Partitions

$\leq$ 45 minute adventure

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So What Are Partitions?

- Partitions are fundamentally the number of ways to break apart numbers into parts, with or without restrictions

$$
\begin{aligned}
5 &= 1+1+1+1+1 \\
&= 2+1+1+1 \\
&= 2+2+1 \\
&= 3+1+1 \\
&= 3+2 \\
&= 4+1 \\
&= 5
\end{aligned}
$$

How Dire Are Things Looking

Leonhard Euler

Srinavasa Ramanujan

Euler's Product

Euler's Product

$$(1-x)(1-x^2)(1-x^3)\dots$$

Partition jumpscare!

$$\frac{1}{1-x}\cdot\frac{1}{1-x^2}\cdots=(1+x+x^2+\dots)(1+x^2+x^4+\dots)(1+x^3+x^6+\dots)\dots$$

Partitions

Definition

A *partition* is a representation of a given positive integer n as a sum of integers from some given set A, say $A = \{a_1, a_2, a_3, \ldots\}$.

When A consists of all the positive integers, repetition is allowed, and the order of the summands is not taken into account, the number of ways n can be written as a sum of positive integers $\leq n$, that is, the number of solutions of

$$n = a_{i_1} + a_{i_2} + \cdots ,$$

denoted $p(n)$, is called the *partition function*.

Generating functions

Definition

A function $F(x)$ defined by a power series $F(x) = \sum f(n)x^n$ is called a *generating function* of the coefficients $f(n)$.

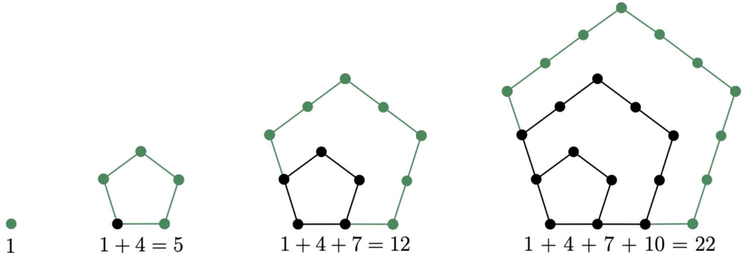
Recap: we've seen an informal argument for the identity

$$\prod_{m=1}^{\infty} \frac{1}{1-x^m} = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)\cdots$$
$$= \sum_{n=0}^{\infty} p(n)x^n,$$

where $p(0) = 1$, ignoring questions of convergence. What about the reciprocal $\prod_{m=1}^{\infty}(1-x^m)$ of this product?

Pentagonal numbers

The numbers $1, 5, 12, 22, \ldots$ are related to the pentagons shown below.



Pentagonal numbers

Generalised pentagonal numbers

These numbers are finite sums of the arithmetic progression:

$$1, 4, 7, 10, \ldots, 3n+1, \ldots$$

If $\omega(n)$ denotes the sum of the first n terms in this progression, then

$$\omega(n) = \sum_{k=0}^{n-1} (3k+1) = \frac{3n^2 - n}{2}.$$

We call the numbers $\omega(n)$ and $\omega(-n) = \frac{3n^2+n}{2}$ the *pentagonal numbers*.

$$\omega(n) : 1, 5, 12, 22, \ldots$$

$$\omega(-n) : 2, 7, 15, 26, \ldots$$

Another generating function

Recap: we've seen an informal argument for the identity

$$\prod_{m=1}^{\infty}(1-x^m) = 1 + \sum_{n=1}^{\infty}(p_e(n) - p_o(n))x^n,$$

where $p_e(n)$ denotes the number of partitions of n into an even number of unequal parts, and $p_o(n)$ denotes the number of partitions of n into an odd number of unequal parts.

Euler's pentagonal number theorem

We've seen

$$\prod_{m=1}^{\infty}(1-x^m) = 1+\sum_{n=1}^{\infty}(p_e(n)-p_o(n))x^n.$$

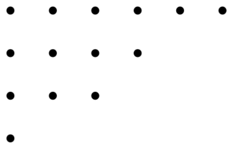
Theorem

$$\prod_{m=1}^{\infty}(1-x^m) = 1-x-x^2+x^5+x^7-x^{12}-x^{15}+\cdots$$

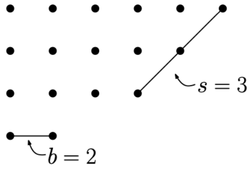
$$= 1+\sum_{n=1}^{\infty}(-1)^n\{x^{\omega(n)+\omega(-n)}\} = \sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)}$$

Geometric representations of partitions

A graph of a partition into unequal parts is in *standard form* if the parts are arranged in decreasing order.

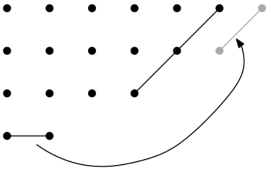


The longest line segment connecting the points in the last row of a graph is called the *base* of the graph, and the longest 45° line segment joining the last point in the first row with other points in the graph is called the *slope*.

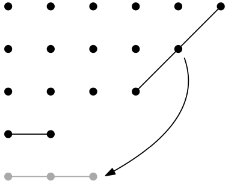


A remarkable combinatorial proof

Define two operations on the graph:



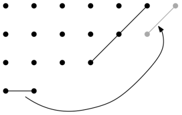
Up (U)



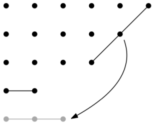
Down (D)

Note that U decreases the number of parts (rows) by 1, and D increases the number of parts (rows) by 1.

A remarkable combinatorial proof



Up (U)



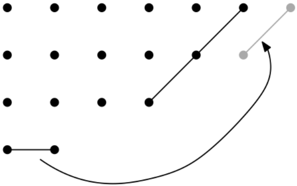
Down (D)

An operation on a graph is *permissible* if the resulting graph is also in standard form.

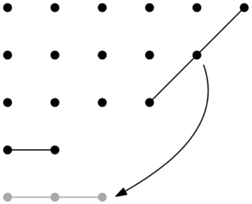
The key insight: If exactly one of U or D is permissible for every partition of some n, then there will be a one-to-one correspondence between partitions of n into odd and even unequal parts, so $p_e(n) = p_o(n)$ for this n. Incidentally, this happens for all n except the pentagonal numbers!

A remarkable combinatorial proof

Case 1: $|\text{base}| < |\text{slope}|$



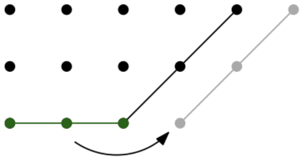
U ✓



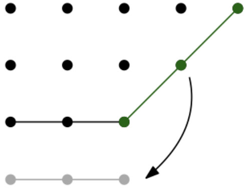
D ✗

A remarkable combinatorial proof

Case 2: $|\text{base}| = |\text{slope}|$



U ✓

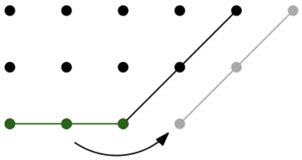


D ✗

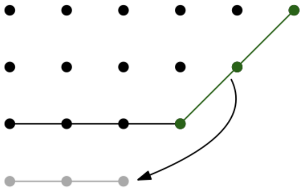
except when the base and slope intersect

A remarkable combinatorial proof

Case 3: $|\text{base}| > |\text{slope}|$



U ✗



D ✓

except when $|\text{base}| = |\text{slope}| + 1$ and the base and slope intersect

A remarkable combinatorial proof

To summarize, exactly one of U and D is permissible for all partitions of n into unequal parts with two exceptions:

- $|\text{base}| = |\text{slope}|$, and the base and slope intersect. If there are k parts (rows), then $|\text{base}| = k$ and

$$\begin{aligned} n &= k + (k+1) + \cdots + (2k-1) \\ &= \frac{3k^2 - k}{2} \\ &= \omega(k) \end{aligned}$$

- $|\text{base}| = |\text{slope}| + 1$, and the base and slope intersect. If there are k parts (rows), then

$$\begin{aligned} n &= \frac{3k^2 - k}{2} + k \\ &= \frac{3k^2 + k}{2} \\ &= \omega(-k) \end{aligned}$$

A remarkable combinatorial proof

To summarize, exactly one of U and D is permissible for all partitions of n into unequal parts with two exceptions:

- $n = \omega(k)$
- $n = \omega(-k)$

Since neither U nor D is permissible in these cases, we have an extra partition into even parts if k is even and an extra partition into odd parts if k is odd. So

$$p_e(n) - p_o(n) = \begin{cases} (-1)^k & \text{if } n \text{ is pentagonal} \\ 0 & \text{otherwise} \end{cases}.$$

This proves

$$\prod_{m=1}^{\infty}(1 - x^m) = 1 + \sum_{n=1}^{\infty}(p_e(n) - p_o(n))x^n = \sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)}.$$

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Jacobi's Triple Product Identity

Jacobi's Triple Product Identity

For any $|x| < 1$ and $z \neq 0$,

$$G_x(z) := \prod_{n=1}^{\infty}(1 - x^{2n})(1 + x^{2n-1}z^2)(1 + x^{2n-1}z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2} z^{2m}$$

$G_x(z)$ satisfies the functional equation

$$xz^2 G_x(xz) = G_x(z)$$

Writing $G_x(z)$ as a Laurent series in z,

$$G_x(z) = \sum_{m=-\infty}^{\infty} a_m(x) z^{2m}$$

Applying the functional equation yields $a_m(x) = a_0(x)x^{m^2}$ and a limiting argument shows that $a_0(x) = 1$.

Jacobi's Triple Product Identity

Euler's Pentagonal Number Theorem

By considering $G_{x^{3/2}}(ix^{1/4})$,

$$\varphi(x) = \prod_{n=1}^{\infty}(1 - x^n) = \sum_{m=-\infty}^{\infty} (-1)^m x^{\omega(m)}$$

where $\omega(m) = (3m^2 - m)/2$.

Identity for $\varphi(x)^3$

By considering $G_{x^{1/2}}(ix^{1/4}z^{1/2})$,

$$\varphi(x)^3 = \prod_{m=1}^{\infty}(1 - x^m)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{(n^2+n)/2}$$

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Back to Ramanujan

$$
\begin{aligned}
p(5n+4) &\equiv 0 \pmod{5} \\
p(7n+5) &\equiv 0 \pmod{7} \\
p(11n+6) &\equiv 0 \pmod{11}
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(5n+4)x^n &= 5\frac{\varphi(x^5)^5}{\varphi(x)^6} \\
\sum_{n=0}^{\infty} p(7n+5)x^n &= 7\frac{\varphi(x^7)^3}{\varphi(x)^4} + 49x\frac{\varphi(x^7)^7}{\varphi(x)^8}
\end{aligned}
$$

Let's Quickly Prepare For This Journey

- Adrian will derive a new formulation of the partition function
- I will discuss the main idea of this proof (intelligent series manipulations)
- Tushar will wrap up with the big conclusion

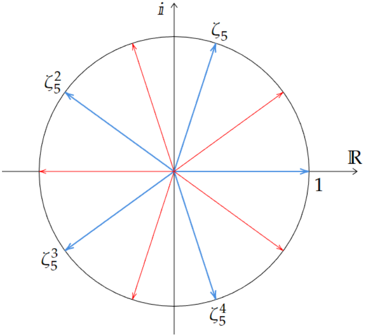
Rewriting the partition generating function

If $m, n \in \mathbb{N}$ with $(m, n) = d$, then

$$\prod_{k=1}^{n} (1 - x\zeta_n^{mk}) = (1 - x^{n/d})^d$$

The roots of the left side are

$$\left\{\zeta_n^{-mk} : 1 \leq k \leq n\right\} = \left\{\zeta_{n/d}^{-mk/d} : 1 \leq k \leq n\right\}$$



Rewriting the partition generating function

Then for some $a \neq 0$

$$\prod_{k=1}^{n}(1 - x\zeta_n^{mk}) = a\prod_{k=0}^{n/d}(x - \zeta_{n/d}^{k})^d = a(x^{n/d} - 1)^d$$

It follows that $a = (-1)^d$ and so

$$\prod_{k=1}^{n}(1 - x\zeta_n^{mk}) = (1 - x^{n/d})^d$$

In the case of $n = p$ prime, sending $x \mapsto x^m$ gives

$$\prod_{k=1}^{p}\left(1 - x^m\zeta_p^{mk}\right) = \begin{cases}(1 - x^m)^p & \text{if } p|m \\ 1 - x^{pm} & \text{otherwise}\end{cases}$$

Rewriting the partition generating function

Then

$$\prod_{m=1}^{\infty}\prod_{k=1}^{p}\left(1 - x^m \zeta_p^{mk}\right) = \frac{\varphi\left(x^p\right)^{p+1}}{\varphi\left(x^{p^2}\right)}$$

Setting $p = 5$ and rearranging gives

$$\frac{\varphi\left(x^{25}\right)}{\varphi\left(x^5\right)^6}\prod_{k=1}^{4}\prod_{m=1}^{\infty}\left(1 - x^m \zeta_5^{mk}\right) = \prod_{m=1}^{\infty}\frac{1}{1 - x^m}$$

which is the generating function of the partition function.

The Rise of the Series

Type r series

A type r series mod p has the form

$$\sum_{n=0}^{\infty} a(n)x^{pn+r}$$

for $0 \leq r < p$, and some arithmetic function $a(n)$.

Simple Observations:

- The sum of a series of type a and a series of type b is $a + b$
- Any relevant power series can be decomposed into series of type k for k in the entire residue class for a choice of prime, p

Return of Euler's Pentagonal Number Theorem

We will first make use of the famed

$$\varphi(x) = \sum_{n=-\infty}^{\infty} (-1)^n x^{\omega(n)}$$

to obtain

$$\begin{aligned}
\varphi(x) = & \sum_{\substack{n=-\infty \\ \omega(n)\equiv 0 \bmod 5}}^{\infty} (-1)^n x^{\omega(n)} \\
& + \sum_{\substack{n=-\infty \\ \omega(n)\equiv 1 \bmod 5}}^{\infty} (-1)^n x^{\omega(n)} \\
& + \sum_{\substack{n=-\infty \\ \omega(n)\equiv 2 \bmod 5}}^{\infty} (-1)^n x^{\omega(n)} \qquad = I_0 + I_1 + I_2
\end{aligned}$$

Just One Slide With Lots of Equations

Lemma

For $\zeta_5 = e^{2\pi i/5}$

$$\prod_{k=1}^{4} \prod_{m=1}^{\infty} (1 - x^m \zeta_5^{mk}) = \prod_{k=1}^{4} (I_0 + I_1 \zeta_5^{k} + I_2 \zeta_5^{2k})$$

where I_k *are the series of type* k *from before*

Theorem 5 can now be simplified a little bit

$$\sum_{n=0}^{\infty} p(n) x^n = \frac{\varphi(x^{25})}{\varphi(x^5)^6} \prod_{k=1}^{4} \prod_{m=1}^{\infty} (1 - x^m \zeta_5^{mk})$$

$$\sum_{n=0}^{\infty} p(n) x^n = \frac{\varphi(x^{25})}{\varphi(x^5)^6} \prod_{k=1}^{4} (I_0 + I_1 \zeta_5^{k} + I_2 \zeta_5^{2k})$$

I Lied Here's Another One

The type 4 component of

$$\sum_{n=0}^{\infty} p(n)x^n \longrightarrow \sum_{n=0} p(5n+4)x^{5n+4}$$

The type 4 component of

$$\frac{\varphi(x^{25})}{\varphi(x^5)^6} \prod_{k=1}^{4} (I_0 + I_1\zeta_5^k + I_2\zeta_5^{2k}) \longrightarrow$$

$$\frac{\varphi(x^{25})}{\varphi(x^5)^6} \left(3I_0I_1^2I_2(\alpha^4+\alpha^3+\alpha^2+\alpha) + I_0^2I_2^2(\alpha^4+\alpha^3+\alpha^2+\alpha+2) + I_1^4\right)$$

$$= \frac{\varphi(x^{25})}{\varphi(x^5)^6}(I_1^4 + I_0^2I_2^2 - 3I_0I_1^2I_2)$$

but this is good!

Using The Cube of the Euler Product

Using the corollary from Jacobi's identity

$$\phi(x)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) x^{(n^2+n)/2} = W_0 + W_1 + W_3$$

Then we can use

$$(I_0 + I_1 + I_2)^3 = W_0 + W_1 + W_3 \implies I_0 I_2 + I_1^2 = 0$$

and finally

$$I_0 I_2 = -I_1^2$$

I_1 Is Simple As Well

Since $\omega(n) \equiv 1 \iff n \equiv 1 \pmod 5$

$$\begin{aligned} I_1 &= -\sum_{n=-\infty}^{\infty} x^{(3(5n+1)^2-(5n+1))/2} \\ &= -x\sum_{n=-\infty}^{\infty} x^{25(3n^2+n)/2} \\ &= -x\sum_{n=-\infty}^{\infty} x^{25(3n^2-n)/2} \\ &= -x\varphi(x^{25}) \end{aligned}$$

Now Take It Away Tushar!

I_1 Is Simple As Well

Since $\omega(n) \equiv 1 \iff n \equiv 1 \pmod 5)$

$$
\begin{aligned}
I_1 &= -\sum_{n=-\infty}^{\infty} x^{(3(5n+1)^2-(5n+1))/2} \\
&= -x \sum_{n=-\infty}^{\infty} x^{25(3n^2+n)/2} \\
&= -x \sum_{n=-\infty}^{\infty} x^{25(3n^2-n)/2} \\
&= -x\varphi(x^{25})
\end{aligned}
$$

Now Take It Away Tushar!

V_4 Is Simple As Well

Thanks Lekh! Now we observe that the product $\prod_{h=1}^{4}(l_0 l_1 \alpha^h + l_2\alpha^{2h})$ is a homogeneous polynomial in l_0, l_1, and l_2 of degree 4, so the terms contributing to the type 4 mod 5 part V_4 of this product are $l_1^4, l_0 l_1^2 l_2, l_0^2 l_2^2$. Expanding the product and using the handy $l_0 l_2 = -l_1^2$, we have

$$\begin{aligned} V_4 &= l_1^4 + 3(\alpha^4+\alpha^3+\alpha^2+\alpha) l_0 l_1^2 l_2 + (\alpha^4+\alpha^3+\alpha^2+\alpha+2) l_0^2 l_2^2 \\ &= l_1^4 - 3(\alpha^4+\alpha^3+\alpha^2+\alpha) l_1^4 + (\alpha^4+\alpha^3+\alpha^2+\alpha+2) - l_1^4 \\ &= c l_1^4, \end{aligned}$$

where $c = 3 - 2\alpha - 2\alpha^2 - 2\alpha^3 - 2\alpha^4$ is a constant.
Since $\alpha = e^{2\pi i/5}$, we find

$$c = 5 - 2(1+\alpha+\alpha^2+\alpha^3+\alpha^4) = 5$$

The Final Stretch

We've already seen that

$$\sum_{m=0}^{\infty} p(5m+4)x^{5m+4} = V_4 \frac{\varphi(x^{25})}{\varphi(x^5)^6},$$

so $V_4 = 5I_1^4$ and $I_1 = -x\varphi(x^{25})$ imply

$$\sum_{m=0}^{\infty} p(5m+4)x^{5m+4} = 5I_1^4 \frac{\varphi(x^{25})}{\varphi(x^5)^6} = 5x^4 \frac{\varphi(x^{25})^5}{\varphi(x^5)^6}.$$

Finally, cancelling x^4 from both sides and replacing x^5 with x gives Ramanujan's identity

$$\sum_{m=0}^{\infty} p(5m+4)x^m = 5\frac{\varphi(x^5)^5}{\varphi(x)^6}. \quad \blacksquare$$

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Conclusion

D. Kruyswijk

O. Kolberg

Louis Mordell