Selected Solutions to Cox, Little, and O'Shea's Ideals, Varieties, and Algorithms

Tushar Muralidharan

February 20 2023

Contents

Chapter 1. Geometry, Algebra, and Algorithms **2**
§2. Affine Varieties 2
§3. Parametrizations of Affine Varieties 3
§4. Ideals 5
§5. Polynomials of One Variable 5

Chapter 2. Gröbner Bases **6**
§2. Orderings on the Monomials in $k[x_1, \ldots, x_n]$ 6
§3. A Division Algorithm in $k[x_1, \ldots, x_n]$ 6
§4. Monomial Ideals and Dickson's Lemma 6
§5. The Hilbert Basis Theorem and Gröbner Bases 6

Chapter 1

Geometry, Algebra, and Algorithms

§2 Affine Varieties

2. (Tushar) The polynomial vanishes when $y^2 = x(x-1)(x-2)$. There are two possible values for y whenever $x(x-1)(x-2) \geq 0$. The curve is symmetric about the x-axis since the corresponding values for y are $\pm\sqrt{x(x-1)(x-2)}$.

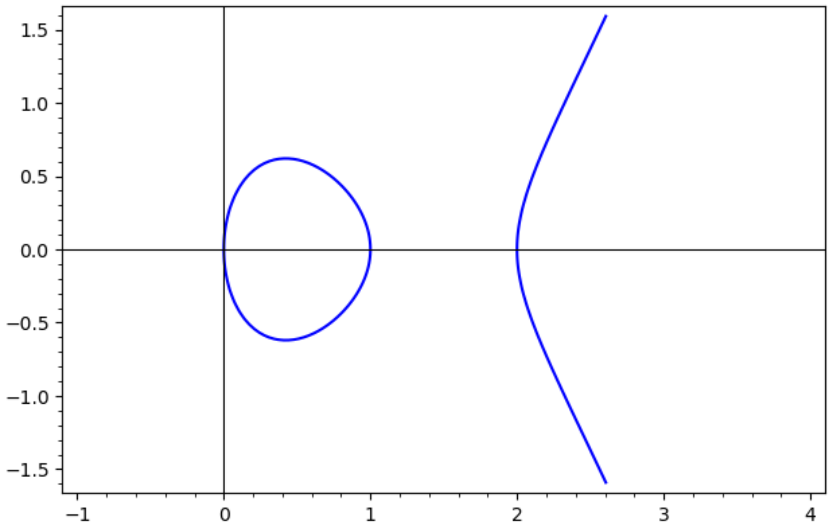

Figure 1.1: Plot of $\mathbf{V}(y^2 - x(x-1)(x-2))$

6. (Tushar)

   a. We can force each x_i to equal a_i by including the equation $x_i - a_i = 0$. Hence the single point $(a_1, \ldots, a_n) \in k^n$ is the affine variety defined by $\mathbf{V}(x_1 - a_1, \ldots, x_n - a_n)$. $\square$

b. We proceed by induction on the cardinality of the subset. The base case is proved by part (a). Assume that any finite subset of k^n with cardinality n is an affine variety. Then for any finite subset A of k^n with cardinality $n+1$, choose a point $(a_1, \ldots, a_n) \in k$. The induction hypothesis gives that $A \setminus (a_1, \ldots, a_n)$ is an affine variety. By part (a), the single point $(a_1, \ldots, a_n)$ is also an affine variety. Then Lemma 2 shows that the union $(A \setminus (a_1, \ldots, a_n)) \cup (a_1, \ldots, a_n) = A$ is also an affine variety. $\square$

8. (Tushar) We know in the proof of Proposition 5 of §1 that a nonzero polynomial in $k[x]$ of degree m has at most m distinct roots. Moreover, we have that $g(t) = f(t,t) = 0$ for all $t \neq 1$. Since $g \in \mathbb{R}[t]$ and $\mathbb{R}$ is infinite, this means that g has infinitely many distinct roots. But this implies that g is the zero polynomial and so $g(1) = f(1,1) = 0$, as required. $\square$

§3 Parametrizations of Affine Varieties

4. (Tushar)

a. Solving for t in the first equation gives

$$t = \frac{x}{1-x}.$$

Substituting this into the second equation then gives

$$y = 1 - \left(\frac{1-x}{x}\right)^2 = \frac{x^2 - (1-x)^2}{x^2} = \frac{2x-1}{x^2}.$$

This defines the affine variety $\mathbf{V}(x^2y - 2x + 1)$.

b. We want to show that for all $(x,y) \neq (1,1)$ satisfying $x^2y - 2x + 1 = 0$, there exists t such that $x = \frac{t}{1+t}$ and $y = 1 - \frac{1}{t^2}$. If $x = 1$, then the equation $x^2y - 2x + 1 = 0$ forces $y = 1$, which we are disregarding. Assuming $x \neq 1$, we can take $t = \frac{x}{1-x}$, and it can easily be checked that this t satisfied the required properties. $\square$

6. (Tushar)

a. The line connecting the north pole, which has $z = 1$, and any other point on the sphere, which must have $z < 1$, must cross the plane $z = 0$ at some point $(u, v, 0)$. On the other hand, the line connecting any point $(u, v, 0)$ and the north pole can be parameterized by $x = u + at, y = v + bt, z = ct$ and substituting this into the equation of the sphere shows that there are at most two possible solutions for t.

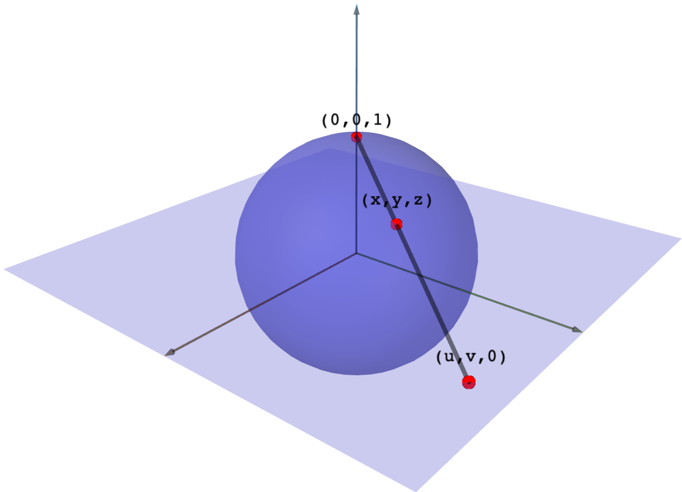

Figure 1.2: Parameterization of the sphere $x^2 + y^2 = z^2 = 1$

b. The line passes through both $(0,0,1)$ and $(u,v,0)$ at $t = 0$ and $t = 1$, respectively. It is clear that the function is a line from the form of the parameterization. $\square$

c. Substituting gives $t^2u^2 + t^2v^2 + 1 - 2t + t^2 = 1$. This yields $(u^2 + v^2 + 1)t^2 - 2t = 0$, so we obtain the solutions $t = 0$ and $t = \frac{2}{u^2+v^2+1}$. Since the point at $t = 0$ is the north pole, we are looking for the other point, where $t = \frac{2}{u^2+v^2+1}$. Thus we obtain

$$x = tu = \frac{2u}{u^2+v^2+1}$$
$$y = tv = \frac{2v}{u^2+v^2+1}$$
$$z = 1 - t = \frac{u^2+v^2-1}{u^2+v^2+1}.$$

8.

9. (Tushar)

a. Note that $a^2 - x^2 = a^2 - a^2\sin^2 t = a^2\cos^2 t$. Hence

$$y = a\tan t(1 + \sin t) = \frac{a\sin t}{a\cos t}(a + a\sin t) = \frac{x}{\pm\sqrt{a^2 - x^2}}(a + x).$$

Squaring both sides eliminates the plus-or-minus sign, which gives

$$y^2 = \frac{x^2(a+x)^2}{a^2 - x^2}.$$

This expression causes y to be undefined when $x = -a$. Thus we must eliminate the removable discontinuity:

$$y^2 = \frac{x^2(a+x)^2}{a^2-x^2} = \frac{x^2(a+x)^2}{(a+x)(a-x)} = \frac{x^2(a+x)}{a-x}$$

so that the final equation is

$$(a-x)y^2 = x^2(a+x).$$

b. Consider the line $y = tx$ for various values of t. This line intersects the strophoid when $(a-x)t^2x^2 = x^2(a+x)$. At the intersection point where $x \neq 0$, we have $(a-x)t^2 = a+x$. Solving for x and using the equation for y then yields the parameterization

$$x = a\frac{t^2-1}{t^2+1}$$
$$y = tx = ta\frac{t^2-1}{t^2+1}.$$

§4 Ideals

7. (Tushar) We note that $\mathbf{V}(x^n, y^m) = \{(0,0)\}$. Thus we aim to show that $\mathbf{I}(\{(0,0)\}) = \langle x, y\rangle$. Any polynomial of the form $f(x,y)x + g(x,y)y$ vanishes at $(0,0)$. This shows that $\langle x, y\rangle \subset \mathbf{I}(\{(0,0)\})$. Now suppose that $f = \sum_{i,j} a_{ij}x^iy^j$ vanishes at $(0,0)$. Then $a_{00} = f(0,0) = 0$, and we can factor a y out of the monomials with y only, and an x out of the remaining monomials so that $f \in \langle x, y\rangle$. This proves that $\mathbf{I}(\{(0,0)\}) \subset \langle x, y\rangle$. □

8. (Tushar)

   a. Suppose that $(a_1, \ldots, a_n) \in V$ for some variety $V \subset k^n$ and field k. If $f^m \in \mathbf{I}(V)$, then $(f(a_1, \ldots, a_m))^m = 0$. Since k is a field, this implies that $f(a_1, \ldots, a_m) = 0$. Since $(a_1, \ldots, a_n) \in V$ was arbitrary, we must have that f vanishes on all of V and hence $f \in \mathbf{I}(V)$. □

   b. We have that $x^2 \in \langle x^2, y^2\rangle$ but $x \notin \langle x^2, y^2\rangle$ since for polynomials of the form $h_1(x,y)x^2 + h_2(x,y)y^2$, every monomial has total degree at least two. Thus $\langle x^2, y^2\rangle$ is not a radical ideal. □

12. (Tushar)

15. (Tushar)

§5 Polynomials of One Variable

5.

8.

11. (Tushar)

12. (Tushar)

14. (Tushar)

17.

Chapter 2

Gröbner Bases

§2 Orderings on the Monomials in $k[x_1, \ldots, x_n]$

11.

§3 A Division Algorithm in $k[x_1, \ldots, x_n]$

10.

§4 Monomial Ideals and Dickson's Lemma

8.

§5 The Hilbert Basis Theorem and Gröbner Bases

3. (Tushar)

10.

15.

17.