Elementary proofs of Gauss' triangular number theorem and a partition identity of Ramanujan

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Abstract

This paper explores partitions in a framework of analytic number theory by covering the essential results regarding partitions and proving Gauss's triangular number theorem (about generating functions) and the partition identity $p(5m+4) \equiv 0 \mod 5$ of Ramanujan. The foundation for this exploration is drawn from Tom M. Apostol's book *Introduction to Analytic Number Theory* [1]. The proof of Gauss's triangular number theorem employs a method developed by Daniel Shanks [3], and the proof of the partition identity relies on a method devised by Kruyswijk [2]. The results obtained not only contribute to the understanding of partitions but also demonstrate the applicability of different techniques in analytic number theory.

1 Introduction

The theory of partitions has been instrumental in advancing combinatorial analysis and the study of modular functions. In the branch of additive number theory, a basic problem is that of expressing a given positive integer n as a sum of integers from some given set A, say

$$A = \{a_1, a_2, \ldots\},$$

where the elements a_i are special numbers such as primes, squares, cubes, triangular numbers, etc. Each representation of n as a sum of elements of A is called a *partition* of n. The definitions and theorems in this section are sourced from Apostol [1]. Therein are the proofs (omitted) of the theorems we state in this section.

One of the most fundamental problems in additive number theory is that of unrestricted partitions. In this case, the set of summands consists of all positive integers, and the partition function to be studied is the number of ways n can be written as a sum of positive integers $\leq n$, that is, the number of solutions of

$$n = a_{i_1} + a_{i_2} + \cdots$$

The number of summands is unrestricted, repetition is allowed, and the order of the summands is not taken into account. The corresponding partition function is denoted by $p(n)$ and is called the *unrestricted partition function*, or simply the *partition function*. The summands are called *parts*.

The concept of generating functions arises as a valuable tool in analyzing the partition function $p(n)$, offering a concise representation of its coefficients. A function $F(s)$ defined by a Dirichlet series $F(s) = \sum f(n)n^{-s}$ is called a *generating function* of the coefficients $f(n)$. The following theorem exhibits a generating function for the partition function $p(n)$.

Theorem 1 (Euler). *For $|x| < 1$ we have*

$$\prod_{m=1}^{\infty} \frac{1}{1-x^m} = \sum_{n=0}^{\infty} p(n)x^n,$$

where $p(0) = 1$.

We also examine the partition function generated by the product $\prod(1-x^m)$, the reciprocal of the generating function of $p(n)$. Write

$$\prod_{m=1}^{\infty}(1-x^m) = 1 + \sum_{n=1}^{\infty} a(n)x^n.$$

To express $a(n)$ as a partition function, we note that every partition of n into unequal parts produces a term x^n on the right with a coefficient $+1$ or -1. The coefficient is $+1$ if x^n arises from the product of an even number of terms and -1 otherwise. Therefore,

$$a(n) = p_e(n) - p_o(n),$$

where $p_e(n)$ is the number of partitions of n into an even number of unequal parts, and $p_o(n)$ is the number of partitions into an odd number of unequal parts. Euler proved that $p_e(n) = p_o(n)$ for all n except those belonging to a special set called pentagonal numbers.

The pentagonal numbers $1, 5, 12, 22, \ldots$ are related to the pentagons shown in Figure 1.

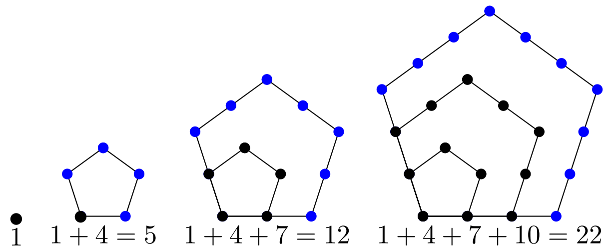

Figure 1

These numbers are also partial sums of the terms in the arithmetic progression

$$1, 4, 7, 10, 13, \ldots, 3n+1, \ldots$$

If we denote the sum of the first n terms of this progression by $\omega(n)$, then

$$\omega(n) = \sum_{k=0}^{n-1}(3k+1) = \frac{3n(n-1)}{2} + n = \frac{3n^2 - n}{2}.$$

The numbers $\omega(n)$ and $\omega(-n) = (3n^2+n)/2$ are called the *pentagonal numbers*. The next theorem unveils a fascinating pattern in the product $\prod(1-x^m)$ and provides a representation of this product in terms of the pentagonal numbers.

Theorem 2 (Euler's pentagonal-number theorem)**.** *If* $|x| < 1$ *we have*

$$\prod_{m=1}^{\infty}(1-x^m) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots$$
$$= 1 + \sum_{n=1}^{\infty}(-1)^n\{x^{\omega(n)} + x^{\omega(-n)}\} = \sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)}$$

Euler's pentagonal number theorem and many other partition identities occur as special cases of a famous identity of Jacobi from the theory of theta functions.

Theorem 3 (Jacobi's triple product identity)**.** *For complex* x *and* z *with* $|x| < 1$ *and* $z \neq 0$ *we have*

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z^2)(1+x^{2n-1}z^{-2}) = \sum_{m=-\infty}^{\infty} x^{m^2}z^{2m}.$$

Jacobi's formula leads to another important formula for the cube of Euler's product, which we isolate as a theorem.

Theorem 4. *If* $|x| < 1$ *we have*

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{m=-\infty}^{\infty}(-1)^m m x^{(m^2+m)/2}$$
$$= \sum_{m=0}^{\infty}(-1)^m(2m+1)x^{(m^2+m)/2}$$

These results form the essence of the theory of partitions and comprise the basis for our upcoming proofs.

2 Gauss' triangular-number theorem

In 1796, Gauss proved that every positive integer is a sum of at most three triangular numbers. Gauss noted his achievement by writing in his diary, "EYPHKA! Num = $\triangle + \triangle + \triangle$." This result is commonly known as Gauss' triangular number theorem. We prove a generating function identity that often borrows the same name.

If $x \neq 1$ let $Q_0(x) = 1$ and for $n \geq 1$ define

$$Q_n(x) = \prod_{r=1}^{n}\frac{1-x^{2r}}{1-x^{2r-1}}.$$

Theorem 5.

(a) *We have the following finite identities of Shanks* [3]:

$$\sum_{m=1}^{2n} x^{m(m-1)/2} = \sum_{s=0}^{n-1}\frac{Q_n(x)}{Q_s(x)}x^{s(2n+1)}, \tag{1}$$

$$\sum_{m=1}^{2n+1} x^{m(m-1)/2} = \sum_{s=0}^{n}\frac{Q_n(x)}{Q_s(x)}x^{s(2n+1)}. \tag{2}$$

(b) (Gauss' triangular-number theorem) Hence for $|x|<1$,

$$\sum_{m=1}^{\infty} x^{m(m-1)/2} = \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^{2n-1}}.$$

Proof. We follow the proof given by Shanks [3].

(a) Let

$$A_n = \sum_{s=0}^{n-1} \frac{Q_n(x)}{Q_s(x)} x^{s(2n+1)} \quad \text{and} \quad S_n = \sum_{m=1}^{2n} x^{m(m-1)/2}.$$

We readily verify the algebraic identity

$$(1-x^{2n})x^{s(2n+1)} = (1-x^{2n-1})x^{s(2n-1)} + (1-x^{2s+1})x^{(s+1)(2n-1)} - (1-x^{2s})x^{s(2n-1)},$$

and multiplying by $\frac{Q_{n-1}(x)}{Q_s(x)(1-x^{2n-1})}$ gives

$$\frac{Q_n(x)}{Q_s(x)} x^{s(2n+1)} = \frac{Q_{n-1}(x)}{Q_s(x)} x^{s(2(n-1)+1)} + \alpha_{s,n} - \beta_{s,n} \tag{3}$$

where

$$\alpha_{s,n} = \frac{1-x^{2s+1}}{1-x^{2n-1}} \frac{Q_{n-1}(x)}{Q_s(x)} x^{(s+1)(2n-1)}$$

and

$$\beta_{s,n} = \frac{1-x^{2s}}{1-x^{2n-1}} \frac{Q_{n-1}(x)}{Q_s(x)} x^{s(2n-1)}.$$

Observe that

$$\beta_{s+1,n} = \alpha_{s,n} \quad \text{for } s = 0, 1, \ldots, n-2$$

and

$$\beta_{0,n} = 0 \quad \text{and} \quad \alpha_{n-1,n} = x^{n(2n-1)},$$

and hence the sum of (3) from $s=0$ to $s=n-1$ telescopes and leaves

$$A_n = A_{n-1} + x^{(n-1)(2n-1)} + x^{n(2n-1)}.$$

This may be written $A_n - A_{n-1} = S_n - S_{n-1}$, and by induction we have

$$A_n - S_n = A_{n-1} - S_{n-1} = \cdots = A_1 - S_1 = \frac{1-x^2}{1-x} - (1+x) = 0.$$

This proves (1), and we obtain (2) by adding $x^{n(2n+1)}$ to both sides of (1).

(b) Since the $s=0$ term of A_n is $Q_n(x)$ and the remaining terms (for $s = 1, 2, \ldots, n-1$) are of order x^{2n+1} and higher, the power series of the function $Q_n(x)$ must agree with that of $A_n(x) = S_n(x)$ up to terms of order x^{2n}. By induction, the function $Q_\infty(x)$ must have the power series S_∞. □

In fact, $Q_\infty(x)$ is a quotient of the generating functions

$$\prod_{r=1}^{\infty} \frac{1}{1-x^{2r-1}} = \sum_{n=0}^{\infty} p_o(n)x^n \quad \text{and} \quad \prod_{r=1}^{\infty} \frac{1}{1-x^{2r}} = \sum_{n=0}^{\infty} p_e(n)x^n,$$

found in Table 14.1 of Apostol [1]. We note that cubing both sides of Theorem 5(b) gives a generating function for the number of partitions of n into three triangular numbers. However, there is more work to do to arrive at the familiar result about integers.

3 A partition identity of Ramanujan

Ramanujan made some striking discoveries about the divisibility properties of $p(n)$ by examining MacMahon's table of the partition function. For instance, he proved that

$$p(5m+4) \equiv 0 \pmod 5, \tag{4}$$

$$p(7m+5) \equiv 0 \pmod 7, \tag{5}$$

$$p(11m+6) \equiv 0 \pmod{11}. \tag{6}$$

In connection with these discoveries, he also stated without proof two remarkable identities that explicitly describe generating functions for the values in (4) and (5),

$$\sum_{m=0}^{\infty} p(5m+4)x^m = 5\frac{\varphi(x^5)^5}{\varphi(x)^6}, \tag{7}$$

and

$$\sum_{m=0}^{\infty} p(7m+5)x^m = 7\frac{\varphi(x^7)^3}{\varphi(x)^4} + 49x\frac{\varphi(x^7)^7}{\varphi(x)^8}, \tag{8}$$

where

$$\varphi(x) = \prod_{n=1}^{\infty}(1-x^n).$$

Since the functions on the right-hand sides of (7) and (8) have power series expansions with integer coefficients (as seen in Theorem 2), Ramanujan's identities immediately imply the congruences (4) and (5).

Several mathematicians, including Darling, Mordell, Rademacher, and Zuckerman, subsequently discovered proofs of (7) and (8) using the theory of modular functions. Alternative proofs independent of the theory of modular functions were provided by Kruyswijk [2] and later Kolberg. Kolberg's method gives not only the Ramanujan identities but also many new ones.

The following theorems give a complete proof of Ramanujan's partition identity

$$\sum_{m=0}^{\infty} p(5m+4)x^m = 5\frac{\varphi(x^5)^5}{\varphi(x)^6}, \quad \text{where } \varphi(x) = \prod_{n=1}^{\infty}(1-x^n),$$

by a method of Kruyswijk [2] not requiring the theory of modular functions.

Theorem 6.

(a) *Let $\varepsilon = e^{2\pi i/k}$ where $k \geq 1$. Then for all x we have*

$$\prod_{h=1}^{k}(1-x\varepsilon^h) = 1-x^k$$

(b) *More generally, if $(n,k)=d$ we have*

$$\prod_{h=1}^{k}(1-x\varepsilon^{nh}) = (1-x^{k/d})^d,$$

and hence

$$\prod_{h=1}^{k}(1-x^n e^{2\pi inh/k}) = \begin{cases} 1-x^{nk} & \text{if } (n,k)=1, \\ (1-x^n)^k & \text{if } k \mid n. \end{cases}$$

Proof.

(a) The polynomial $x^k - 1$ factors as $\prod_{h=1}^{k}(x - \varepsilon^h)$, so

$$\begin{aligned}
\prod_{h=1}^{k}(1 - x\varepsilon^h) &= \prod_{h=1}^{k}\varepsilon^h \prod_{h=1}^{k}(\varepsilon^{-h} - x) \\
&= \varepsilon^{k(k+1)/2}\prod_{h=1}^{k}(\varepsilon^h - x) \\
&= (-1)^k e^{(k+1)\pi i}\prod_{h=1}^{k}(x - \varepsilon^h) \\
&= -\prod_{h=1}^{k}(x - \varepsilon^h) \\
&= 1 - x^k.
\end{aligned}$$

(b) If $(n,k) = d, m = n/d$ and $\delta = \varepsilon^d = e^{2\pi i d/k}$, we have

$$\prod_{h=1}^{k}(1 - x\varepsilon^{nh}) = \prod_{h=1}^{k}(1 - x\delta^{mh}) = \prod_{h=1}^{k/d}(1 - x\delta^{mh})^d$$

since δ^{mh} has multiplicative order k/d. Since $(m, k/d) = 1$, as mh runs through a reduced residue system mod k, so does h. Thus by part (a), we have

$$\prod_{h=1}^{k/d}(1 - x\delta^{mh})^d = \prod_{h=1}^{k/d}(1 - x\delta^h)^d = (1 - x^{k/d})^d.$$

Now consider the function $\prod_{h=1}^{k}(1 - x^n e^{2\pi i nh/k}) = \prod_{h=1}^{k}(1 - x^n\varepsilon^{nh})$.
If $(n,k) = 1 = d$, then

$$\prod_{h=1}^{k}(1 - x^n e^{2\pi i nh/k}) = (1 - (x^n)^{k/1})^1 = 1 - x^{nk}.$$

If $k \mid n$, then $(n,k) = k$ and so

$$\prod_{h=1}^{k}(1 - x^n e^{2\pi i nh/k}) = (1 - (x^n)^{k/k})^k = (1 - x^n)^k. \qquad \square$$

We relate the previous theorem to Euler's product $\varphi(x)$ as follows:

Theorem 7.

(a) *For prime q and $|x| < 1$ we have*

$$\prod_{n=1}^{\infty}\prod_{h=1}^{q}(1 - x^n e^{2\pi i nh/q}) = \frac{\varphi(x^q)^{q+1}}{\varphi(x^{q^2})},$$

and hence

(b) *we have the identity*

$$\sum_{m=0}^{\infty} p(m)x^m = \frac{\varphi(x^{25})}{\varphi(x^5)^6} \prod_{h=1}^{4} \prod_{n=1}^{\infty} (1 - x^n e^{2\pi i n h/5}).$$

Proof.

(a) Using Theorem 6(b), we have

$$\begin{aligned}
\prod_{n=1}^{\infty} \prod_{h=1}^{q} (1 - x^n e^{2\pi i n h/q}) &= \prod_{\substack{n \geq 1 \\ (n,q)=1}} (1 - x^{nq}) \prod_{\substack{n \geq 1 \\ q|n}} (1 - x^n)^q \\
&= \prod_{r=1}^{q-1} \prod_{m=1}^{\infty} (1 - x^{(mq+r)q}) \prod_{m=1}^{\infty} (1 - x^{mq})^q \\
&= \frac{\prod_{r=0}^{q-1} \prod_{m=1}^{\infty} (1 - x^{(mq+r)q})}{\prod_{m=1}^{\infty} (1 - x^{mq^2})} \prod_{m=1}^{\infty} (1 - x^{mq})^q \\
&= \frac{\varphi(x^q)}{\varphi(x^{q^2})} \varphi(x^q)^q \\
&= \frac{\varphi(x^q)^{q+1}}{\varphi(x^{q^2})}.
\end{aligned}$$

(b) Taking $q = 5$ in part (a) gives

$$\frac{\varphi(x^{25})}{\varphi(x^5)^6} \prod_{n=1}^{\infty} \prod_{h=1}^{5} (1 - x^n e^{2\pi i n h/5}) = 1,$$

and dividing both sides by the term obtained when $h = 5$ and using Theorem 1 yields the result. $\square$

We now introduce power series of specific types to extend the results from the previous theorem using Euler's pentagonal number theorem (Theorem 2). If q is a prime and $0 \leq r < q$, a power series of the form

$$\sum_{n=0}^{\infty} a(n)x^{qn+r}$$

is said to be of *type* r mod q.

Theorem 8.

(a) *The product* $\varphi(x)$ *is a sum of three power series,*

$$\varphi(x) = \prod_{n=1}^{\infty} (1 - x^n) = I_0 + I_1 + I_2,$$

where I_k *denotes a power series of type* k *mod 5.*

(b) *Let* $\alpha = e^{2\pi i/5}$. *Then*

$$\prod_{h=1}^{4}\prod_{n=1}^{\infty}(1 - x^n\alpha^{nh}) = \prod_{h=1}^{4}(I_0 + I_1\alpha^h + I_2\alpha^{2h}).$$

(c) *We have*

$$\sum_{m=0}^{\infty} p(5m+4)x^{5m+4} = V_4\frac{\varphi(x^{25})}{\varphi(x^5)^6},$$

where V_4 *is the power series of type 4 mod 5 obtained from the product in part (b).*

Proof.

(a) By checking cases, we verify that $\omega(n) = (3n^2 - n)/2 \equiv 0, 1,$ or $2 \pmod 5$. By Euler's pentagonal number theorem (Theorem 2), we may take

$$I_k = \sum_{\omega(n)\equiv k \pmod 5} (-1)^n x^{\omega(n)}$$

for $k = 0, 1, 2$.

(b) We may write

$$\prod_{h=1}^{4}\prod_{n=1}^{\infty}(1 - x^n\alpha^{nh}) = \prod_{h=1}^{4}\varphi(x\alpha^h),$$

where

$$\varphi(x\alpha^h) = I'_0 + I'_1 + I'_2$$

and I'_k is equal to I_k with x replaced with $x\alpha^h$. For each I'_k, we have $\omega(n) \equiv k \pmod 5$ and hence $\alpha^{h\omega(n)} = \alpha^{hk}$. Factoring α^{hk} out of each I'_k, we obtain

$$\varphi(x\alpha^h) = I_0 + I_1\alpha^h + I_2\alpha^{2h}.$$

(c) Note that the product of a series of type r mod q and a series of type s mod q is a series of a type congruent to $r + s$ mod q. By Theorem 2, $\varphi(x^{25})$ is a series of type 0 mod 5 and by Theorem 1, $1/\varphi(x^5)^6$ is a series of type 0 mod 5. Hence $\varphi(x^{25})/\varphi(x^5)^6$ is a series of type 0 mod 5. Using Theorem 7(b) in combination with part (b), we have

$$\sum_{m=0}^{\infty} p(m)x^m = \frac{\varphi(x^{25})}{\varphi(x^5)^6}\prod_{h=1}^{4}(I_0 + I_1\alpha^h + I_2\alpha^{2h})$$

Hence, equating the series of type 4 mod 5 on both sides of this equation gives the result. □

The subsequent theorem connects the power series in Theorem 8 and the coefficients of $\varphi(x)^3$.

Theorem 9.

(a) *The cube of Euler's product is a sum of three power series,*

$$\varphi(x)^3 = W_0 + W_1 + W_3,$$

where W_k *denotes a power series of type* k *mod 5.*

(b) *The power series in Theorem 8(a) satisfy the relation*

$$I_0 I_2 = -I_1^2.$$

(c) $I_1 = -x\varphi(x^{25})$.

Proof.

(a) Using Theorem 4, we have

$$\prod_{n=1}^{\infty}(1-x^n)^3 = \sum_{m=0}^{\infty}(-1)^m(2m+1)x^{(m^2+m)/2}.$$

By checking cases, we verify that $(m^2+m)/2 \equiv 0, 1, \text{ or } 3 \pmod 5$, so we may take

$$W_k = \sum_{(m^2+m)/2} (-1)^m(2m+1)x^{(m^2+m)/2}$$

for $k = 0, 1, 3$.

(b) We have the identity $W_0 + W_1 + W_3 = (I_0 + I_1 + I_2)^3$. Equating the series of type 2 mod 5 on both sides gives

$$0 = I_0 I_2 + I_1^2.$$

(c) Note that $\omega(n) \equiv 1 \pmod 5$ if and only if $n \equiv 1 \pmod 5$. Hence

$$\begin{aligned} I_1 &= \sum_{n\equiv 1 \pmod 5} (-1)^n x^{\omega(n)} = \sum_{n=-\infty}^{\infty}(-1)^{5n+1}x^{\omega(5n+1)} \\ &= -\sum_{n=-\infty}^{\infty}(-1)^n x^{1+25(3n^2+n)/2} = -x\sum_{n=-\infty}^{\infty}(-1)^n x^{25\omega(-n)} \\ &= -x\varphi(x^{25}), \end{aligned}$$

by Euler's pentagonal number theorem (Theorem 2). □

Finally, we establish Ramanujan's partition identity. Observe that the product $\prod_{h=1}^{4}(I_0 + I_1\alpha^h + I_2\alpha^{2h})$ is a homogeneous polynomial in I_0, I_1, I_2 of degree 4, so the terms contributing to the series of type 4 mod 5 come from the terms $I_1^4, I_0 I_1^2 I_2$ and $I_0^2 I_2^2$.

Theorem 10.

(a) *There exists a constant c such that*

$$V_4 = cI_1^4,$$

where V_4 is the power series in Theorem 8(c), and hence

$$\sum_{m=0}^{\infty} p(5m+4)x^{5m+4} = cx^4\frac{\varphi(x^{25})^5}{\varphi(x^5)^6}.$$

(b) (Ramanujan's partition identity) *We have $c = 5$ and hence*

$$\sum_{m=0}^{\infty} p(5m+4)x^m = 5\frac{\varphi(x^5)^5}{\varphi(x)^6}.$$

Proof.

(a) Extracting the type 4 mod 5 terms from the product $\prod_{h=1}^{4}(I_0 + I_1\alpha^h + I_2\alpha^{2h})$, we verify

$$\begin{aligned} V_4 &= I_1^4 + 3(\alpha^4+\alpha^3+\alpha^2+\alpha)I_0I_1^2I_2 + (\alpha^4+\alpha^3+\alpha^2+\alpha+2)I_0^2I_2^2 \\ &= I_1^4 - 3(\alpha^4+\alpha^3+\alpha^2+\alpha)I_1^4 + (\alpha^4+\alpha^3+\alpha^2+\alpha+2)I_1^4 \\ &= cI_1^4, \end{aligned}$$

where $c = 3 - 2\alpha - 2\alpha^2 - 2\alpha^3 - 2\alpha^3 - 2\alpha^4$. Using Theorem 9(c), we find

$$\sum_{m=0}^{\infty} p(5m+4)x^{5m+4} = V_4\frac{\varphi(x^{25})}{\varphi(x^5)^6} = cI_1^4\frac{\varphi(x^{25})}{\varphi(x^5)^6} = c(-x\varphi(x^{25}))^4\frac{\varphi(x^{25})}{\varphi(x^5)^6} = cx^4\frac{\varphi(x^{25})^5}{\varphi(x^5)^6}.$$

(b) Since $1+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0$, we have

$$c = 5 - 2(1+\alpha+\alpha^2+\alpha^3+\alpha^4) = 5.$$

Thus

$$\sum_{m=0}^{\infty} p(5m+4)x^{5m+4} = 5x^4\frac{\varphi(x^{25})^5}{\varphi(x^5)^6}.$$

Hence, dividing both sides by x^4 and substituting x^5 for x gives the result. □

Some reflection is in order. Why was this method of proof successful? We were "lucky" that we were able to eliminate the W_k in the identity $W_0 + W_1 + W_3 = (I_0 + I_1 + I_2)^3$ to obtain the relation $I_0I_2 = -I_1^2$ in Theorem 9(b). We were also "lucky" that this relation allowed us to describe V_4 using $\varphi(x)$ in Theorem 10(a). It happens that Kruyswijk's method can find a generating function for $p(7n+4)$. Unfortunately, these instances of good fortune do not occur in every case, but future work introducing the theory of modular functions was a successful workaround.

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