Computer Algorithms

Unit-3

Dynamic Programming

The General Method

- An algorithm design method that can be used
- When the solution to a problem can be viewed
- as the result of a sequence of decisions.
- Step wise decisions can be made for the problem like – knapsack problem, job sequencing with deadline, optimal merge pattern etc.
- Which can be solved using Greedy Approach

The General Method

- For some problems it is not possible to make a sequence of stepwise decisions, which will give optimal decision sequence.
- Solution to such problems is – to try all possible decision sequences
- Enumerate all decision sequences and then pick out best.
- But the time and space requirement may be prohibitive.

The General Method

- Dynamic Programming drastically reduces the amount of time and space required.
- It avoids the enumeration of some decision sequences , that cannot possibly be optimal

Difference between Greedy Method and Dynamic Programming

- In Greedy Method, only one decision sequence is even generated.
- In Dynamic Programming, many decision sequences can be generated
- But sequences containing suboptimal subsequences can not be optimal
- And so will not be generated

Multistage Graph

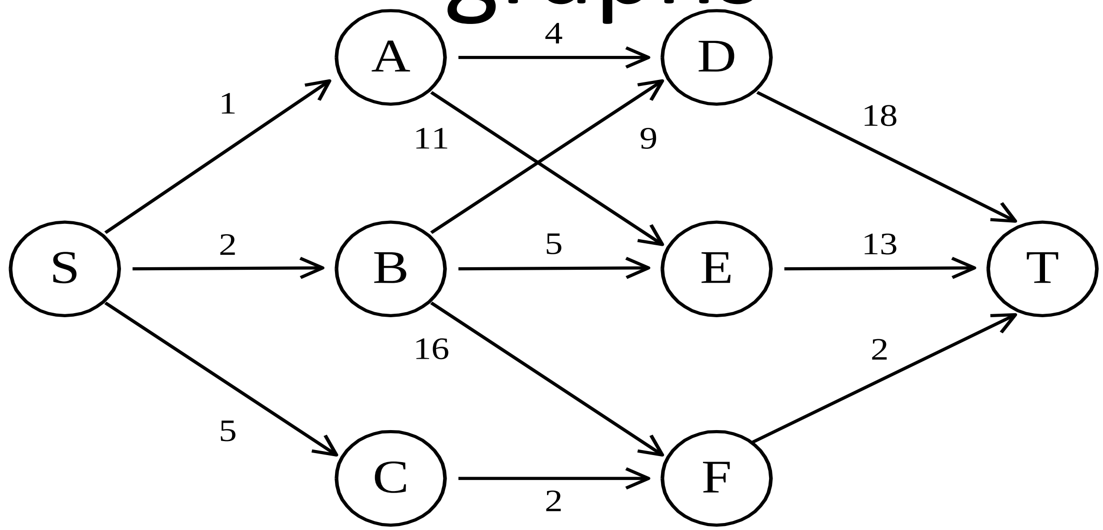
- A multistage graph G=(V, E) is a directed graph, in which
- The vertices are partitioned into $k \geq 2$ disjoint sets $V_{i,}$ $1 \leq i < k$
- Sets V_1 and V_k are such that $|V_1| = |V_k| = 1$
- Let s and t be the vertices in V_1 and V_k respectively.
- s is source and t is sink (destination)

Multistage Graph

- Let c(i, j) be the cost of edge (i, j)
- The cost of a path from s to t is the sum of the costs of all the edges on the path
- The multistage graph problem is to find a minimum-cost path from s to t
- Each stage V_i defines a stage in the graph
- Constraint is, every path from s to t starts in stage 1, goes to stage 2 and so on
- And terminates in k stage

The shortest path in multistage graphs

- e.g.



- (S, A, D, T) 1+4+18 = 23.
- The real shortest path is:

  (S, C, F, T) 5+2+2 = 9.

Multistage Graph

- Dynamic Programming Approach
- Such problems can be solved using 2 approaches
- Forward Approach
  - Backward Reasoning
- Backward Approach
  - Forward Reasoning

Multistage Graph- Example

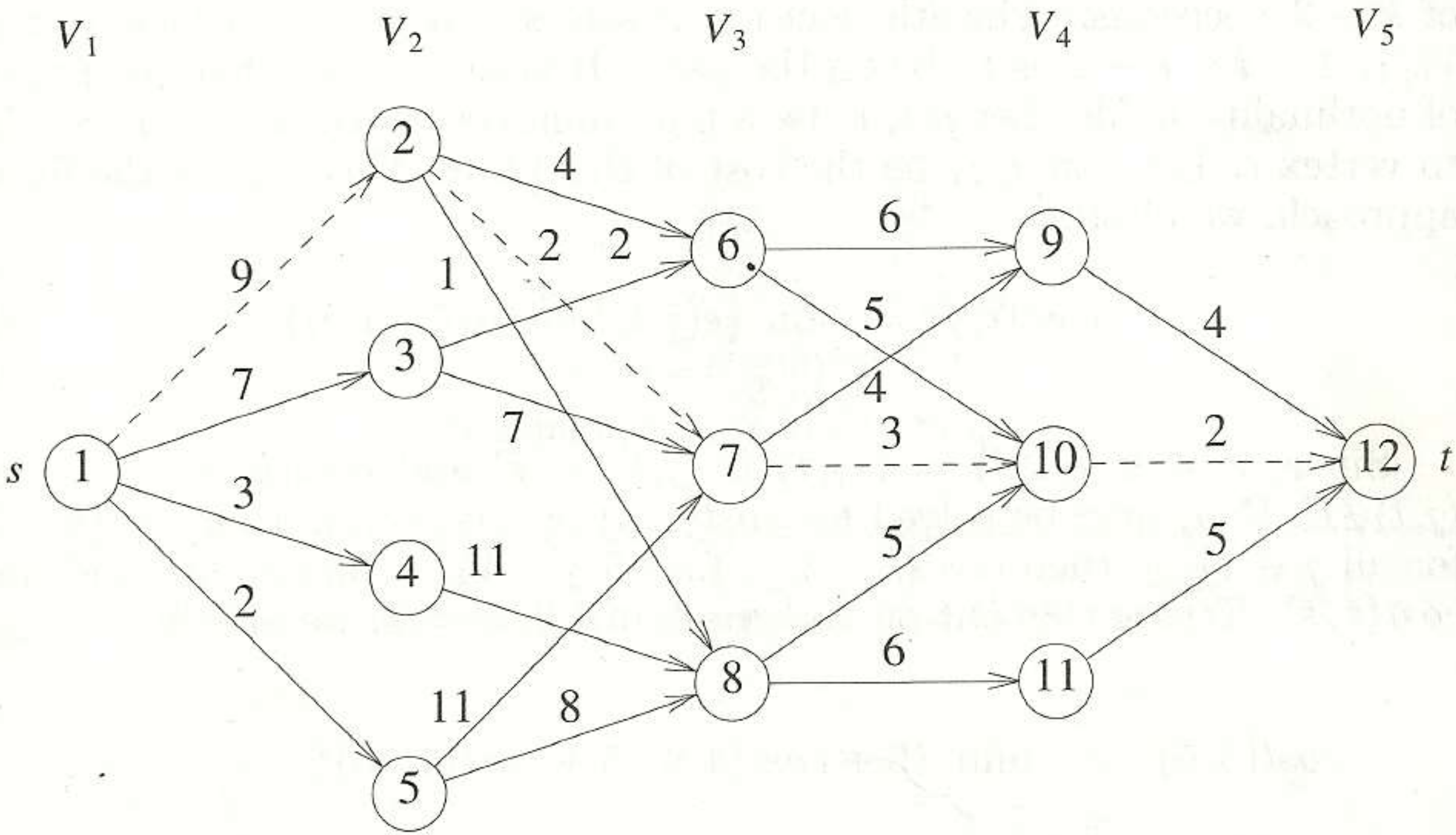


Figure 5.2 Five-stage graph

Multistage Graph- Example

- Every s to t path is the result of a sequence of k-2 decisions
- i^{th} decision involves determining which vertex in V_{i+1} is to be on the path
- cost(i, j)=min {c(j, l) + cost (i+1, l)}
- Cost(1,1)= min{9+cost(2,2), 7+cost(2,3), 3+cost(2,4), 2+cost(2,5)}
- Forward Approach - backward reasoning.
- Calculate Distance from Target as reference

Multistage Graph- Example

- Cost(1,1)= min{9+cost(2,2), 7+cost(2,3), 3+cost(2,4), 2+cost(2,5)}
- Cost(4,9)=c(9,12)+cost(5,12)

  =4+0 = 4
- Cost(4,10)=c(10,12)+cost(5,12)

  =2+0 = 2
- Cost(4,11)=c(11,12)+ cost(5,12)

  =5+0 =5

Multistage Graph- Example

- cost(3,6) = min{6+cost(4,9), 5+cost(4,10)}

  =min(6+4, 5+2) = min(10,7)=7

- cost(3,7) = min{4+cost(4,9), 3+cost(4,10)}

  =min(4+4, 3+2)=min(8,5)=5

- cost(3,8) = min{5+cost(4,10), 6+cost(4,11)}

  =mint(5+2, 6+5)=min(7, 11)=7

Multistage Graph- Example

- cost(2,2)

  =min{4+cost(3,6), 2+cost(3,7), 1+cost(3,8)}

  = min(4+7, 2+5, 1+7) = 7

- cost(2,3)

  =min{2+cost(3,6), 7+cost(3, 7)}

  =min(2+7, 7+5)= 9

Multistage Graph- Example

- cost(2,4)

  =min{11+cost(3,8)}

  =min(11+7)=18

- cost(2,5)

  =mint{11+cost(3,7), 8+cost(3,8)}

  =min(11+5, 8+7)=15

Multistage Graph- Example

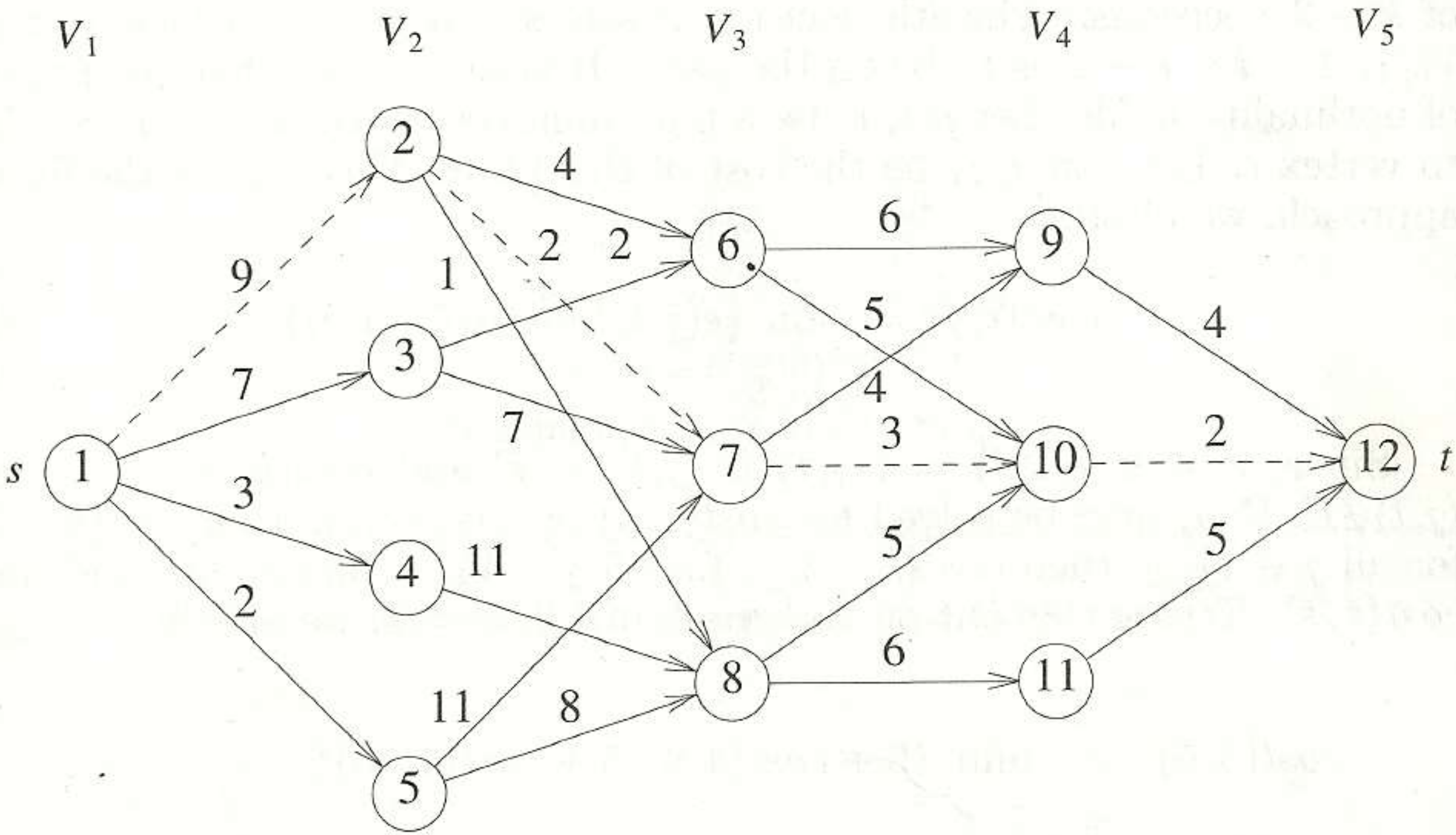


Figure 5.2 Five-stage graph

Multistage Graph- Example

- Cost(1,1)

= min{9+cost(2,2), 7+cost(2,3), 3+cost(2,4), 2+cost(2,5)}

=min(9+7, 7+9, 3+18, 2+15)

=16

- A minimum cost s to t path has a cost 16
- This path can be determined easily if we record the decision made at each state (vertex)

Multistage Graph- Example

- Let d(i, j) be the value of l (l is a node in next level) that minimizes {c(j, l) + cost (i+1, l)}

- d(3,6)=10 d(3,7)=10 d(3,8)=10
- d(2,2)=7 d(2,3)=6 d(2,4)=8
- d(2,5)=8 d(1,1)=2 or 3

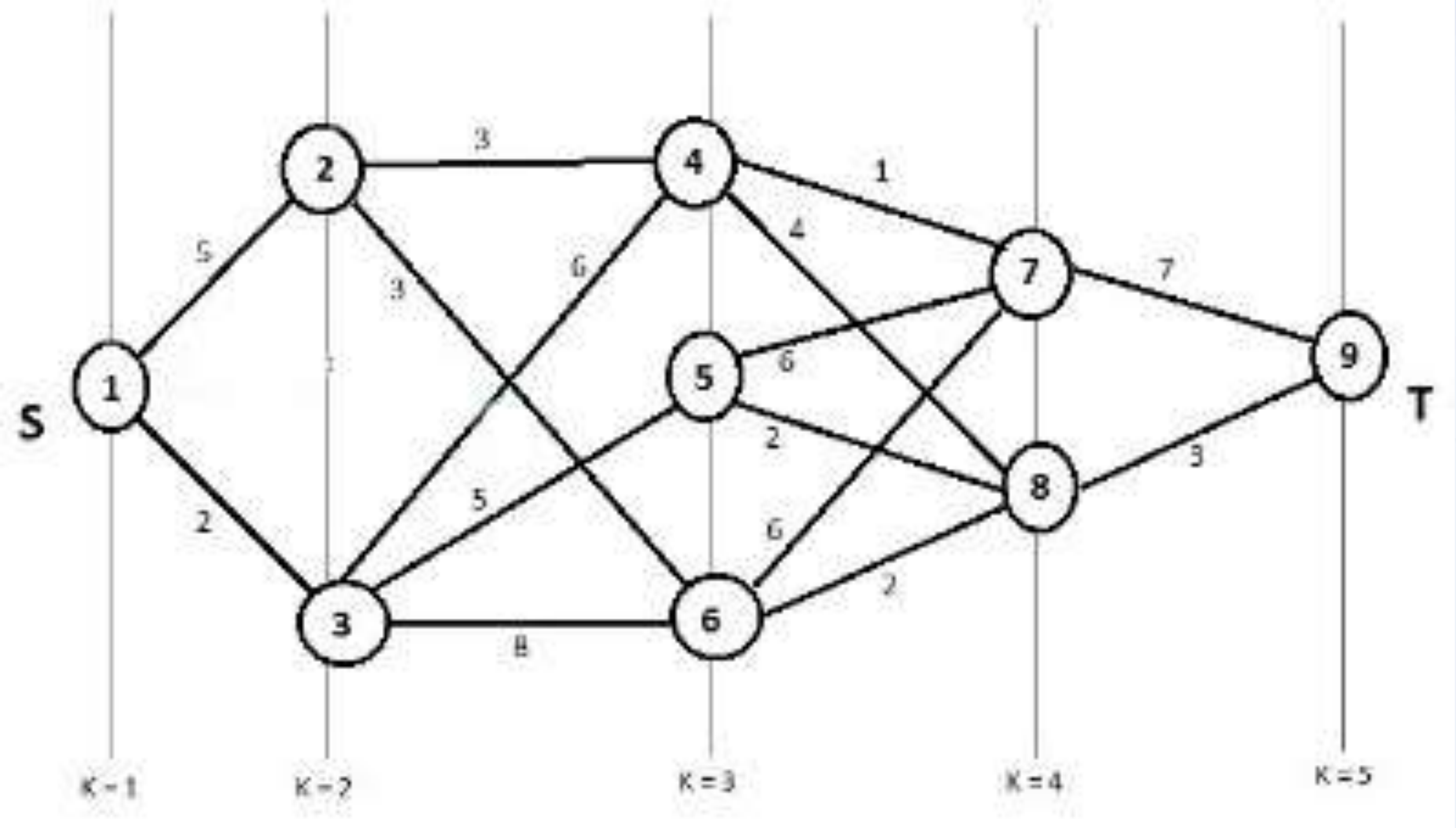
Multistage Graph- Example

- Let the minimum-cost path be
- $s, v_2, v_3, ..v_{k-1}, t$
- $V_2=d(1,1)=2$ $v_3=d(2,2)=7$
- $v_4=d(3,7)=10$
- So the path is 1, 2, 7, 10, 12
- Cost= 16

Multistage Graph- Example

- Let the minimum-cost path be
- $s, v_2, v_3, \ldots v_{k-1}, t$
- $V_2 = d(1,1) = 3$ $v_3 = d(2,3) = 6$
- $v_4 = d(3,6) = 10$
- So the path is 1, 3, 6, 10, 12
- Cost= 16

Multistage Graph- Example



Multistage Graph- Backward Approach

- backward Approach - Forward reasoning.
- Calculate Distance from Source as reference
- bcost(i, j) = min(bcost(i-1, l)+c(l, j))
- bcost(5,12)= min{4+cost(4,9), 2+cost(4,10), 5+cost(4,11)}

Multistage Graph- Example

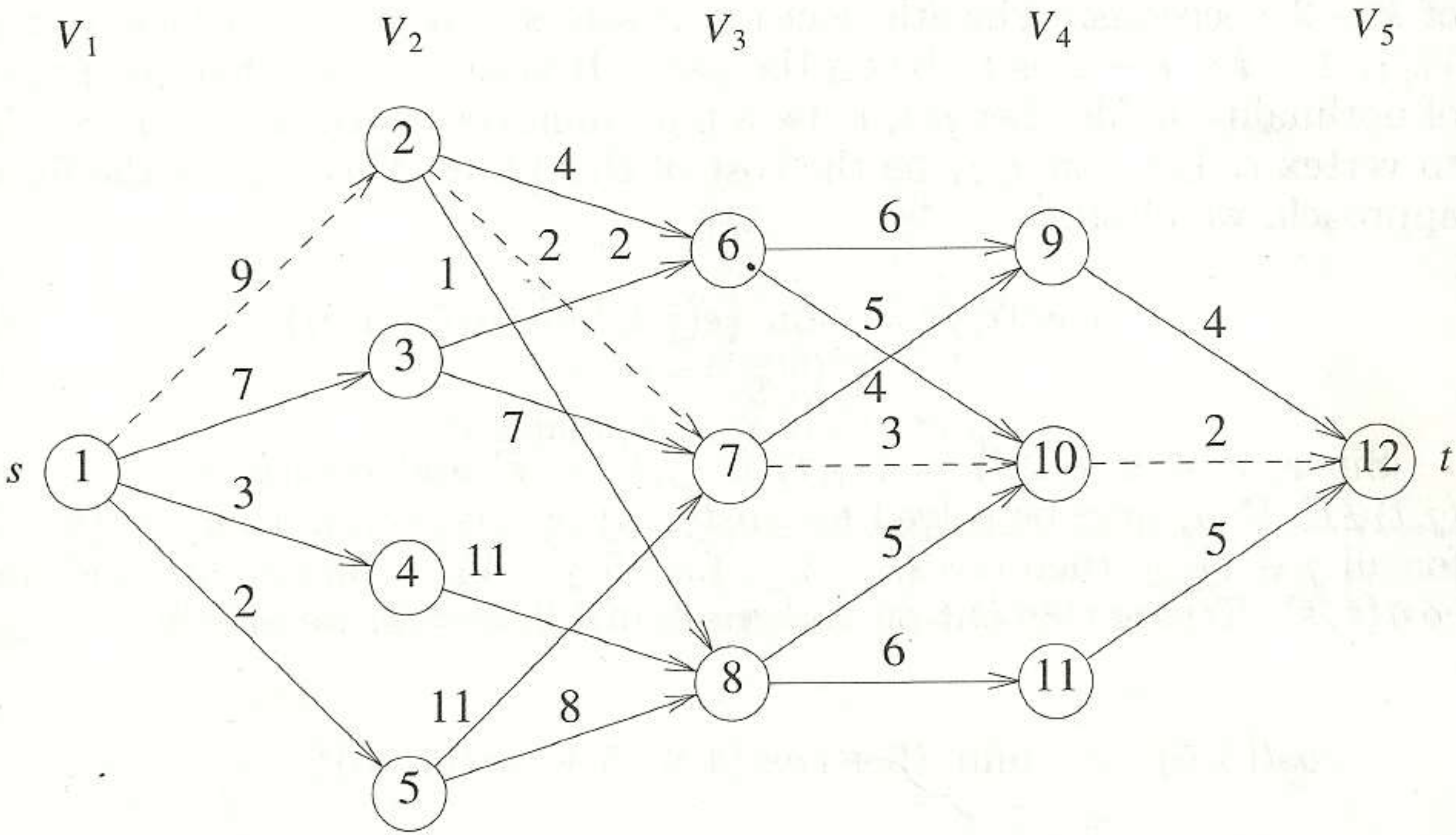


Figure 5.2 Five-stage graph

Multistage Graph- Backward Approach

- bcost(i, j) = min(bcost(i-1, l)+c(l, j))
- bcost(2, 2)=9
- bcost(2, 3)=7
- bcost(2, 4)=3
- bcost(2, 5)=2
- bcost(3, 6)= min{ bcost(2, 2)+c(2, 6)

  bcost(2, 3)+c(3, 6)}

  =min{9+4, 7+2} = 9

Multistage Graph- Backward Approach

- bcost(3, 7)=11
- bcost(3, 8)=10
- bcost(4, 9)=15
- bcost(4, 10)=14
- bcost(4, 11)=16
- bcost(5, 12)=16
-

Multistage Graph

- Example-
- Consider a resource allocation problem
- n units of resource are to be allocated to r projects
- If j, $0 \le j \le n$, units of the resource are allocated to project i,
- Then the resulting net profit is N(i, j)

Resource allocation problem

- The problem is to allocate the resource to the r projects
- in such a way as to maximize total net profit
- This problem can be formulated as an r+1 stage graph problem
- Stage i, $1 \le i \le r$ represents project i
- There are n+1 vertices V(i, j), $0 \le j \le n$ associated with stage i, $2 \le i \le r$

Resource allocation problem

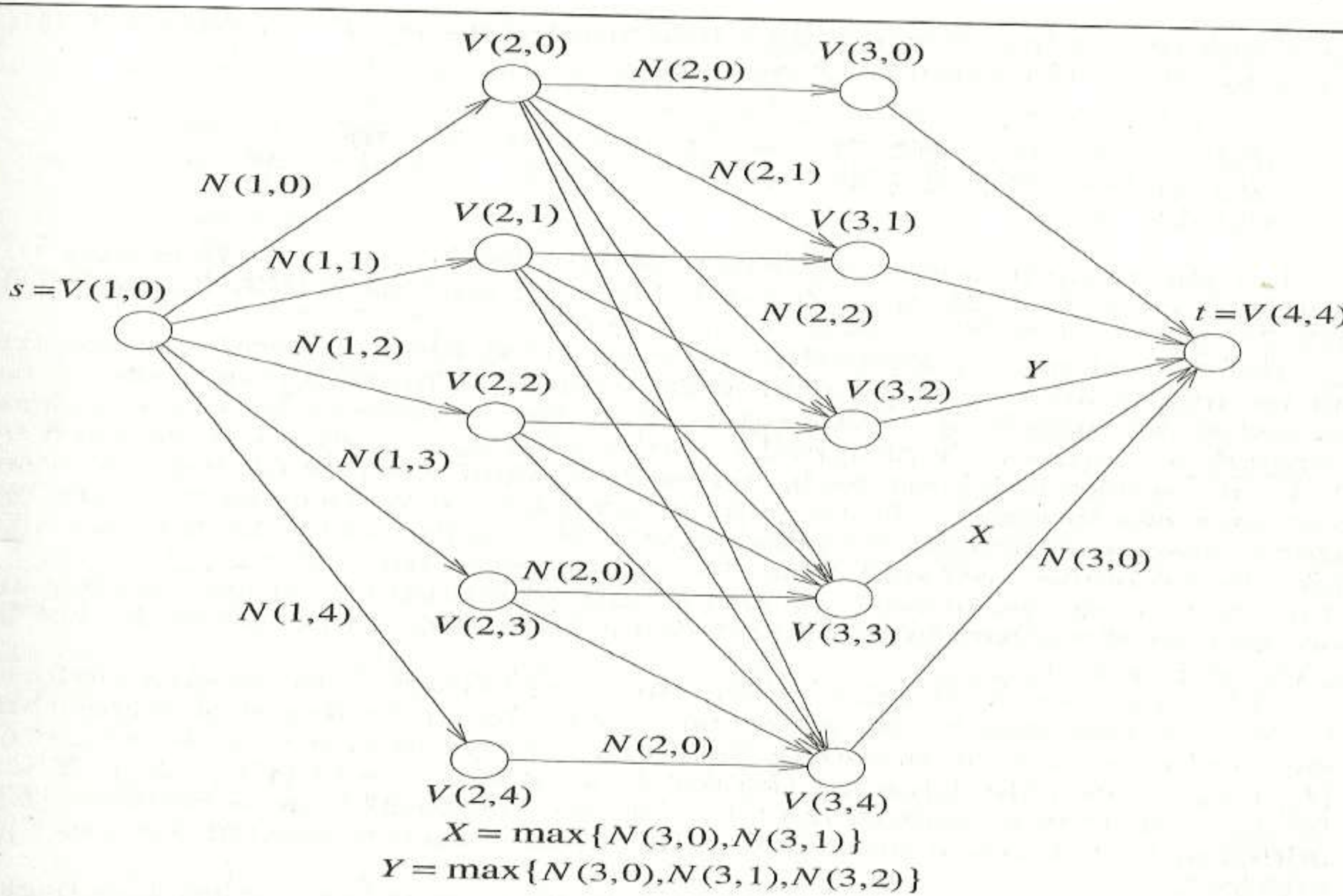


Figure 5.3 Four-stage graph corresponding to a three-project problem

All-Pairs Shortest Paths (APSP)

- Let G=(V,E) be a directed graph with n vertices.
- Let cost be a adjacency matrix for G, such that cost(i, i)=0 ,1≤ i ≤ n
- cost(i, j) is the length (or cost) of edge (i,j) if (i, j) Є E(G)

All-Pairs Shortest Paths (APSP)

- All pair shortest path problem is to determine a matrix A
- such that A(i, j) is the length of shortest path from i to j.
- A can be obtained by solving n Single source shortest path problems

– Complexity $n*n^2$ i.e. $O(n^3)$

All-Pairs Shortest Paths (APSP)

Shortest path from i to j, $i \neq j$

- Path may go through some intermediate vertex
- If k is the index of this intermediate vertex then path (i,k) and (k,j) must be shortest path
- If k is the intermediate vertex with highest index then
  – i to k path is the shortest i to k path going through no vertex with index greater than k-1
  – So is the k to j path
- We need to decide highest index intermediate vertex k

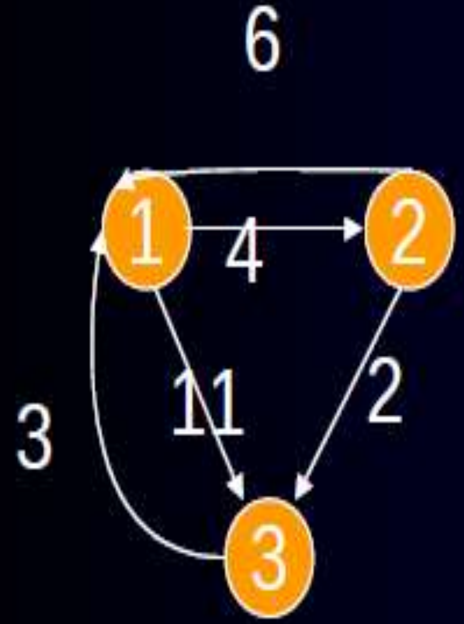
All-Pairs Shortest Paths (APSP)

- Let $A^k(i, j)$ be length of shortest path from i to j going through no vertex of index greater than k

- $A(i, j) = \min\{ \min_{1 \le k \le n} \{A^{k-1}(i, k) + A^{k-1}(k, j)\}, cost(i,j)\}$

- A shortest path from i to j may or may not go through highest index vertex
- $A^k(i, j) = \min\{ A^{k-1}(i, j), A^{k-1}(i, k) + A^{k-1}(k, j)\}$
- $K \ge 1$

$$A^k(i,j)=\min\{A^{k-1}(i,j), A^{k-1}(i,k)+A^{k-1}(k,j)\}, k\geq 1$$



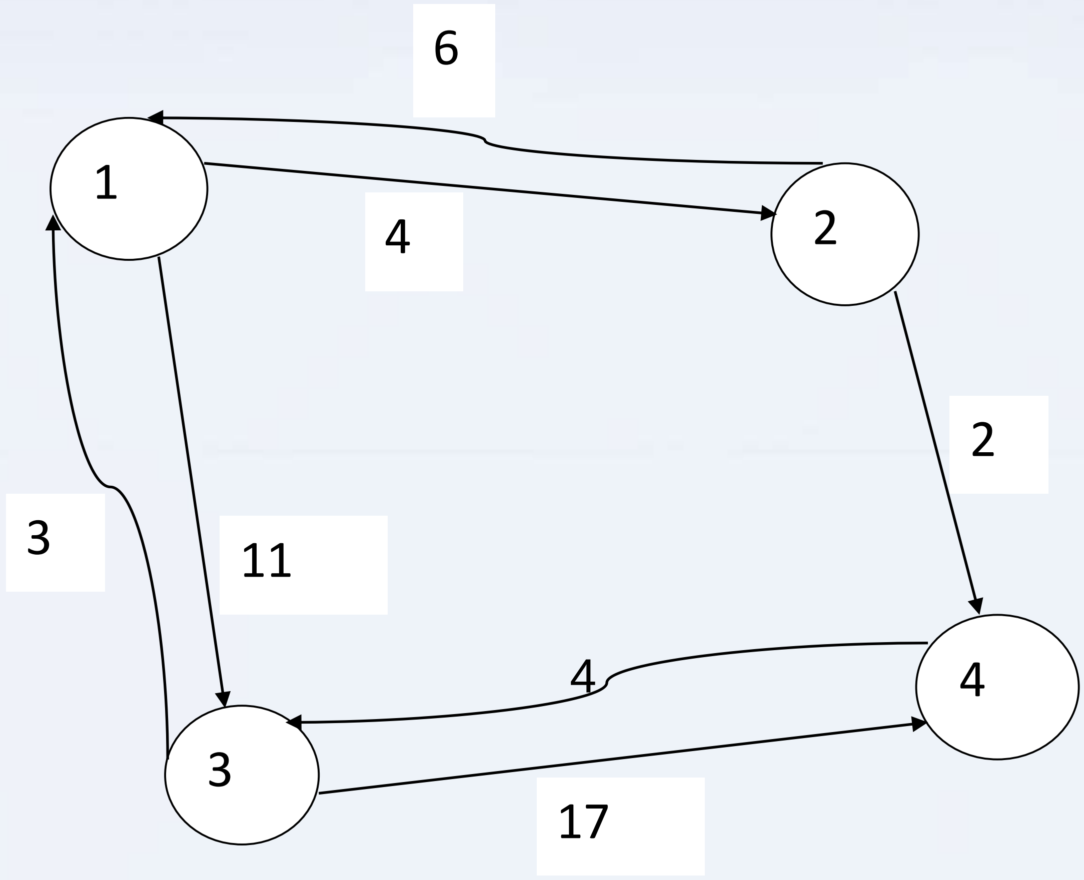
A^0	1	2	3
1	0	4	11
2	6	0	2
3	3	∞	0

A^1	1	2	3
1	0	4	11
2	6	0	2
3	3	7	0

A^2	1	2	3
1	0	4	6
2	6	0	2
3	3	7	0

A^3	1	2	3
1	0	4	6
2	5	0	2
3	3	7	0

All-Pairs Shortest Paths (APSP)



All-Pairs Shortest Paths (APSP)

- A^0

	1	2	3	4
1	0	4	11	∞
2	6	0	∞	2
3	3	∞	0	17
4	∞	∞	4	0

All-Pairs Shortest Paths (APSP)

- A^1

	1	2	3	4
1	0	4	11	∞
2	6	0	17	2
3	3	7	0	17
4	∞	∞	4	0

All-Pairs Shortest Paths (APSP)

- A^2

	1	2	3	4
1	0	4	11	6
2	6	0	17	2
3	3	7	0	9
4	∞	∞	4	0

All-Pairs Shortest Paths (APSP)

- A^3

	1	2	3	4
1	0	4	11	6
2	6	0	17	2
3	3	7	0	9
4	7	11	4	0

All-Pairs Shortest Paths (APSP)

- A^4

	1	2	3	4
1	0	4	10	6
2	6	0	6	2
3	3	7	0	9
4	7	11	4	0

Optimal Binary Search Tree

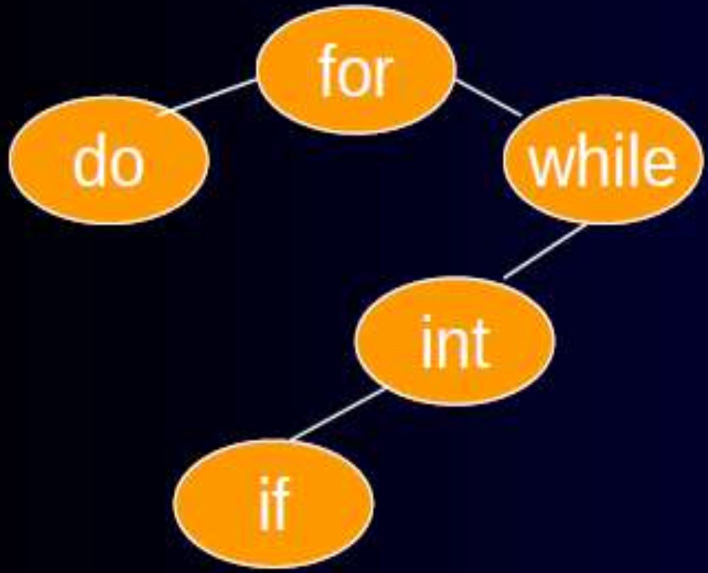
- Given set of identifiers, different binary trees could be formed.
- Average number of comparisons needed for finding the identifiers will be different for different trees.
- In the simplest form we assume

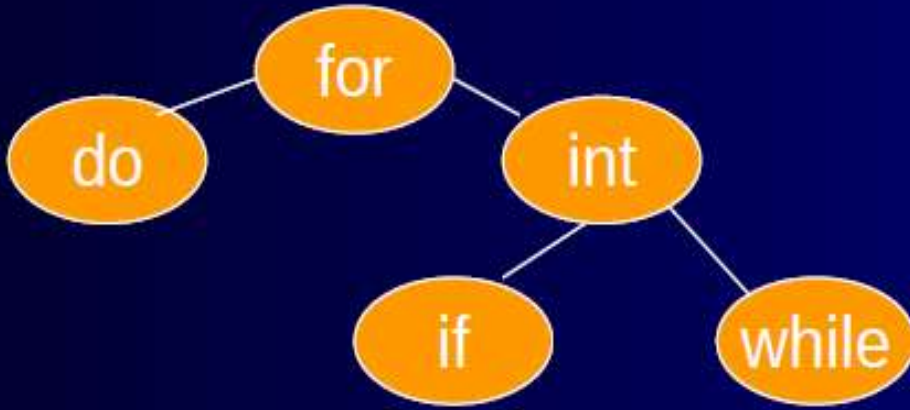
– Probability of search of each element is equal

– No unsuccessful searches made

Optimal Binary Search Tree

- Let set of identifiers be {for, do, while, int, if} with do <for < if < int < while
- Possible trees are





Comparison =1+2+2+3+4/5
=12/5

Comparison =1+2+2+3+3/5
=11/5

Optimal Binary Search Tree

- In general situation different identifiers can be searched with different probabilities
- and unsuccessful searches can also be made
- Let set of identifiers be $\{a_1, a_2, .., a_n\}$ with $a_1 < a_2 < .. < a_n$
- Let $p(i)$ be the probability with which we search a_i.

Optimal Binary Search Tree

- Let q(i) be the probability with which we search x such that $a_i < x < a_{i+1}$, , $0 \le i \le n$.

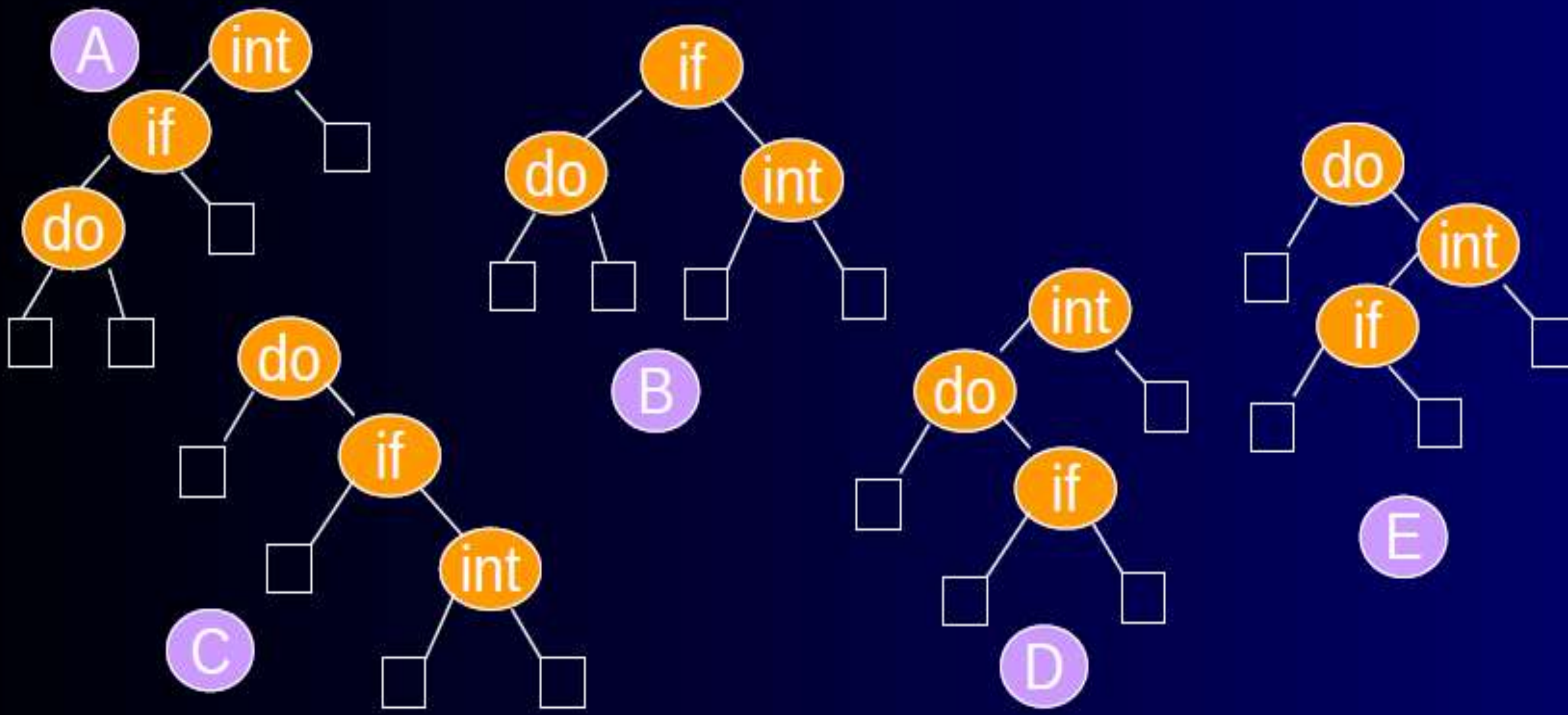
– Assume $a_0 = -\infty$ and $a_{n+1} = \infty$.

- Then
- $\sum_{0 \le i \le n} q(i)$ is the probability of unsuccessful search

- Clearly
- $\sum_{1 \le i \le n} p(i) + \sum_{0 \le i \le n} q(i) = 1$

Optimal Binary Search Tree

$$cost = \sum_{1 \le i \le n} p(i) * level(a_i) + \sum_{0 \le i \le n} q(i) * (level(E_i) - 1)$$



Optimal Binary Search Tree

- (a1, a2, a3) = (do, if, int)
- If all searches (successful and un-successful) equally probable (1/7)
- the cost will be
- Cost(tree A)=1/7+2/7+3/7+1/7+2/7+3/7+3/7
- =15/7

Cost(B)=13/7
Cost(C)=15/7
Cost(D)=15/7
Cost(E)=15/7

- Tree B is optimal

Optimal Binary Search Tree

- (a1, a2, a3) = (do, if, int)
- p(1)=0.5, p(2)=0.1, p(3)=0.05 and
- q(0)=0.15, q(1)=0.1, q(2)=0.05, q(3)=0.05
- the costs will be
- cost(A)=((0.5 X 3) + (0.1 X 2)+ (0.05 X 1)) +

  ((0.15 X 3)+(0.1 X 3)+ (0.05 X2)+(0.05 X1))
- = 1.5+0.2+0.05+0.45+0.3+0.1+0.05
- = 2.65

Optimal Binary Search Tree

- cost(B)=((0.5 X 2)+(0.1 X 1)+(0.05 X 2)) +

  ((0.15 X 2)+(0.1 X 2)+ (0.05 X2)+(0.05 X2))

  =1.0+0.1+0.1+0.30+0.2+0.1+0.1

  =1.9

- cost(C)=((0.5 X 1)+(0.1 X 2)+(0.05 X 3)) +

  ((0.15 X1)+(0.1 X 2)+ (0.05 X3)+(0.05 X3))

  =0.5+0.2+0.15+0.15+0.2+0.15+0.15

  =1.5

Optimal Binary Search Tree

- cost(D)=((0.5 X 2)+(0.1 X 3)+(0.05 X 1)) +

  ((0.15 X 2)+(0.1 X 3)+ (0.05 X3)+(0.05 X1))

  =1.0+0.3+0.05+0.30+0.3+0.15+0.05

  =2.15

- cost(E)=((0.5 X 1)+(0.1 X 3)+(0.05 X 2)) +

  ((0.15 X1)+(0.1 X 3)+ (0.05 X3)+(0.05 X2))

  =0.5+0.3+0.10+0.15+0.3+0.15+0.10

  =1.6

Optimal Binary Search Tree

- p(1)=0.5, p(2)=0.1, p(3)=0.05 and
- q(0)=0.15, q(1)=0.1, q(2)=0.05, q(3)=0.05
- the costs will be
- cost(A)=2.65
- cost(B)=1.9
- cost(C)=1.5
- cost(D)=2.15
- cost(E)=1.6
- Tree C is optimal

Applying Dynamic Programming Approach

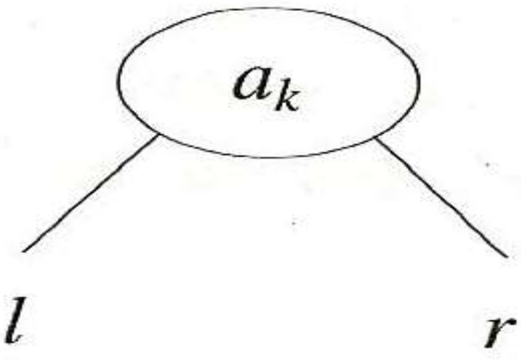
- Construct tree as a result of sequence of decisions
- Make decision as to which of the a_i should be assigned to the root node of the tree
- If we choose a_k as a root, then
- Internal nodes $a_{1,}$ $a_{2,\ldots,}$ a_{k-1} and external nodes $E_{1,}$ $E_{2,\ldots,}$ E_{k-1} will lie in left sub-tree l of root
- The remaining nodes will lie in right sub-tree r

Applying Dynamic Programming Approach

$$cost(l) = \sum_{1 \le i < k} p(i) * \text{level}(a_i) + \sum_{0 < i < k} q(i) * (\text{level}(E_i) - 1)$$

$$cost(r) = \sum_{k < i \le n} p(i) * \text{level}(a_i) + \sum_{k < i \le n} q(i) * (\text{level}(E_i) - 1)$$

Root of sub-tree is assumed to be present at level 1



Applying Dynamic Programming Approach

Using $w(i,j)$ to represent the sum $q(i) + \sum_{l=i+1}^{j}(q(l) + p(l))$, we obtain the following as the expected cost of the search tree

$$p(k) + cost(l) + cost(r) + w(0, k-1) + w(k, n)$$

- If the tree is optimal, then this value must be minimum
- Hence cost(l) and cost(r) must be minimum

Applying Dynamic Programming Approach

- If we use c(i, j) to represent the cost of an optimal binary search tree t_{ij} containing $a_{i+1,}$ $a_{i+2,\ldots,}$ a_j and $E_{i,}$ $E_{i+1,\ldots,}$ E_j
- Then for the tree to be optimal
- cost(l)=c(0, k-1) and cost(r)=c(k, n)
- k must be chosen such that

$$p(k) + c(0, k-1) + c(k, n) + w(0, k-1) + w(k, n)$$

is minimum

Applying Dynamic Programming Approach

- Hence for c(0, n) we obtain

$$c(0,n) = \min_{1 \le k \le n} \{c(0,k-1) + c(k,n) + p(k) + w(0,k-1) + w(k,n)\}$$

- We can generalize it to obtain for any c(i, j)

$$c(i,j) \;=\; \min_{i<k\le j} \{c(i,k-1) + c(k,j) + p(k) + w(i,k-1) + w(k,j)\}$$

$$c(i,j) \;=\; \min_{i<k\le j} \{c(i,k-1) + c(k,j)\} + w(i,j)$$

Applying Dynamic Programming Approach

- This equation can be solved for c(0, n)
- By first computing all c(i, j), such that j-i =1
- c(i, i)=0 w(i, i)=q(i) $0 \leq i \leq n$
- Next computing all c(i, j), such that j-i =2 and then computing all c(i, j), such that j-i =3 so on
- During this computation we record the root r(i, j) of each tree t_{ij} ,
- then optimal binary search tree can be constructed from these r(i, j)

Example 5.18 Let $n = 4$ and $(a_1, a_2, a_3, a_4) =$ (**do, if, int, while**). Let $p(1:4) = (3,3,1,1)$ and $q(0:4) = (2,3,1,1,1)$. The p's and q's have been multiplied by 16 for convenience. Initially, we have $w(i,i) = q(i), c(i,i) = 0$ and $r(i,i) = 0, 0 \leq i \leq 4$. Using Equation 5.12 and the observation $w(i,j) = p(j) + q(j) + w(i,j-1)$, we get

$$
\begin{aligned}
w(0,1) &= p(1) + q(1) + w(0,0) = 8 \\
c(0,1) &= w(0,1) + \min\{c(0,0) + c(1,1)\} \;=\; 8 \\
r(0,1) &= 1 \\
w(1,2) &= p(2) + q(2) + w(1,1) \;=\; 7 \\
c(1,2) &= w(1,2) + \min\ \{c(1,1) + c(2,2)\} \;=\; 7 \\
r(0,2) &= 2 \\
w(2,3) &= p(3) + q(3) + w(2,2) \;=\; 3 \\
c(2,3) &= w(2,3) + \min\ \{c(2,2) + c(3,3)\} \;=\; 3 \\
r(2,3) &= 3 \\
w(3,4) &= p(4) + q(4) + w(3,3) \;=\; 3 \\
c(3,4) &= w(3,4) + \min\ \{c(3,3) + c(4,4)\} \;=\; 3 \\
r(3,4) &= 4
\end{aligned}
$$

	0	1	2	3	4
0	$w_{00} = 2$ $c_{00} = 0$ $r_{00} = 0$	$w_{11} = 3$ $c_{11} = 0$ $r_{11} = 0$	$w_{22} = 1$ $c_{22} = 0$ $r_{22} = 0$	$w_{33} = 1$ $c_{33} = 0$ $r_{33} = 0$	$w_{44} = 1$ $c_{44} = 0$ $r_{44} = 0$
1	$w_{01} = 8$ $c_{01} = 8$ $r_{01} = 1$	$w_{12} = 7$ $c_{12} = 7$ $r_{12} = 2$	$w_{23} = 3$ $c_{23} = 3$ $r_{23} = 3$	$w_{34} = 3$ $c_{34} = 3$ $r_{34} = 4$	
2	$w_{02} = 12$ $c_{02} = 19$ $r_{02} = 1$	$w_{13} = 9$ $c_{13} = 12$ $r_{13} = 2$	$w_{24} = 5$ $c_{24} = 8$ $r_{24} = 3$		
3	$w_{03} = 14$ $c_{03} = 25$ $r_{03} = 2$	$w_{14} = 11$ $c_{14} = 19$ $r_{14} = 2$			
4	$w_{04} = 16$ $c_{04} = 32$ $r_{04} = 2$				

Applying Dynamic Programming Approach

- w(i, j)= p(j)+q(j)+w(i, j-1)

$$c(i,j) = \min_{i<k\leq j}\{c(i,k-1)+c(k,j)\}+w(i,j)$$

- w(0, 2) =p(2)+q(2)+w(0,1)

  =3+1+8

  =12

c(0,2)=min{c(0,0)+c(1,2), c(0,1)+c(2,2)}+w(0,2)

=min{0+3, 7+0}+12

=3+12= 15

r(0, 2) = 1;

Applying Dynamic Programming Approach

- w(i, j)= p(j)+q(j)+w(i, j-1)

$$c(i,j) = \min_{i<k\leq j} \{c(i,k-1) + c(k,j)\} + w(i,j)$$

- w(1, 3) =p(3)+q(3)+w(1,2)

  =1+1+7

  =9

c(1,3)=min{c(1,1)+c(2,3), c(1,2)+c(3,3)}+w(1,3)

=min{0+3, 7+0}+9

=3+9= 12

r(1, 3) = 2;

Applying Dynamic Programming Approach

- w(i, j)= p(j)+q(j)+w(i, j-1)

$$c(i,j) = \min_{i<k\leq j}\{c(i,k-1)+c(k,j)\}+w(i,j)$$

- w(2, 4) =p(4)+q(4)+w(2,3)

  =1+1+3

  =5

c(2,4)=min{c(2,2)+c(3,4), c(2,3)+c(4,4)}+w(2,4)

=min{0+3, 3+0}+5

=3+5= 8

r(2, 4) = 3;

Applying Dynamic Programming Approach

$$w(0, 3)=p(3)+q(3)+w(0,2)$$

$$=1+1+12$$

$$=14$$

$$c(0,3)=\min\{c(0,0)+c(1,3),\ c(0,1)+c(2,3),$$

$$c(0,2)+c(3,3)\} +w(0,3)$$

$$=\min\{0+12,\ 8+3,\ 19+0\}+14$$

$$=11+14= 25$$

$$r(0, 3) = 2;$$

Applying Dynamic Programming Approach

$w(1, 4)=p(4)+q(4)+w(1,3)$

$=1+1+9$

$=11$

$c(1,4)=\min\{c(1,1)+c(2,4),\ c(1,2)+c(3,4),$

$c(1,3)+c(4,4)\} +w(1,4)$

$=\min\{0+8,\ 7+3,\ 12+0\}+11$

$=8+11= 19$

$r(1, 4) = 2;$

Applying Dynamic Programming Approach

$w(0, 4)=p(4)+q(4)+w(0,3)$

$=1+1+14$

$=16$

$r(0,4)=\min\{c(0,0)+c(1,4), c(0,1)+c(2,4),$

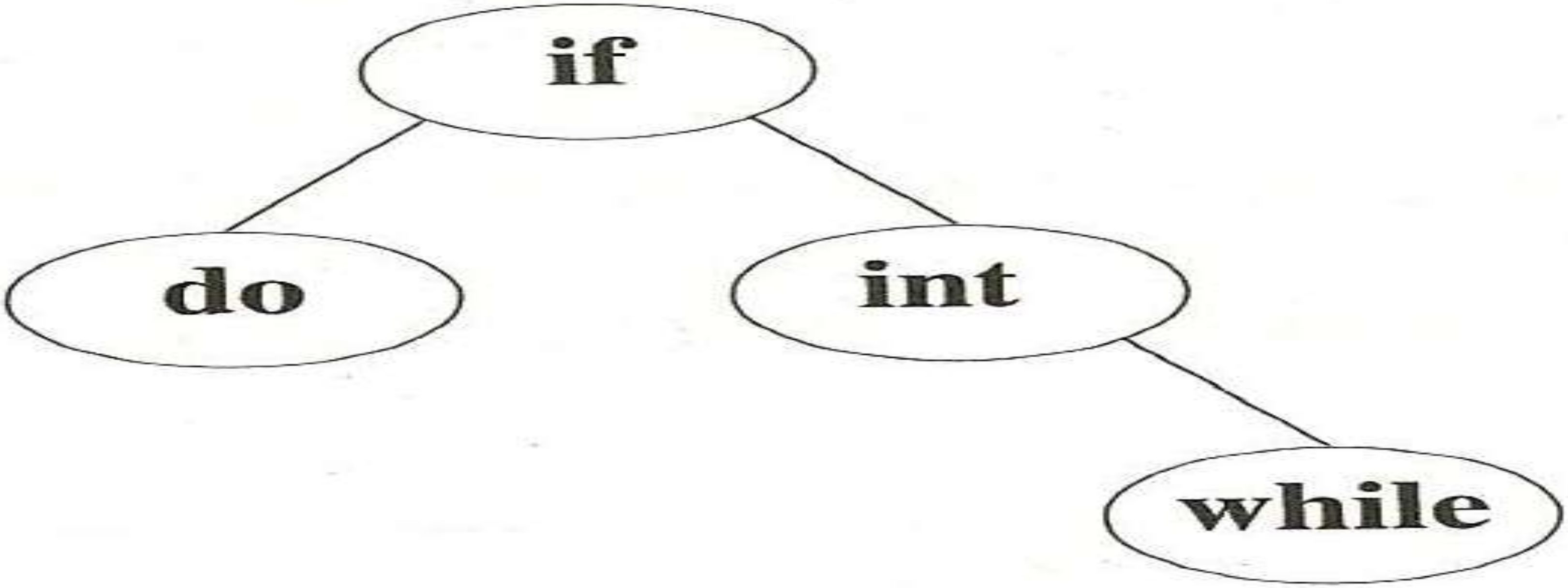
$c(0,2)+c(3,4), c(0,3)+c(4,4)\} +w(0,4)$

$=\min\{0+19, 8+8, 19+3, 25+0 \}+16$

$=16+16= 32$

$r(0, 4) = 2;$

	0	1	2	3	4
0	$w_{00} = 2$ $c_{00} = 0$ $r_{00} = 0$	$w_{11} = 3$ $c_{11} = 0$ $r_{11} = 0$	$w_{22} = 1$ $c_{22} = 0$ $r_{22} = 0$	$w_{33} = 1$ $c_{33} = 0$ $r_{33} = 0$	$w_{44} = 1$ $c_{44} = 0$ $r_{44} = 0$
1	$w_{01} = 8$ $c_{01} = 8$ $r_{01} = 1$	$w_{12} = 7$ $c_{12} = 7$ $r_{12} = 2$	$w_{23} = 3$ $c_{23} = 3$ $r_{23} = 3$	$w_{34} = 3$ $c_{34} = 3$ $r_{34} = 4$	
2	$w_{02} = 12$ $c_{02} = 19$ $r_{02} = 1$	$w_{13} = 9$ $c_{13} = 12$ $r_{13} = 2$	$w_{24} = 5$ $c_{24} = 8$ $r_{24} = 3$		
3	$w_{03} = 14$ $c_{03} = 25$ $r_{03} = 2$	$w_{14} = 11$ $c_{14} = 19$ $r_{14} = 2$			
4	$w_{04} = 16$ $c_{04} = 32$ $r_{04} = 2$				



compute $w(i,j)$, $r(i,j)$, and $c(i,j)$, $0 \leq i < j \leq 4$, for the identifier set (a_1, a_2, a_3, a_4) = (**cout**, **float**, **if**, **while**) with $p(1) = 1/20$, $p(2) = 1/5$, $p(3) = 1/10$, $p(4) = 1/20$, $q(0) = 1/5$, $q(1) = 1/10$, $q(2) = 1/5$, $q(3) = 1/20$, and $q(4) = 1/20$. Using the $r(i,j)$'s, construct the optimal binary search tree.

0 / 1 Knapsack

Statement: Same as knapsack except that no fraction of item is allowed (0 or 1 is allowed).

- Solution obtained by sequence of decision $x_1, x_2 \ldots x_n$ (x= o/1).
- Let fi(y) be the value of optimal solution to KNAP(1,i,y)
  - $f_i(y)=\max\{f_{i-1}(y), f_{i-1}(y-w_i)+p_i\}$.
- S^i be the pair (P,W) where $P=f_i(y_j)$ and $W=y_j$
- S^{i+1} can be computed from S^i by computing $S^i_1 = \{(P,W)|(P-pi,W-wi) \in S^i\}$
- S^{i+1} can be computed by merging the pairs S^i and S^i_1 stogether.

0 / 1 Knapsack

- If S^{i+1} contains two pairs (P_j, W_j) and (P_k, W_k) with the values $P_j \leq P_k$ and $W_j \geq W_k$
- Then pair (P_j, W_j) can be discarded
- It is called Dominance Rule/ Purging Rule.

Example 5.21 Consider the knapsack instance $n = 3$, $(w_1, w_2, w_3) = (2, 3, 4)$, $(p_1, p_2, p_3) = (1, 2, 5)$, and $m = 6$. For these data we have

$$\begin{aligned}
S^0 &= \{(0,0)\}; S_1^0 = \{(1,2)\} \\
S^1 &= \{(0,0),(1,2)\}; S_1^1 = \{(2,3),(3,5)\} \\
S^2 &= \{(0,0),(1,2),(2,3),(3,5)\}; S_1^2 = \{(5,4),(6,6),(7,7),(8,9)\} \\
S^3 &= \{(0,0),(1,2),(2,3),(5,4),(6,6),(7,7),(8,9)\}
\end{aligned}$$

Note that the pair (3, 5) has been eliminated from S^3 as a result of the purging rule stated above. □

0 / 1 Knapsack

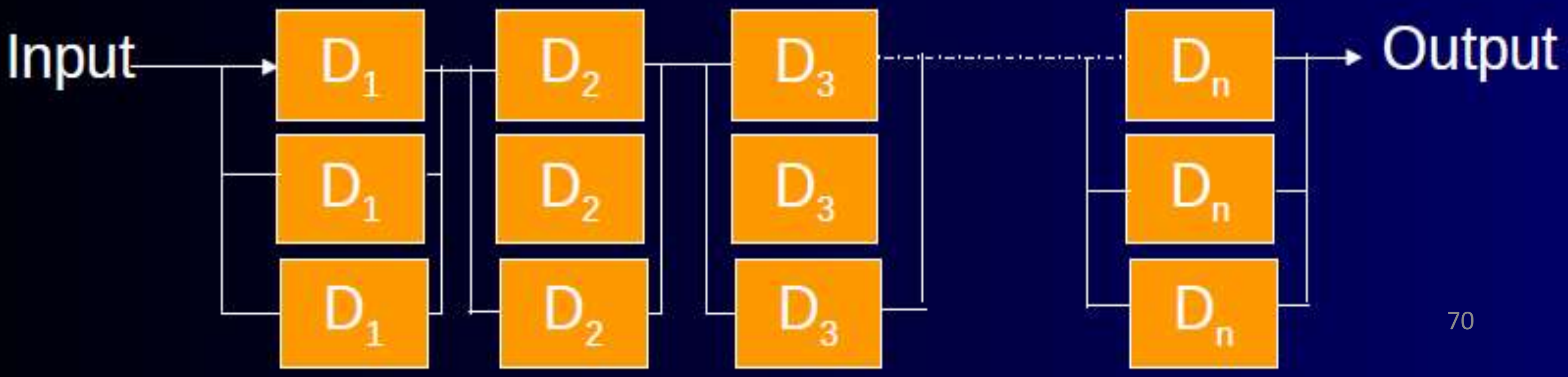
Example 5.22 With $m = 6$, the value of $f_3(6)$ is given by the tuple (6, 6) in S^3 (Example 5.21). The tuple $(6, 6) \notin S^2$, and so we must set $x_3 = 1$. The pair (6, 6) came from the pair $(6 - p_3, 6 - w_3) = (1, 2)$. Hence $(1, 2) \in S^2$. Since $(1, 2) \in S^1$, we can set $x_2 = 0$. Since $(1, 2) \notin S^0$, we obtain $x_1 = 1$. Hence an optimal solution is $(x_1, x_2, x_3) = (1, 0, 1)$. □

Generate the sets S^i , $0 \leq i \leq 4$, when
$(w_1, w_2, w_3, w_4) = (10, 15, 6, 9)$ and
$(p_1, p_2, p_3, p_4) = (2, 5, 8, 1)$
$m=25$

Reliability Design

Problem statement: To design a system that is composed of several devices connected in series.

If each device is associated with the reliability (r_1, r_2, $r_{3,\dots}$ $r_{n)}$, the reliability of entire system will be ¶ r_n. This can be improved by connecting more number of devices in parallel.



Reliability Design: Statement

- If m_i no of devices are connected in parallel then the reliability will be $1-(1-r_i)^{mi}$
- Let reliability of i^{th} stage is given as $ø_i(m_i)$
- Reliability of the system is

$$\prod_{1\leq i\leq n} \phi_i(m_i)$$

- Let c_i be cost of each unit and c total cost.
- We want to maximize the above expression subject to

$$\sum_{1\leq i\leq n} c_i(m_i) \leq c$$

Reliability Design: Solution

$$f_i(x) = \max_{1 \le m_i \le u_i} \{\Phi_i(m_i) f_{i-1}(x - c_i m_i)\}$$

- If $f_0(x)=1$ for all x, $0 \le x \le c$.
- Let S^i consists of tuples from (f,x), with $f=f_i(x)$
- There is at most one tuple for each different x that result from sequence of decision on $m_1, m_2, \ldots m_n$
- Dominance rule

  (f_1,x_1) dominates (f_2,x_2) iff $f_1 \ge f_2$ and $x_1 \le x_2$

  Dominated tuples can be descarded

Reliability Design: Example

- The devices be D_1,D_2,D_3 with costs as {30,15,20} with reliability {0.9,0.8,0.5} and total cost 105.
- Compute m_1,m_2,m_3 and reliability ???

$c_1= 30$, $c_2= 15$, $c_3= 20$, $c=105$, $r_1= 0.9$, $r_2= 0.8$, $r_3= 0.5$

- $u_1= 2$, $u_2= 3$, $u_3= 3$
- Let S^i represents set of all undominated tuples (f,x) that may result from various decision sequences $m_1,m_2,\ldots m_n$
- S^i is computed from S^{i-1}
- S^i_j represents all tuple obtainable from S^{i-1} by choosing $m_i= j$

Reliability Design: Solution

- $c_1 = 30$, $c_2 = 15$, $c_3 = 20$, $c = 105$
- $r_1 = 0.9$, $r_2 = 0.8$, $r_3 = 0.5$
- $u_1 = 2$, $u_2 = 3$, $u_3 = 3$

- $S^0 = \{(1,0)\}$
- $S^1_1 = \{(.9,30)\}$, $S^1_2 = \{(.99,60)\}$
- $S^1 = \{(.9,30),(.99,60)\}$
- $S^2_1 = \{(.72,45),(.792,75)\}$, $S^2_2 = \{(.864,60), \underline{(.954,90)}\}$, $S^2_3 = \{(.8928,75)\}$,
  - Tuple $\{(.954,90)\}$ which comes from $(.99,60)$ has been eliminated from S^2_2 as it leaves only Rs 10, not sufficient to have $m_3=1$.
- So $S^2 = \{(.72,45),(.864,60)\},(.8928,75)\}$
  - Tuple $(.792,75)$ is dominated by $(.864,60)$.
- $S^3_1 = \{(.36,65),(.423,80),(.4464,95)\}$, $S^3_2 = \{(.54,85),(.648,100)\}$, $S^3_3 = \{(.63,105)\}$
- $S^3 = \{(.36,65),(.423,80),(.54,85),(.648,100)\}$
- Best Design with reliability .648 and cost 100
- Tracking back through S^i we get $m_1,=1, m_2=2, m_3=2$

Reliability Design

- **$s^0 = \{(1, 0)\}$** – Initial Set Before adding any device
- Add 1 device of type D1
- $s^1_1 = \{(0.9, 30)\}$
- Add 2 devices of type D1
- Reliability of 2 devices of type D1 connected in parellel
- $1-(1-r)^m = 1-(1-0.9)^2 = 1- 0.1^2 = 0.99$
- $s^1_2 = \{(0.99, 60)\}$
- **$s^1 = \{(0.9, 30), (0.99, 60)\}$**

Reliability Design

- Add 1 device of type D2
- s^2_1 ={(0.72, 45), (0.792, 75)}
- Add 2 devices of type D2
- Reliability of 2 devices of type D2 connected in parellel
- $1-(1-r)^m = 1-(1-0.8)^2 = 1- 0.2^2= 0.96$
- s^2_2 ={(0.864, 60), (0.9504, 90)}

Reliability Design

- Add 3 devices of type D2
- Reliability of 3 devices of type D2 connected in parellel
- $1-(1-r)^m = 1-(1-0.8)^3 = 1- 0.2^3 = 0.992$
- s^2_3 ={ (0.8928, 75) }
- **s^2 ={** (0.72, 45), (0.792, 75), (0.864, 60), (0.9504, 90), (0.8928, 75) }
- **s^2 ={**(0.72, 45), (0.864, 60), (0.8928, 75) }

Reliability Design

- Add 1 device of type D3
- s^3_1 ={(0.36, 65), (0.432, 80), (0.4464, 95)}
- Add 2 devices of type D3
- Reliability of 2 devices of type D3 connected in parellel
- $1-(1-r)^m = 1-(1-0.5)^2 = 1- 0.5^2 = 0.75$
- s^3_2 ={(0.54, 85), (0.648, 100)}

Reliability Design

- Add 3 devices of type D3
- Reliability of 3 devices of type D3 connected in parellel
- $1-(1-r)^m = 1-(1-0.5)^3 = 1- 0.5^3 = 0.875$
- s^3_3 ={ (0.63, 105) }
- s^3 ={(0.36, 65), (0.432, 80), (0.54, 85), (0.648, 100)}(0.63, 105) }
- Optimal - (0.648, 100)
- 1,2,2

Reliability Design- Example 2

• $C1=40$	$c2=25$	$c3=35$
• $r1=0.75$	$r2=0.85$	$r3=0.6$
• $u1=3$	$u2=3$	$u3=2$
• $c=175$		

- Till now we have seen how dynamic programming problem could be applicable to subset selection problem

- Now will see some permutation problem with n! complexity

Traveling Salesperson Problem

- Statement: Let G(V,E) be a directed graph with edge cost c_{ij}.
- $c_{ij} > 0$, and $c_{ij} = \infty$ if $\langle i, j \rangle \notin E$.
- A tour of G is directed simple cycle that includes every vertex in V.
- Cost of tour is sum of cost of edges on the tour

Traveling Salesperson Problem

- Traveling salesman problem is to find tour of minimum cost.

Traveling Salesman Problem: Applications

- Postal van picking mails from postal mail boxes.

Traveling Salesman Problem: Solution

- Let the tour starts at vertex 1 and the next vertex visited is k.
- Tour will consists of edge <1,k> for some k ε V-{1}and path from vertex k to 1.
- This path from k to 1 goes through each vertex in V-{1,k} exactly once.
- If the tour is optimal, the path from k to 1 must be shortest k to 1 path, going through all vertex in V-{1,k}.
- Let g(i,S) be the length of shortest path starting at vertex i, going through all vertex in S and terminating at vertex 1.

Traveling Salesman Problem: Solution

- The function g(1, V-{1,k}) is the length an optimal salesperson tour.

$$g(1, V-\{1\}) = \min_{2 \le k \le n} \{c_{1k} + g(k, V-\{1,k\})\}$$

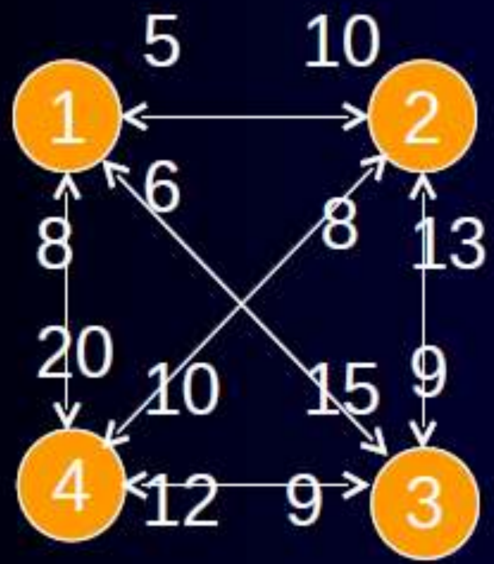
- Generalizing

$$g(i, V-\{1\}) = \min_{j \in S} \{c_{ij} + g(i, S-\{j\})\}$$

- $g(i,ø)=c_{i1,}$

Traveling Salesman Problem: Example

$$g(i, V-\{1\}) = \min_{j \in S} \{c_{ij} + g(i, S-\{j\})\}$$



- g(2,ø)=c21=5
- g(2,{3})=c23+g(3,ø)=15
  g(2,{4})=? g(3,{4})=?
- g(2,{3,4})= min {c23+g(3,{4}), c24+g(4,{3})}=25
- g(3,{2,4})=?
- g(1,{2,3,4})= 35

$$\begin{bmatrix} 0 & 10 & 15 & 20 \\ 5 & 0 & 9 & 10 \\ 6 & 13 & 0 & 12 \\ 8 & 8 & 9 & 0 \end{bmatrix}$$

Traveling Salesman Problem

- $g(2,\Phi)=C_{21}=5$
- $g(3,\Phi)=C_{31}=6$
- $g(4,\Phi)=C_{41}=8$

- $g(2, \{3\})= C_{23} + g(3,\Phi) = 9+6=15$
- $g(2, \{4\})= C_{24} + g(4,\Phi) = 10+8=18$
- $g(3, \{2\})= C_{32} + g(2,\Phi) = 13+5=18$
- $g(3, \{4\})= C_{34} + g(4,\Phi) = 12+8=20$
- $g(4, \{2\})= C_{42} + g(2,\Phi) = 8+5=13$
- $g(4, \{3\})= C + g(3,\Phi) = 9+6=15$

Traveling Salesman Problem

- $g(2, \{3, 4\})=$
- $= \min \{C_{23} + g(3,\{4\}), \underline{C_{24} + g(4,\{3\})}\}$
- $= \min \{9+20, 10+15\} = \min\{29, 25\} = 25$

- $g(3, \{2, 4\})=$
- $= \min \{C_{32} + g(2,\{4\}), \underline{C_{34} + g(4,\{2\})}\}$
- $= \min \{13+18, 12+13\} = \min\{31, 25\} = 25$

- $g(4, \{2, 3\})=$
- $= \min \{\underline{C_{42} + g(2,\{3\})}, C_{43} + g(3,\{2\})\}$
- $= \min \{8+15, 9+18\} = \min\{23, 27\} = 23$

Traveling Salesman Problem

- $g(1, \{2, 3, 4\})=$
- $= \min \{\underline{C_{12} + g(2,\{3, 4\}}),$

  $C_{13} + g(3,\{2, 4\}),$

  $C_{14} + g(4,\{2, 3\})\}$
- $= \min \{10+25, 15+25, 20+23\}$
- $= \min\{35, 40, 43\}$
- $=35$

v1= 1 v2= 2 v3=4 v4= 3 v5=1

- 1,2,4,3,1

Flow Shop Scheduling

Problem Statement

- n jobs requiring m tasks
  $T_{1i}, T_{2i}, T_{3i}, \ldots. T_{mi}$, $1 \leq i \leq n$
- Task T_{ji} is to be performed on Processor P_j, $1 \leq i \leq m$
- Time required to complete T_{ji} is t_{ji}
- Schedule for n jobs is an assignment of task to time interval on the processor.

Problem statement: Conditions

- Task T_{ji} must be asigned to the Processor P_j
- No processor may have more than one task assigned to it in any time interval.
- For any job i the Processing of task T_{ji}, $j \geq 1$, cannot be started until task $T_{j-1, i}$ is completed.

Example

- Two jobs have to be scheduled on three processors.
- Task time is given by matrix

	Job1	Job 2
Processor 1	2	0
Processor 2	3	3
Processor 3	5	2

Schedule 1 (Non Preemptive)

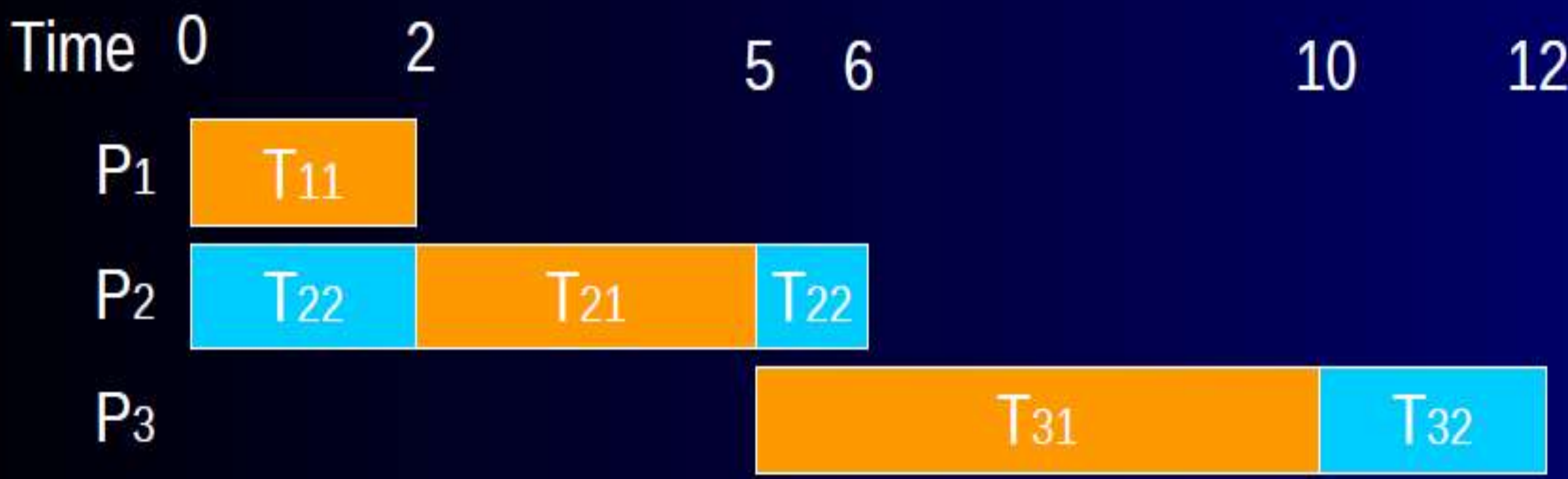
Job 1

Job 2

	J1	J2
P1	2	0
P2	3	3
P3	5	2

Schedule 2(Preemptive)

	J1	J2
P1	2	
0		
P2	3	
3		
P3	5	
2		



Schedule

- Non-preemptive schedule is a schedule in which processing of task on any processor is not terminated until it is complete. (Schedule 1)
- A schedule in which it is not true is preemptive schedule. (Schedule 2)

Parameters

- Finish time $f_i(s)$ of job i is the time at which all tasks of job i have been completed in schedule s.
- Finish time $F(s)$ of the schedule s is given by

  $F(s) = \max\{ f_i(s) \}$, amongst all jobs.
- Mean flow time $MFT(s)$ is defined as

  $MFT(s) = 1/n \sum f_i(s)$
- OFT (Optimal Finish Time) schedule
- POFT (Preemptive Optimal Finish Time) schedule

Flow Shop Scheduling

Example 5.28 Let $n = 4, (a_1, a_2, a_3, a_4) = (3, 4, 8, 10)$, and $(b_1, b_2, b_3, b_4) = (6, 2, 9, 15)$. The sorted sequence of a's and b's is $(b_2, a_1, a_2, b_1, a_3, b_3, a_4, b_4) = (2, 3, 4, 6, 8, 9, 10, 15)$. Let $\sigma_1, \sigma_2, \sigma_3$, and σ_4 be the optimal schedule. Since the smallest number is b_2, we set $\sigma_4 = 2$. The next number is a_1 and we set $\sigma_1 = a_1$. The next smallest number is a_2. Job 2 has already been scheduled. The next number is b_1. Job 1 has already been scheduled. The next is a_3 and we set σ_3. This leaves σ_3 free and job 4 unscheduled. Thus, $\sigma_3 = 4$. □

Thank You