# Exploring Projective Algebraic Geometry

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**Abstract**

This report provides an introduction to projective algebraic geometry, covering key concepts such as the projective plane, projective space, projective varieties, and the projective closure of an affine variety. The projective algebra-geometry dictionary is also presented, which provides a translation between geometric concepts and algebraic equations. This report provides a close walkthrough of the results and select exercises in Chapter 8 of Cox, Little and O'Shea (2015), while introducing additional topics such as the Zariski topology on projective space, the affine cone over a projective variety, dual projective spaces, the vanishing ideal of a subset of projective space, and the implementation of an algorithm for computing the projective closure in Sage. This report serves as a useful introduction for those who are familiar with the first four chapters of Cox, Little and O'Shea (2015) and are seeking to explore projective algebraic geometry and its applications.

## 1 Introduction

One of the blemishes of Euclidean geometry is that any two distinct lines intersect at exactly one point, *except* when they are parallel. The projective plane addresses this exception by viewing parallel lines as meeting at some sort of point at $\infty$. For this concept to be useful in light of the first sentence, parallel lines must intersect at exactly one point at $\infty$.

Most of the theorems, definitions, corollaries, and lemmas in this report come from [1] and the exercises therein. In order to avoid redundancy, proofs that are found in full in the book are omitted, so that the proofs that are included in this report are *not* found in the book.

## 2 The Projective Plane

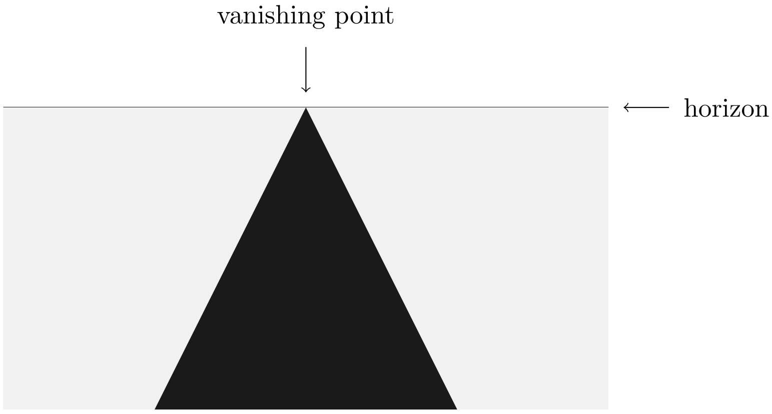
We introduce an equivalence relation on the lines in the plane $\mathbb{R}^2$ by setting $L_1 \sim L_2$ if $L_1$ and $L_2$ are parallel. Then an equivalence class $[L]$ consists of all lines parallel to a given line $L$.

**Definition 1.** The **projective plane** over $\mathbb{R}$, denoted $\mathbb{P}^2(\mathbb{R})$, is the set

$$\mathbb{P}^2(\mathbb{R}) = \mathbb{R}^2 \cup \{\text{one point at } \infty \text{ for each equivalence class of parallel lines}\}.$$

Let $[L]_\infty$ denote the common point at $\infty$ of all lines parallel to $L$. Then we call the set $\overline{L} = L \cup [L]_\infty \subseteq \mathbb{P}^2(\mathbb{R})$ the *projective line* corresponding to $L$.

To visualize points at $\infty$, imagine you are standing on a long straight road, looking out into the distance. The road stretches out in front of you, and you notice that the edges of the road appear to converge into a single point in the distance. We call this point the vanishing point.




When we talk about parallel lines in the projective plane, we mean lines that do not intersect even when extended to infinity. Just like the edges of the road, these lines appear to converge towards a vanishing point as they extend further and further away from us. Even though they never intersect in reality, they will appear to intersect at the vanishing point on the horizon. This is the notion of infinity in the projective plane, and lines that extend to infinity will eventually meet at this point. Since we want lines to intersect at exactly one point, we want exactly one vanishing point for each pair of parallel lines, regardless of the direction the lines are traversed.

We turn our attention back to formalizing the projective plane. There is another way to formulate the projective plane that will be useful in allowing us to specify points. Define an equivalence relation on the nonzero points of $\mathbb{R}^3$ by setting

$$(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$$

if there is a nonzero real number $\lambda$ such that $(x_1, y_1, z_1) = \lambda(x_2, y_2, z_2)$. One can easily check that $\sim$ is an equivalence relation on $\mathbb{R}^3 \setminus \{0\}$ (where as usual 0 refers to the origin $(0, 0, 0)$ in $\mathbb{R}^3$). Then we can redefine projective space as follows.

**Definition 2.** $\mathbb{P}^2(\mathbb{R})$ is the set of equivalence classes of $\sim$ on $\mathbb{R}^3 \setminus \{0\}$. Thus, we can write

$$\mathbb{P}^2(\mathbb{R}) = (\mathbb{R}^3 \setminus \{0\})/ \sim .$$

Given a triple $(x, y, z) \in \mathbb{R}^3 \setminus \{0\}$, its equivalence class $p \in \mathbb{P}^2(\mathbb{R})$ will be denoted $p = (x : y : z)$, and we say that $(x : y : z)$ are **homogeneous coordinates** of $p$. Thus

$$(x_1 : y_1 : z_1) = (x_2 : y_2 : z_2) \iff (x_1, y_1, z_1) = \lambda(x_2, y_2, z_2) \text{ for some } \lambda \in \mathbb{R} \setminus \{0\}.$$

This definition highlights that the projective plane has the structure of a three-dimensional space, although it is often represented visually as a two-dimensional plane with points at infinity.

To relate our two definitions of projective plane, we will use the map

$$\mathbb{R}^2 \to \mathbb{P}^2(\mathbb{R}) \tag{1}$$

defined by sending $(x, y) \in \mathbb{R}^2$ to the point $p \in \mathbb{P}^2(\mathbb{R})$ whose homogeneous coordinates are $(x : y : 1)$. This map has the following properties.

**Proposition 1.** *The map (1) is one-to-one and the complement of its image is the projective line $H_\infty$ defined by $z = 0$.*

This proposition shows that $\mathbb{R}^2$ has a one-to-one embedding in the projective plane. We will call $H_\infty$ the *line at* $\infty$. We conclude this section by showing that the projective plane we have defined solves the problem of non-intersecting parallel lines in $\mathbb{R}^2$. Using $\mathbb{P}^2(\mathbb{R})$ as in Definition 1, we have seen that the projective lines in $\mathbb{P}^2(\mathbb{R})$ are exactly the projective lines $\overline{L} = L \cup [L]_\infty$ and the line at $\infty$.

**Theorem 1.** *Let $\mathbb{P}^2(\mathbb{R})$ denote the projective plane over $\mathbb{R}$. Then:*

(i) *Any two distinct points in $\mathbb{P}^2(\mathbb{R})$ determine a unique projective line.*

(ii) *Any two distinct projective lines in $\mathbb{P}^2(\mathbb{R})$ meet at a unique point.*

**Proof.** We start with (i). If neither point is a point at $\infty$, then $\overline{L} = L \cup [L]_\infty$, where $L$ is the line connecting the two points in $\mathbb{R}^2$, is the unique projective line containing the two points. If exactly one of the two points is a point at $\infty$, then $\overline{L} = L \cup [L]_\infty$, where $L$ is the line passing through whichever point is in $\mathbb{R}^2$ and whose slope is such that its point at infinity $[L]_\infty$ is the other point, is the unique projective line passing through the two points. Finally, if both points are points at $\infty$, then the line at $\infty$ is the unique projective line passing through the two points.

We now prove (ii). If neither projective line is the line at $\infty$, then if the lines are not parallel in $\mathbb{R}^2$, the two lines either meet in $\mathbb{R}^2$, or if they are parallel, the two lines meet exactly at their common point at $\infty$. If one of the two projective lines is the line at $\infty$, then they meet exactly at the point at $\infty$ of the other line. $\square$

This means that two distinct projective lines in $\mathbb{P}^2(\mathbb{R})$ determine a unique point, and two distinct points in $\mathbb{P}^2(\mathbb{R})$ determine a unique projective line. This is a characterization of the *incidence geometry* of the projective plane. The projective plane is just one example of a projective space, which is a more general concept that can be defined for any dimension over any field.

# 3 Projective Space and Projective Varieties

We focus on constructing projective spaces of any dimension $n$ over any field $k$. We define an equivalence relation $\sim$ on the nonzero points of $k^{n+1}$ by setting

$$(x'_0, \ldots, x'_n) \sim (x_0, \ldots, x_n)$$

if there is a nonzero element $\lambda \in k$ such that $(x'_0, \ldots, x'_n) = \lambda(x_0, \ldots, x_n)$. If we let 0 denote the origin $(0, \ldots, 0)$ in $k^{n+1}$, then we define projective space as follows.

**Definition 3.** ***n*-dimensional projective space** over the field $k$, denoted $\mathbb{P}^n(k)$, is the set of equivalence classes of $\sim$ on $k^{n+1} \setminus \{0\}$. Thus,

$$\mathbb{P}^n(k) = (k^{n+1} \setminus \{0\})/\sim .$$

Given an $(n+1)$-tuple $(x_0, \ldots, x_n) \in k^{n+1} \setminus \{0\}$, its equivalence class $p \in \mathbb{P}^n(k)$ will be denoted $(x_0 : \cdots : x_n)$, and we will say that $(x_0 : \cdots : x_n)$ are **homogeneous coordinates** of $p$. Thus

$$(x'_0 : \cdots : x'_n) = (x_0 : \cdots : x_n) \iff (x'_0, \ldots, x'_n) = \lambda(x_0, \ldots, x_n) \text{ for some } \lambda \in k \setminus \{0\}.$$

When $n = 2$ and $k = \mathbb{R}$, the $n$-dimensional projective space over $k$ is the projective plane. Similar to Proposition 1, which concerns the projective plane, we can embed $k^n$ in $\mathbb{P}^n(k)$.

**Proposition 2.** *Let*

$$U_0 = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(k) \mid x_0 \neq 0\}.$$

*Then the map* $\phi$ *that sends* $(a_1, \ldots, a_n) \in k^n$ *to* $(1 : a_1 : \cdots : a_n) \in \mathbb{P}^n(k)$ *is a one-to-one correspondence between* $k^n$ *and* $U_0 \subseteq \mathbb{P}^n(k)$.

Note that the position of the 1 in the map $\phi$ is irrelevant. In Proposition 1, we decided to append 1 to the last coordinate of a point in $\mathbb{R}^2$. For projective spaces in an arbitrary dimension, we do not know the final position, so we choose to prepend 1 instead.

Now recall that when $n = 2$, we regarded the complement of the map (1) as the line at $\infty$. We do something similar in the general case. By the definition of $U_0$, we see that $\mathbb{P}^n(k) = U_0 \cup H$, where

$$H = \{p \in \mathbb{P}^n(k) \mid p = (0 : x_1 : \cdots : x_n)\}. \tag{2}$$

If we identify $U_0$ with the affine space $k^n$, we can think of $H$ as the *hyperplane at infinity*. It follows that the points in $H$ are in one-to-one correspondence with nonzero $n$-tuples $(x_1, \ldots, x_n)$, where two $n$-tuples represent the same point of $H$ if one is a nonzero scalar multiple of the other (just ignore the first component of points in $H$). In other words, $H$ is a "copy" of $\mathbb{P}^{n-1}(k)$, the projective space of one smaller dimension. Identifying $U_0$ with $k^n$ and $H$ with $\mathbb{P}^{n-1}(k)$, we can write

$$\mathbb{P}^n(k) = k^n \cup \mathbb{P}^{n-1}(k).$$

The following corollary ensures that $U_0$ is not the only way to embed $k^n$ in $\mathbb{P}^n(k)$.

**Corollary 1.** *For each* $i = 0, \ldots, n$, *let*

$$U_i = \{(x_0 : \cdots : x_n) \in \mathbb{P}^n(k) \mid x_i \neq 0\}.$$

(i) *The points of each* $U_i$ *are in one-to-one correspondence with the points of* $k^n$.
(ii) *The complement* $\mathbb{P}^n(k) \setminus U_i$ *may be identified with* $\mathbb{P}^{n-1}(k)$.
(iii) *We have* $\mathbb{P}^n(k) = \bigcup_{i=0}^n U_i$.

**Proof.** To prove (i), the map $\phi_i$ from $k^n$ to $\mathbb{P}^n(k)$ that sends

$$(a_1, \ldots, a_n) \mapsto (a_1 : \cdots : a_i : 1 : a_{i+1} : \cdots : a_n)$$

is a one-to-one correspondence between $k^n$ and $U_i$. This is proved analogously to Proposition 2 in [1].

The comments before the statement of the corollary are easily modified by using an arbitrary index $i$ instead of 0 to prove (ii).

Finally, we will prove (iii). The inclusion $\bigcup_{i=0}^n U_i \subseteq \mathbb{P}^n(k)$ is clear. Suppose that $p = (a_0 : \ldots : a_n) \in \mathbb{P}^n(k)$. Since we never allow all homogeneous coordinates to vanish simultaneously, there exists an $i$ such that $a_i \neq 0$. This means that $p \in U_i \subseteq \bigcup_{i=0}^n U_i$, and the result follows. □

We desire to extend the notion of varieties from affine space to projective space. However, a polynomial that vanishes on one of the homogeneous coordinates of a point might not vanish on all of them. We avoid this problem by considering homogeneous polynomials. A polynomial is *homogeneous* of total degree $d$ if every term appearing in $f$ has total degree $d$. We note the following property of homogeneous polynomials.

**Proposition 3.** *Let* $f \in k[x_0, \ldots, x_n]$ *be a homogeneous polynomial. If* $f$ *vanishes on any one set of homogeneous coordinates for a point* $p \in \mathbb{P}^n(k)$, *then* $f$ *vanishes for all homogeneous coordinates of* $p$. *In particular* $\mathbf{V}(f) = \{p \in \mathbb{P}^n(k) \mid f(p) = 0\}$ *is a well-defined subset of* $\mathbb{P}^n(k)$.

This allows us to make the following definition.

**Definition 4.** Let $k$ be a field and let $f_1, \dots, f_s \in k[x_0, \dots, x_n]$ be homogeneous polynomials. We set

$$\mathbf{V}(f_1, \dots, f_s) = \{(a_0 : \cdots : a_n) \in \mathbb{P}^n(k) \mid f_i(a_0, \dots, a_n) = 0 \text{ for all } 1 \leq i \leq s\}.$$

We call $\mathbf{V}(f_1, \dots, f_s)$ the **projective variety** defined by $f_1, \dots, f_s$.

The definition of a projective variety is analogous to that of an affine variety, but with the important difference being that the defining polynomials are required to be homogeneous. This allows us to work with equivalence classes of points in projective space.

In $\mathbb{P}^n(k)$, any nonzero homogeneous polynomial of degree 1,

$$\ell(x_0, \dots, x_n) = c_0 x_0 + \cdots + c_n x_n,$$

defines a projective variety $\mathbf{V}(\ell)$ called a *hyperplane*. One example we have seen is the hyperplane at infinity in (2), which was defined as $H = \mathbf{V}(x_0)$. When $n = 2$, we call $\mathbf{V}(\ell)$ a projective line, or more simply a *line* in $\mathbb{P}^2(k)$. Similarly, when $n = 3$, we call a hyperplane a *plane* in $\mathbb{P}^3(k)$.

If we take a projective variety $V$ and intersect it with one of the $U_i \cong k^n$ as in Proposition 1, it makes sense to ask whether we obtain an affine variety. The answer to this question is always *yes*, and the defining equations of the variety $V \cap U_i$ may be obtained by a process called *dehomogenization.*

**Proposition 4.** *Let $V = \mathbf{V}(f_1, \dots, f_s)$ be a projective variety. Then $W = V \cap U_0$ can be identified with the affine variety $\mathbf{V}(g_1, \dots, g_s) \subseteq k^n$, where $g_i(x_1, \dots, x_n) = f_i(1, x_1, \dots, x_n)$ for each $1 \leq i \leq s$.*

This proposition allows us to study a projective variety by looking at its affine pieces. The following proposition describes the reverse process.

**Proposition 5.** *Let $g(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ be a polynomial of total degree $d$.*

(i) *Let $g = \sum_{i=0}^{d} g_i$ be the expansion of $g$ as the sum of its homogeneous components where $g_i$ has total degree $i$. Then*

$$\begin{aligned} g^h(x_0, \dots, x_n) &= \sum_{i=0}^{d} g_i(x_1, \dots, x_n) x_0^{d-i} \\ &= g_d(x_1, \dots, x_n) + g_{d-1}(x_1, \dots, x_n) x_0 \\ &\quad + \cdots + g_0(x_1, \dots, x_n) x_0^d \end{aligned}$$

*is a homogeneous polynomial of total degree $d$ in $k[x_0, \dots, x_n]$. We will call $g^h$ the* **homogenization** *of $g$.*

(ii) *The homogenization of $g$ can be computed using the formula*

$$g^h = x_0^d \cdot g\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right).$$

(iii) *Dehomogenizing $g^h$ yields $g$, i.e., $g^h(1, x_1, \dots, x_n) = g(x_1, \dots, x_n)$.*

(iv) *Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial and let $x_0^e$ be the highest power of $x_0$ dividing $F$. If $f = F(1, x_1, \dots, x_n)$ is a dehomogenization of $F$, then $F = x_0^e \cdot f^h$.*

**Proof.** To prove (i), we must show that every term appearing in $g^h$ has total degree $d$. This follows easily since each $g_i(x_1,\ldots,x_n)x_0^{d-i}$ has total degree $i+(d-i)=d$.

For part (ii), let $g=\sum_{i=0}^{d} g_i$ be as in (i). We compute

$$\begin{aligned}
g^h(x_0,\ldots,x_n) &= \sum_{i=0}^{d} g_i(x_1,\ldots,x_n)x_0^{d-i} \\
&= x_0^d \sum_{i=0}^{d} \frac{1}{x_0^i}\cdot g_i\left(x_1,\ldots,x_n\right) \\
&= x_0^d \sum_{i=0}^{d} g_i\left(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0}\right) \\
&= x_0^d \cdot g\left(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0}\right).
\end{aligned}$$

As for (iii), we have

$$\begin{aligned}
g^h(1,x_1,\ldots,x_n) &= \sum_{i=0}^{d} g_i(x_1,\ldots,x_n)1^{d-i} \\
&= \sum_{i=0}^{d} g_i(x_1,\ldots,x_n) \\
&= g(x_1,\ldots,x_n).
\end{aligned}$$

Finally, for part (iv), note that if $F$ has total degree $d$, then we can write $F(x_0,\ldots,x_n)=x_0^e\cdot G(x_0,\ldots,x_n)$ where $G$ is a homogeneous polynomial with total degree $d-e$ and $x_0$ does not divide some term of $G$. In other words, $G$ contains a term involving only $x_1,\ldots,x_n$ with total degree $d-e$. It follows that $f=F(1,x_1,\ldots,x_n)=G(1,x_1,\ldots,x_n)$ has total degree $d-e$, so that $f^h=x_0^{d-e}f(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0})$. Furthermore, since $G$ is homogeneous, its homogenization is itself. This means that $x_0^{d-e}\cdot G(1,\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0})=G(x_0,\ldots,x_n)$. Then we have

$$\begin{aligned}
F(x_0,\ldots,x_n) &= x_0^e\cdot G(x_0,\ldots,x_n) \\
&= x_0^e\cdot x_0^{d-e}G\left(1,\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0}\right) \\
&= x_0^e\cdot x_0^{d-e}F\left(1,\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0}\right) \\
&= x_0^e\cdot x_0^{d-e}f\left(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0}\right) \\
&= x_0^e\cdot f^h.
\end{aligned}$$

□

By Proposition 5, given any affine variety $W=\mathbf{V}(g_1,\ldots,g_s)\subseteq k^n$, we can homogenize the defining equations of $W$ to obtain a projective variety $V=\mathbf{V}(g_1^h,\ldots,g_s^h)\subseteq\mathbb{P}^n(k)$. Moreover, by part (iii) of Proposition 5 and Proposition 4, we see that $V\cap U_0=W$. Thus, our original affine variety $W$ is the *affine portion* of the projective variety $V$.

We now will give a more geometric way to describe the construction of $\mathbb{P}^n(k)$. Let $\mathcal{L}$ denote the set of lines through the origin in $k^{n+1}$.

**Proposition 6.** *There is a one-to-one correspondence between $\mathbb{P}^n(k)$ and $\mathcal{L}$.*

**Proof.** Given *any* line $L$ through the origin in $k^{n+1}$, a vector $(x_0, \ldots, x_n)$ on $L \setminus \{0\}$ is a nonzero vector such that any other point on $L \setminus \{0\}$ is a nonzero multiple of $(x_0, \ldots, x_n)$. This shows that every element of $\mathcal{L}$ can be represented as the set of scalar multiples of some nonzero vector in $k^{n+1}$.

We identify a nonzero vector $(x_0, \ldots, x_n)$ in $k^{n+1}$ with the element of $\mathcal{L}$ defined by the scalar multiples of $(x_0, \ldots, x_n)$. Suppose that $v'$ and $v$ are nonzero vectors in $k^{n+1}$ that define the same element of $\mathcal{L}$. In particular, this line passes through $v'$ and $v$. This means that $v'$ is a scalar multiple of $v$, so that $v' \sim v$ as in Definition 3. Conversely, suppose that $v' \sim v$. Observe that this means that the vectors $v'$ and $v$ lie on the same line through the origin in $k^{n+1}$. The requirement in Definition 3 that $v', v \neq 0$ guarantees that $v'$ and $v$ are nonzero. Hence, $v'$ and $v$ are two nonzero vectors that define the same element of $\mathcal{L}$. This shows that two nonzero vectors $v'$ and $v$ in $k^{n+1}$ define the same element of $\mathcal{L}$ if and only if $v' \sim v$.

This shows that we have a one-to-one correspondence between $\mathbb{P}^n(k)$ and $\mathcal{L}$ by identifying a point $(x_0 : \cdots : x_n) \in \mathbb{P}^n(k)$ with the element of $\mathcal{L}$ defined by the scalar multiples of $(x_0, \ldots, x_n)$. □

Thus, projective space can be viewed as a space of lines rather than points. Concretely, each point in $\mathbb{P}^n(k)$ corresponds to a line in $k^{n+1}$.

A homogeneous polynomial $f \in k[x_0, \ldots, x_n]$ can also be used to define the *affine* variety $C = \mathbf{V}_a(f)$ in $k^{n+1}$, where the subscript denotes we are working in affine space. We call $C$ the *affine cone* over the projective variety $V = \mathbf{V}(f) \subseteq \mathbb{P}^n(k)$.

**Proposition 7.** *Suppose that a point $P \in k^{n+1} \setminus \{0\}$ gives homogeneous coordinates for a point $p \in \mathbb{P}^n(k)$. Then $p$ is in the projective variety $V$ if and only if the line through the origin determined by $P$ is contained in $C$.*

**Proof.** We prove the following lemma.

**Lemma 1.** *If $C$ contains the point $P \neq (0, \ldots, 0)$, then $C$ contains the whole line through the origin in $k^{n+1}$ spanned by $P$.*

**Proof of Lemma.** If $P \in C$, then $f(P) = 0$. Therefore, $f(\lambda P) = \lambda^d f(P) = 0$ where $d$ is degree of $f$, so $\lambda P \in C$ for all $\lambda \in k^{n+1}$. □

We continue the proof of the proposition as follows. Suppose that $p$ is in the projective variety $V$. Then *all* the homogeneous coordinates of $p$ lie in $C$. In particular, $P$ is in $C$. It follows from Lemma 1 that the line through the origin determined by $P$ is contained in $C$.

Conversely, suppose that the line through the origin determined by $P$ is contained in $C$. Then $f$ vanishes at $P$, which implies that $p$ is in the projective variety $V$. □

**Corollary 2.** *$C$ is the union of the collection of lines through the origin in $k^{n+1}$ corresponding to the points in $V$ via Proposition 6.*

**Proof.** This is an immediate consequence of Proposition 7 and Proposition 6. □

This explains the reason for the "cone" terminology since an ordinary cone is also a union of lines through the origin.

We will now consider the hyperplanes of $\mathbb{P}^n(k)$ in greater detail, starting with a useful result.

**Theorem 2.** *Two homogeneous linear polynomials,*

$$0 = a_0x_0 + \cdots + a_nx_n,$$
$$0 = b_0x_0 + \cdots + b_nx_n, \tag{3}$$

*define the same hyperplane in $\mathbb{P}^n(k)$ if and only if there is $\lambda \neq 0$ in $k$ such that $b_i = \lambda a_i$ for all $i = 0, \dots, n$.*

**Proof.** Suppose that there is $\lambda \neq 0$ in $k$ such that $b_i = \lambda a_i$ for all $i = 0, \dots, n$. Then (3) gives

$$\begin{aligned} 0 = a_0x_0 + \cdots + a_nx_n &\iff 0 = \lambda(a_0x_0 + \cdots + a_nx_n) \\ &\iff 0 = \lambda a_0x_0 + \cdots + \lambda a_nx_n \\ &\iff 0 = b_0x_0 + \cdots + b_nx_n, \end{aligned}$$

which shows that the two homogeneous linear polynomials define the same hyperplane in $\mathbb{P}^n(k)$.

Conversely, suppose that the homogeneous linear polynomials in (3) define the same hyperplane in $\mathbb{P}^n(k)$. Take $n$ distinct points $(v_0^{(i)} : \cdots : v_n^{(i)})$ for $i = 1, \dots, n$ on the hyperplane $0 = a_0x_0 + \cdots + a_nx_n$. Then the vectors $(v_0^{(i)}, \dots, v_n^{(i)})$ are linearly independent [since otherwise we would have $(v_0^{(i)} : \cdots : v_n^{(i)}) = (v_0^{(j)} : \cdots : v_n^{(j)})$ for some $i \neq j$]. It follows from linear algebra that the system of equations $c_0v_0^{(1)} + \cdots + c_nv_n^{(1)} = \cdots = c_0v_0^{(n)} + \cdots + c_nv_n^{(n)} = 0$ have a 1-dimensional solution space for the variables $c_0, \dots, c_n$. Hence, there is $\lambda \neq 0$ in $k$ such that $b_i = \lambda a_i$ for all $i = 0, \dots, n$. □

Since hyperplanes in projective space are uniquely determined by their defining homogeneous linear polynomials up to scalar multiples, we can make the following identification.

**Corollary 3.** *The map sending the hyperplane with equation $a_0x_0 + \cdots + a_nx_n = 0$ to the vector $(a_0, \dots, a_n)$ gives a one-to-one correspondence*

$$\phi \colon \{\text{hyperplanes in } \mathbb{P}^n(k)\} \to (k^{n+1} \setminus \{0\})/\sim,$$

*where $\sim$ is the equivalence relation in Definition 3.*

**Proof.** Suppose that $(b_0, \dots, b_n) \sim (a_0, \dots, a_n)$, where $\sim$ is the equivalence relation in Definition 3. This means that $(b_0, \dots, b_n) = \lambda(a_0, \dots, a_n)$ for some $\lambda \in k\setminus\{0\}$. Hence, $b_i = \lambda a_i$ for all $i = 0, \dots, n$. It follows from Theorem 2 that the equations $a_0x_0 + \cdots + a_nx_n = 0$ and $b_0x_0 + \cdots + b_nx_n = 0$ define the same hyperplane in $\mathbb{P}^n(k)$. This proves that the map is injective.

The equivalence class $(c_0 : \cdots : c_n)$ of any $(n+1)$-tuple $(c_0, \dots, c_n) \in k^{n+1} \setminus \{0\}$ is the image of the hyperplane defined by $c_0x_0 + \cdots + c_nx_n = 0$. This proves that the map is surjective. □

The set on the left is called the *dual projective space* and is denoted $\mathbb{P}^n(k)^\vee$. Geometrically, the points of $\mathbb{P}^n(k)^\vee$ are hyperplanes in $\mathbb{P}^n(k)$. We finish this section by describing a subset of the dual projective space.

**Example 1.** We aim to describe the subset of $\mathbb{P}^n(k)^\vee$ corresponding to the hyperplanes containing $p = (1 : 0 : \cdots : 0)$. If a hyperplane with equation $a_0x_0 + \cdots + a_nx_n = 0$ contains $p = (1 : 0 : \cdots : 0)$, then it must have $a_0 = 0$. By Corollary 3, there is a one-to-one correspondence between this subset of hyperplanes and the set $H = \{p \in \mathbb{P}^n(k) \mid p = (0 : a_1 : \cdots : a_n)\}$. This exactly the hyperplane at infinity that we have seen in (2), which we described in the following remarks as $\mathbb{P}^{n-1}(k)$, the projective space of one smaller dimension.

By homogenizing polynomials, we were able to extend the notion of varieties to projective space, which allowed us to define projective varieties as solutions to systems of homogeneous polynomial equations. We take the time now to add to our Algebra-Geometry Dictionary, so that we can translate between our new algebraic and geometric concepts.

# 4 The Projective Algebra-Geometry Dictionary

We begin by defining a special class of ideals in $k[x_0, \ldots, x_n]$.

**Definition 5.** An ideal $I$ in $k[x_0, \ldots, x_n]$ is said to be **homogeneous** if for each $f \in I$, the homogeneous components $f_i$ of $f$ are in $I$ as well.

We have the following equivalent conditions for when an ideal $I$ in $k[x_0, \ldots, x_n]$ is homogeneous.

**Theorem 3.** *Let $I \subseteq k[x_0, \ldots, x_n]$ be an ideal. Then the following are equivalent:*

(i) *$I$ is a homogeneous ideal of $k[x_0, \ldots, x_n]$.*

(ii) *$I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s$ are homogeneous polynomials.*

(iii) *A reduced Gröbner basis of $I$ (with respect to any monomial ordering) consists of homogeneous polynomials.*

To get the hang of homogeneous ideals, we prove the following simple result.

**Proposition 8.** *If $I$ and $J$ are homogeneous ideals in $k[x_0, \ldots, x_n]$, then $I \cap J$ is also a homogeneous ideal.*

**Proof.** Suppose that $f$ is in $I \cap J$. Since $I$ and $J$ are homogeneous, the homogeneous components of $f$ are in $I$ and in $J$. Therefore, the homogeneous components of $f$ are in $I \cap J$, and hence $I \cap J$ is a homogeneous ideal. $\square$

In fact, the same argument as above proves the proposition for an arbitrary intersection of homogeneous ideals. It also turns out that the radical of a homogeneous ideal is also homogeneous.

Similar to the affine case, for any homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$ we may define

$$\mathbf{V}(I) = \{p \in \mathbb{P}^n(k) \mid f(p) = 0 \text{ for all } f \in I\},$$

as in the affine case. The next result is exactly as we would expect.

**Proposition 9.** *Let $I \subseteq k[x_0, \ldots, x_n]$ be a homogeneous ideal and suppose that $I = \langle f_1, \ldots, f_s \rangle$, where $f_1, \ldots, f_s$ are homogeneous. Then*

$$\mathbf{V}(I) = \mathbf{V}(f_1, \ldots, f_s),$$

*so that $\mathbf{V}(I)$ is a projective variety.*

**Proof.** Suppose that $p \in \mathbf{V}(I)$. Then $f(p) = 0$ for all $f \in I$. Since $I$ is generated by $f_1, \ldots, f_s$, we have $f_1(p) = \cdots = f_s(p) = 0$. Therefore, $p$ is in $\mathbf{V}(f_1, \ldots, f_s)$.

Conversely, suppose that $p \in \mathbf{V}(f_1, \ldots, f_s)$. Then $f_1(p) = \cdots = f_s(p) = 0$. We need to show that for any $f \in I$, $f(p) = 0$. Since $I$ is generated by $f_1, \ldots, f_s$, we have $f = h_1 f_1 + \cdots + h_s f_s$ for some polynomials $h_1, \ldots, h_s$. Then $f(p) = h_1(p) f_1(p) + \cdots + h_s(p) f_s(p) = 0$. Therefore, $p$ is in $V(I)$. $\square$

Thus, homogeneous ideals are precisely the ideals that correspond to projective varieties. We now look at some of the properties of projective varieties.

**Proposition 10.** *We have that:*

(i) *$\emptyset$ and $\mathbb{P}^n(k)$ are projective varieties.*

(ii) *If $V_1, \ldots, V_n$ are projective varieties, then $\bigcup_i V_i$ is a projective variety.*

(iii) *If $\{V_\alpha\}$ is an indexed family of projective varieties, then $\bigcap_\alpha V_\alpha$ is a projective variety.*

**Proof.** (i) $\mathbf{V}(1) = \emptyset$ and $\mathbf{V}(0) = \mathbb{P}^n(k)$.

(ii) We will show that if $V, W \subseteq \mathbb{P}^n(k)$ are projective varieties, then so is $V \cup W$. This will imply the result. We claim that $V \cup W = \mathbf{V}(F_iG_j \mid 1 \leq i \leq s, 1 \leq j \leq t)$, which is a projective variety since each $F_iG_j$ is homogeneous. The proof is the same as in the affine case.

(iii) Let $P_\alpha$ be the set of polynomials that defines the projective variety $V_a$. We claim that $\bigcap_\alpha V_\alpha = \mathbf{V}(I)$, where $I$ is the homogeneous ideal generated by $\bigcup_\alpha P_\alpha$. Suppose that $p \in \bigcap_\alpha V_\alpha$. Then all the polynomials in $\bigcup_\alpha P_\alpha$ vanish at $p$. Since any element in $I$ can be written as a finite linear combination of polynomials in $\bigcup_\alpha P_\alpha$, every element in $I$ vanishes at $p$. This shows that $p \in \mathbf{V}(I)$. Conversely, suppose that $p \in \mathbf{V}(I)$. Then since $I$ is generated by $\bigcup_\alpha P_\alpha$, all the polynomials in $\bigcup_\alpha P_\alpha$ vanish at $p$. Thus $p \in \bigcap_\alpha V_\alpha$. That $I$ is a homogeneous ideal can be deduced from the fact that it is generated by homogeneous polynomials. We will see a proof of this fact shortly in Proposition 13. Hence, $\bigcap_\alpha V_\alpha$ is a projective variety.

□

From Proposition 4, we conclude that the projective varieties in $\mathbb{P}^n(k)$ can be taken as the closed sets of a topology on $\mathbb{P}^n(k)$. This topology is called the *Zariski topology* on $\mathbb{P}^n(k)$.

Projective varieties also give rise to homogeneous ideals, as we see via the following proposition.

**Proposition 11.** *Let $V \subseteq \mathbb{P}^n(k)$ be a projective variety and let*

$$\mathbf{I}(V) = \{f \in k[x_0, \ldots, x_n] \mid f(x_0, \ldots, x_n) = 0 \text{ for all } (a_0 : \cdots : a_n) \in V\}.$$

*(This means that $f$ must be zero for all homogeneous coordinates of all $p$.)*
*If $k$ is infinite, then $\mathbf{I}(V)$ is a homogeneous ideal in $k[x_0, \ldots, x_n]$.*

This allows us to establish a correspondence between homogeneous ideals and projective varieties, which becomes one-to-one if we restrict to nonempty projective varieties and certain proper radical homogeneous ideals. Note the similarity with the ideal-variety correspondence in affine space.

**Theorem 4.** *Let $k$ be an infinite field.*

(i) *The maps*

$$\text{projective varieties} \xrightarrow{\mathbf{I}} \text{homogeneous ideals}$$

*and*

$$\text{homogeneous ideals} \xrightarrow{\mathbf{V}} \text{projective varieties}$$

*are inclusion-reversing.*

(ii) *For any projective variety $V$, we have*

$$\mathbf{V}(\mathbf{I}(V)) = V,$$

*so that $\mathbf{I}$ is always one-to-one.*

(iii) *If $k$ is algebraically closed, and if we restrict to nonempty projective varieties and radical homogeneous ideals properly contained in $\langle x_0, \ldots, x_n \rangle$, then the maps*

$$\{\text{nonempty projective varieties}\} \xrightarrow{\mathbf{I}} \left\{ \begin{array}{c} \text{radical homogeneous ideals} \\ \text{properly contained in } \langle x_0, \ldots, x_n \rangle \end{array} \right\}$$

*and*

$$\left\{ \begin{array}{c} \text{radical homogeneous ideals} \\ \text{properly contained in } \langle x_0, \ldots, x_n \rangle \end{array} \right\} \xrightarrow{\mathbf{V}} \{\text{nonempty projective varieties}\}$$

*are inclusion-reversing bijections which are inverses of each other.*

The final results of this section are projective versions of the Nullstellensatz. Unlike in the affine case, it is not true in general that if $k$ is an algebraically closed field and $\mathbf{V}(I) = \emptyset$ is a projective variety, then $I = k[x_0, \ldots, x_n]$. For example, the ideal $\langle x_0, \ldots, x_n \rangle$ defines an empty projective variety (since we do not allow homogeneous coordinates to all vanish simultaneously), but does not contain the nonzero constant polynomials. Thus, we need a modified version of the Weak Nullstellensatz.

**Theorem 5 (The Projective Weak Nullstellensatz).** *Let $k$ be an algebraically closed field and let $I$ be a homogeneous ideal in $k[x_0, \ldots, x_n]$. Then the following are equivalent:*

(i) $\mathbf{V}(I) \subseteq \mathbb{P}^n(k)$ *is empty.*

(ii) *Let $G$ be a reduced Gröbner basis for $I$ (with respect to some monomial ordering). Then for each $0 \leq i \leq n$, there is $g \in G$ such that $\mathrm{LT}(g)$ is a nonnegative power of $x_i$.*

(iii) *For each $1 \leq i \leq n$, there is an integer $m_i \geq 0$ such that $x_i^{m_i} \in I$.*

(iv) *There is some $r \geq 1$ such that $\langle x_0, \ldots, x_n \rangle^r \subseteq I$.*

(v) $I : \langle x_0, \ldots, x_n \rangle^\infty = k[x_1, \ldots, x_n]$.

It follows that another equivalent condition for the Weak Nullstellensatz is that $\sqrt{I} = \langle x_0, \ldots, x_n \rangle$. Observing this, the Strong Nullstellensatz takes the same form as it does in the affine case.

**Theorem 6 (The Projective Strong Nullstellensatz).** *Let $k$ be an algebraically closed field and let $I$ be a homogeneous ideal in $k[x_0, \ldots, x_n]$. If $V = \mathbf{V}(I)$ is a nonempty projective variety $\mathbb{P}^n(k)$, then we have*

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

We finish the section with a generalization of a concept seen in affine space. Whether or not a subset $S \subseteq k^n$ is an affine variety, the set

$$\mathbf{I}(S) = \{f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in S\}$$

is an ideal in $k[x_1, \ldots, x_n]$. In fact, it is radical. By the affine ideal-variety correspondence, $\mathbf{V}(\mathbf{I}(S))$ is an affine variety. Proposition 1 of Chapter 4, §4 in [1] proves that this variety is the smallest affine variety that contains the set $S$. We seek to extend this notion into projective space. Whether or not a subset $S \subseteq \mathbb{P}^n(k)$ is a projective variety, the set

$$\mathbf{I}(S) = \{f \in k[x_0, \ldots, x_n] \mid f(a_0, \ldots, a_n) = 0 \text{ for all } (a_0 : \cdots : a_n) \in S\}$$

is an ideal in $k[x_0, \ldots, x_n]$. In fact, it is a radical homogeneous ideal. By Theorem 4, $\mathbf{V}(\mathbf{I}(S))$ is a projective variety. The following proposition states that this projective variety is the smallest projective variety that contains the set $S$.

**Proposition 12.** *If $S \subseteq \mathbb{P}^n(k)$, the affine variety $\mathbf{V}(\mathbf{I}(S))$ is the smallest projective variety that contains $S$ [in the sense that if $W \subseteq \mathbb{P}^n(k)$ is any projective variety containing $S$, then $\mathbf{V}(\mathbf{I}(S)) \subseteq W$].*

**Proof.** If $W \supseteq S$, then $\mathbf{I}(W) \subseteq \mathbf{I}(S)$ because $\mathbf{I}$ is inclusion-reversing. But then $\mathbf{V}(\mathbf{I}(W)) \supseteq \mathbf{V}(\mathbf{I}(S))$ because $\mathbf{V}$ also reverses inclusions. Since $W$ is a projective variety, $\mathbf{V}(\mathbf{I}(W)) = W$ by Theorem 4, and the result follows. □

Given the fact that we can embed affine varieties into projective varieties via homogenization, it is natural to ask whether we can find the smallest projective variety that contains an affine variety. We will resolve this matter in the next section.

# 5 The Projective Closure of an Affine Variety

There is a way to obtain a homogeneous ideal from a given ideal.

**Definition 6.** Let $I$ be an ideal in $k[x_1,\ldots,x_n]$. We define the **homogenization of $I$** to be the ideal

$$I^h = \langle f^h \mid f \in I\rangle \subseteq k[x_0,\ldots,x_n],$$

where $f^h$ is the homogenization of $f$ as in Proposition 5.

We now show that the homogenization of an ideal has the property we want.

**Proposition 13.** *For any ideal $I \subseteq k[x_1,\ldots,x_n]$, the homogenization $I^h$ is a homogeneous ideal in $k[x_0,\ldots,x_n]$.*

**Proof.** Suppose that $f \in I^h$. Then

$$f = \sum_{i=1}^{s} A_i f_i^h \tag{4}$$

for some $A_i \in k[x_1,\ldots,x_n]$. Suppose we expand each $A_i$ as the sum of its homogeneous components, where each $A_{ij}$ has total degree $j$:

$$A_i = \sum_{j=1}^{d} A_{ij}.$$

If we substitute these expressions into (4) and collect terms of the same total degree, we obtain an expansion of $f$ into its homogeneous components

$$f = \sum_{\ell} \sum_{j+\deg(f_i^h)=\ell} A_{ij} f_i^h,$$

from which it is clear that the homogeneous components $\sum_{j+\deg(f_i^h)=\ell} A_{ij} f_i^h$ lie in the ideal $I^h$. $\square$

The proof for Proposition 13 actually proves the stronger statement that any ideal generated by homogeneous polynomials is homogeneous, which was used in the proof of Proposition 4.

To obtain generators for the homogenized ideal, we introduce a type of monomial order. A graded monomial order in $k[x_1,\ldots,x_n]$ is one that orders first by total degree:

$$x^\alpha > x^\beta$$

whenever $|\alpha| > |\beta|$. Note that grlex and grevlex are graded orders, whereas lex is not. This gives us the following result.

**Theorem 7.** *Let $I$ be an ideal in $k[x_1,\ldots,x_n]$ and let $G = \{g_1,\ldots,g_t\}$ be a Gröbner basis for $I$ with respect to a graded monomial order in $k[x_1,\ldots,x_n]$. Then $G^h = \{g_1^h,\ldots,g_t^h\}$ is a Gröbner basis for $I^h \subseteq k[x_0,\ldots,x_n]$.*

Thus, given a Gröbner basis for an ideal, we get a Gröbner basis of the homogenized ideal by homogenizing each element in the basis. We arrive at the crux of the section.

**Definition 7.** Given an affine variety $W \subseteq k^n$, the **projective closure** of $W$ is the projective variety $\overline{W} = \mathbf{V}(\mathbf{I}_a(W)^h) \subseteq \mathbb{P}^n(k)$, where $\mathbf{I}_a(W)^h \subseteq k[x_0,\ldots,x_n]$ is the homogenization of ideal $\mathbf{I}_a(W) \subseteq k[x_1,\ldots,x_n]$ as in Definition 6.

In other words, the projective closure is defined by homogenizing the ideal of the affine variety and taking the projective variety defined by the homogenized ideal. The projective closure has the following important properties.

**Proposition 14.** *Let $W \subseteq k^n$ be an affine variety and let $\overline{W} \subseteq \mathbb{P}^n(k)$ be its projective closure. Then*

(i) $\overline{W} \cap U_0 = \overline{W} \cap k^n = W$.

(ii) *$\overline{W}$ is the smallest projective variety in $\mathbb{P}^n(k)$ containing $W$.*

(iii) *If $W$ is irreducible, then so it $\overline{W}$.*

(iv) *No irreducible component of $\overline{W}$ lies on the hyperplane at infinity $\mathbf{V}(x_0) \subseteq \mathbb{P}^n(k)$.*

**Proof.** We prove part (iii) as the remaining proofs can be found in [1].

Suppose that $W$ is irreducible. Assume, to the contrary, that $\overline{W}$ is reducible. Then $\overline{W} = W_1 \cup W_2$ for strictly smaller projective varieties $W_1$ and $W_2$. This means that $\overline{W} \not\subseteq W_1$ and $\overline{W} \not\subseteq W_2$. Hence $W = W \cap \overline{W} = W \cap (W_1 \cup W_2) = (W \cap W_1) \cup (W \cap W_2)$. Since $\overline{W} \not\subseteq W_1$ and $\overline{W} \not\subseteq W_2$, we have $W \cap W_1 \subsetneq W$ and $W \cap W_2 \subsetneq W$. This contradicts our assumption that $W$ is irreducible. Therefore, $\overline{W}$ must be irreducible. □

The following theorem allows us to compute the projective closure from the ideal defining the affine variety.

**Theorem 8.** *Let $k$ be an algebraically closed field, and let $I \subseteq k[x_1, \ldots, x_n]$ be an ideal. Then $\mathbf{V}(I^h) \subseteq \mathbb{P}^n(k)$ is the projective closure of $\mathbf{V}_a(I) \subseteq k^n$.*

The vanishing ideal of the projective closure is given by the following proposition, which adapts the proof of part (ii) of Proposition 14 given in [1].

**Proposition 15.** $\mathbf{I}(\overline{W}) = \mathbf{I}_a(W)^h$ *for any affine variety $W \subseteq k^n$.*

**Proof.** Suppose that $F \in \mathbf{I}(\overline{W})$. We consider the case where $F$ is homogeneous. Then $F$ vanishes on $\overline{W}$, so that its dehomogenization $f = F(1, x_1, \ldots, x_n)$ vanishes on $W$ by part (i) of Proposition 14 and Proposition 4. Thus, $f \in \mathbf{I}_a(W)$ and, hence $f^h \in \mathbf{I}_a(W)^h$. But part (iv) of Proposition 5 implies that $F = x_0^e f^h$ for some integer $e$. Thus $F \in \mathbf{I}_a(W)^h$. Since every polynomial can be expanded as a sum of its homogeneous components and ideals are closed under sums, this proves the inclusion.

Conversely, suppose that $F \in \mathbf{I}_a(W)^h$. Then $F = \sum_{i=1}^{s} h_i f_i^h$ where each $f_i$ vanishes on $W$. It follows from Proposition 5 that each $f_i^h$ vanishes on $\overline{W}$. Hence $F \in \mathbf{I}(\overline{W})$. □

Hence, the vanishing ideal of $\overline{W}$ is $\mathbf{I}_a(W)^h$. We conclude the section with our first and only algorithmic method in projective space. If we combine Theorem 7 and Theorem 8, we get an **algorithm for computing the projective closure of an affine variety** over an algebraically closed field $k$: given $W \subseteq k^n$ defined by $f_1 = \cdots = f_s = 0$, compute a Gröbner basis $G$ of $\langle f_1, \ldots, f_s \rangle$ with respect to a graded order, and then the projective closure in $\mathbb{P}^n(k)$ is defined by $g^h = 0$ for $g \in G$. It is important to note that this algorithm only holds for algebraically closed fields $k$, since Theorem 8 only holds for algebraically closed fields. A simple implementation of the algorithm in Sage is shown below. To find the projective closure of the affine twisted cubic $\mathbf{V}(y - x^2, z - x^3) \subseteq \mathbb{C}^3$, we compute:

```
sage: R.<x,y,z> = PolynomialRing(CC,order='degrevlex')
sage: I = Ideal(y - x^2, z - x^3)
sage: [f.homogenize() for f in I.groebner_basis()]
[x^2 - y*h, x*y - z*h, y^2 - x*z]
```

and conclude that $\overline{W} = \mathbb{V}(x^2 - yh, xy - zh, y^2 - xz)$.

# 6 Conclusion

Projective algebraic geometry is a powerful tool for studying varieties. By considering points at infinity, it allows us to fill in the gaps in affine geometry and to study the global properties of varieties. The projective plane and, more generally, projective space, give rise to the theory of projective varieties. The projective algebra-geometry dictionary is then a key tool for translating between the algebraic and geometric aspects of projective varieties. The projective closure of an affine variety is an important construction that allows us to extend the affine variety to a projective variety.

Further study could involve exploring Grassmanians as projective varieties, using Sage to compute the hyperplanes passing through a point for a finite field, relating the projective variety-radical homogeneous ideal correspondence to homogeneous coordinate rings, and examining the Zariski topology for properties such as quasi-compactness.

Projective algebraic geometry is a rich and fascinating subject with numerous applications in mathematics and beyond. It continues to be an active area of research, with many open questions and exciting developments that intersect with many other areas of mathematics.

# References

[1] D. Cox, J. Little, D. O'Shea, *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, 4th edn. (Springer, New York, 2015)