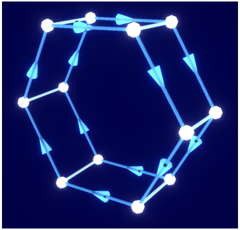


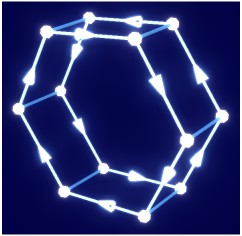
D_6

If we define $\varphi\colon C_2 \to Aut(C_6) \cong C_2$ to be the nontrivial homomorphism, then $C_6 \rtimes_\varphi C_2 \cong D_6$!

Our Original Example



$C_2 \times C_6 \cong C_6 \times C_2$



D_6

If we define $\varphi\colon C_2 \to Aut(C_6) \cong C_2$ to be the nontrivial homomorphism, then $C_6 \rtimes_\varphi C_2 \cong D_6$! In this case, φ acts like a twist that reverses the directions of the arrows on one of the hexagons.

Table of Contents

Why Should We Care?

Are Cayley graphs just for show?

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Are Cayley graphs just for show? No.

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- *Word Problem.* Is it possible to decide which words (i.e. finite products of the generators and their inverses) represent the identity element?
- *Conjugacy Problem.* Is it possible to decide whether two elements are conjugate?
- *Isomorphism Problem.* Is it possible to decide whether two groups are isomorphic?

All three problems are...

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All three problems are... undecidable, in general. However...

The Decision Problems

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If a group G is generated by the set S, then every word w in the generators determines a path in the Cayley graph from $\bullet_e$ to $\bullet_w$. Conversely, every edge path in the Cayley graph determines a word (where we add inverses if the path travels in the opposite direction of an edge).

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The shortest path metric on the Cayley graph is called the *word metric*. Treating Cayley graphs (1-complexes) as geometric objects can illumiate geometric properties that are useful in solving the word problem. For example, short-cuts in Cayley graphs in hyperbolic space can be shown to correspond to length-reducing applications of group relations [3].

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My research involves Cayley graphs embedded in $\mathrm{CAT}(0)$ spaces (complexes), which are defined by non-positive curvature.

A Parting Reflection

So, my interest in symmetry has not been misplaced.

H. S. M. Coxeter,
upon learning that his brain displayed
a high degree of bilateral symmetry

References I

Matt Clay and Dan Margalit.
Office Hours with a Geometric Group Theorist.
Princeton University Press, 2017.

M. Dehn.
Über unendliche diskontinuierliche gruppen.
Mathematische Annalen, 71(1):116–144, Mar 1911.

M. Gromov.
Hyperbolic Groups, pages 75–263.
Springer New York, New York, NY, 1987.

References II

John Meier.
Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups.
London Mathematical Society Student Texts. Cambridge University Press, 2008.

Jules Poon.
Cayley graphs.

Jules Poon.
Cayley graphs and pretty things.

References III

Richard Rennie and Jonathan Law.
A Dictionary of Physics.
Oxford University Press, 2019.