1. Mathematical Logic

> [The Sentence in the Square Bracket is False]
>
> \- Liar Paradox

1. Introduction

Mathematical logic or logic is the discipline that deals with the methods of reasoning. It provides rules and techniques for determining whether a given argument or mathematical proof or conclusion in a scientific theory is valid or not.

Logic is concerned with studying arguments and conclusions.

Logic is used in mathematics to prove theorems, and to draw conclusions from experiments in physical sciences, and in our every day life to solve many types of problems.

Logic is used in computer science to verify the correctness of programs.

The rules of logic or techniques of logic are called *rules of inference*, because the main aim of logic is to draw conclusions, inferences from given set of hypotheses. At this context, the theory of inference needs a language in which these rules of inference can be stated. It is necessary to develop a formal language called the *object language*. A *formal language* is one in which the syntax is well defined. Apart from syntax, symbols will be used in the object languages. Thus, a systematic study of arguments by making extensive use of symbols is known as *Symbolic logic*. Our study of the object language requires the use of another language i.e., a natural language (English) called as *metalanguage*.

In this chapter, we introduce the building blocks of our object language viz., Statements, Truthvalues, Connectives etc., to state and apply rules of valid inference.

2. Statements and Notation

In any theory, assertions are made in the form of sentences. Sentences are usually classified as declarative, exclamatory, interrogative and imperative. In our study of logic, we will confine ourselves to declarative sentences only. i.e., we begin by assuming that the object language contains a set of declarative sentences. A *primary statement* is a declarative sentence which cannot be further broken down or analyzed into simpler sentences. These primary statements are the basic units of the object language. The declarative sentences will be admitted in the object language if they have one and only one of two possible values called "*truth values*". The two truth values are 'TRUE' and 'FALSE' and are denoted by the symbols T and F respectively. They are also denoted by the symbols 1 and 0. Our logic is called as *two-valued logic*, since we have only two possible truth values in our logic.

Declarative sentences in the object language are of two types. The first type includes those sentences which are considered to be *primitive* or *primary* in the object language. These will be denoted by distinct alphabetical capital letters A, B, C, ..., P, Q, ... while declarative sentences of the second type are obtained from the primitive ones by using certain symbols, called *connectives*, and certain punctuation marks, such as parentheses to join primitive sentences.

In any case, all the declarative sentences to which it is possible to assign one and only one of the two possible truth values are called *statements*. These statements which do not contain any of the connectives are called *atomic (primary, primitives)* statements.

Consider the following sentences:

1. The integer 5 is a prime number
2. The integer 25 is a prime number
3. The sum of the angles of a rectangle is 360.
4. MOSCOW is the capital of England
5. This Statement is false.
6. Close the Box
7. Today is Monday
8. Do you speak Telugu?
9. Mathematical logic is a dull subject
10. $1 + 101 = 110$.

The sentences (1), (2), (3), (4), (7), (9), (10) are declarative statements. The statements (1) and (3) have truth value 'TRUE', while the statements (2), (4) have truth value 'FALSE''. Statement (5) is not a statement according to our definition, because we cannot properly assign to it a definite truth value. If we assign the truth value true, then sentence (5) says that statement (5) is false. On the other hand if we assign to it the truth value false, then sentence (5) implies that statement (5) is true. This example illustrates a semantic paradox. Clearly (6) is not a statement, it is a command. The truth value of statement (7) depends upon the day in which the statement is made or said. If the sentence is uttered by some one on a Monday, the statement is true and if it is uttered on any otherday, the statement is false. Clearly (8) is not a statement, it is interrogative sentence. The truth value of the statement (9) depends on the person who utters this statement. Lostly, the truth value of statement (10) depends upon the context; viz. if we are talking about numbers in the decimal system, then it is a false statement. On the other hand, for numbers in binary, it is a true statement.

Definition 2.1 A statement or proposition is a declarative sentence to which it is possible to assign a truth value TRUE or FALSE, but not both simultaneously.

3. Connectives

Till now, we considered atomic or primary statements. But in practice, we often combine, simple (Primary) statements to form compound statements by using certain connecting words known as sentential connectives or simply *connectives*. Thus primary statements are combined by means of connectives: *and, or, if...then* and *if and only if*, lostly '*not*'. These five main types of connectives can be defined in terms of the three: *and, or* and *not*.

Note 3.1 To denote statements we use the capital letters $P, Q, \ldots P_1, P_2 \ldots$.

Ex: P: It is raining today

Here, a statement "P" either denotes a particular statement or serves as a placeholder for the statement.

We proceed to give the definitions of the connectives.

3.1 Negation

The negation of a statement is generally formed by introducing the word "not" at a proper place in the statement or by prefixing the statement with the phrase, "it is not the case that (or) it is not true that."

If "P" denotes a statement, then the negation of "P" is written as "$\neg P$" and read as "not P". If the truth value of "P" is T, then the truth value of $\neg P$ is F. Also if the truth value of "P" is F, then the truth value of $\neg P$ is T. This definition of the negation is summarized as follows by a table:

The truth table of $\neg P$:

P	$\neg P$
T	F
F	T

We now illustrate the formation of the negation of a statement.

Example 3.1.1 Consider the statement

$$P : \text{HYDERABAD is a city}$$

Then $\neg P$ is the statement

$$\neg P : \text{It is not that case that HYDERABAD is a city.}$$

Simply $\neg P$ can be written as

$$\neg P : \text{HYDERABAD is not a city.}$$ □

Example 3.1.2 The negation of the statement

$$P : 2+2>1$$

$$\neg P : 2+2 \not> 1$$

or $$\neg P : 2+2 \leq 1$$

in words "it is not true that $2+2>1$. □

Example 3.1.3 The negation of the statement

P : I went to a movie yesterday is

$\neg$P : I did not go to movie yesterday □

Note 3.1.1 The negation $\neg P$ of P is also denoted by '$\sim P$' or '$\bar{P}$' or 'not P'.

3.2 Conjunction

The conjunction ie., joining of two statements P and Q is the statement $P \wedge Q$ which is read as "P and Q". The statement $P \wedge Q$ has the truth value T whenever both P and Q have the truth value T; otherwise it has the truthvalue F.

The conjunction is defined as follows:

Truth table for conjunction:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

We illustrate the usage of 'and' by the following examples:

Example 3.2.1 The conjunction of the statements

P : It is raining.

$$Q : 2 + 2 = 4.$$

is $P \wedge Q$: It is raining and $2 + 2 = 4$. □

Note 3.2.1 To form $P \wedge Q$, the statements P and Q need not be related to each other in one way or the other. We can form $P \wedge Q$ even if P and Q are totally unrelated to each other. The statements P and Q given in above example have no common property or any relation.

The truthvalue of $P \wedge Q$ is True only when the statements P and Q are both true. Let us consider the case when it is raining. Then P is true. The statement Q is always true. So in the case of raining, as both statements P and Q are true, $P \wedge Q$ is also true. When it is not raining, the statement P is false and by definition $P \wedge Q$ is also false.

Example 3.2.2 Translate the following statement into symbolic form

Ramu and Raghu went to school.

Solution: In order to write it as a conjunction of two statements, it is necessary first to paraphrase the statement as Ramu went to school and Raghu went to school.

Now write: P: Ramu went to school

Q: Raghu went to school

then the given statement can be written in symbolic form as $P \wedge Q$. □

Note 3.2.2 From the definition of $P \wedge Q$, it is clear that the truth value of the conjunction $P \wedge Q$ of two statements P and Q depends upon the truth values of P and Q. There are 2^2 possible combinations of truth values of P and Q that must be considered, because, each one of the statements P and Q can have any one of the two possible truth vlaues true and false. For each such possible combinations of truth values of P and Q, we determine the truth value of $P \wedge Q$. All possible truth values of $P \wedge Q$ can be shown by means of a table (discussed already).

3.3 Disjunction

The disjunction of two statements P and Q is the statement $P \vee Q$ which is read as "P or Q". The statement $P \vee Q$ has the truth value F only when both P and Q have the truth value F otherwise it is true. The disjunction is defined by the following table:

Truth table for disjunction:

Thus $P \vee Q$ is true if either P is true or Q is true (or both P and Q are true).

Note 3.3.1 The connective $\vee$ is not always the same as the word "or" because of the fact that the word "or" in English Language can be used in two different senses:

i) Inclusive OR (one or the other or both) and

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

ii) Exclusive OR (one or the other, but not both)

In logic we use $\vee$ ('or') as inclusive OR.
For example, Consider the following statements:

1. Ramu will take Mpc or Bi.p.c group in intermediate. The above statement says that only one of the group will be taken by Ramu as a main group of study in his intermediate. Here 'or' is used in the sense 'one or the other, but not both', (Exclusive OR).
2. "I will buy a computer or a car next year" The above statement indicates that the speaker may mean that he is trying to make up his mind so as to which one of the two to buy, but he could also mean that he will buy atleast one of them, possibly both. Here 'or' is used in the sense 'one or the other or both' (Inclusive OR).

Example 3.3.1 Consider the following statements

P : I will buy a computer

Q : I will buy a car

Then $P \vee Q$ is the following statement

$P \vee Q$: I will buy a computer or I will buy a car.

3.4 Conditional Statements

If P nd Q are any two statements, then the statement $P \rightarrow Q$ read as "If P, then Q" is called a *conditional* statement. The Statement $P \rightarrow Q$ has a truth value F when Q has the truth value F and P the truth value T; otherwise it has the truth value T. The conditional is defined by the following table:

Truth table for conditional: $P \rightarrow Q$

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The Statement P is called the *antecedent* and Q the *consequent* in $P \rightarrow Q$. The sign "$\rightarrow$" is called the sign of implication we will also write $P \rightarrow Q$, for

P only if Q

Q if P

Q provided that P

P is sufficient condition for Q

Q is necessary condition for P

P implies Q

Q is implied by P

Note 3.4.1 According to the definition, it is not necessary that there be any kind of relation between P and Q in order to form $P \rightarrow Q$.

Note 3.4.2 In general, the use of "If ..., then ..." in English has only partial resemblance to the use of $\rightarrow$ in logic.

Note 3.4.3 The converse of $P \rightarrow Q$ is $Q \rightarrow P$ and the contrapositive of $P \rightarrow Q$ is $\neg Q \rightarrow \neg P$. The inverse of $P \rightarrow Q$ is $\neg P \rightarrow \neg Q$. Also $P \rightarrow Q$ and contrapositive $\neg Q \rightarrow \neg P$ have the same truth values.

Example 3.4.1 Let

P : Amulya works hard

Q : Amulya will pass the exam.

Then $P \rightarrow Q$: If Amulya works hard, then she will pass the exam. □

3.5 Biconditional Statements

If P and Q are any two statements, then the statement $P \rightleftarrows Q$ which is read as "P if and only if Q" and abbreviated as 'P iff Q' is called a biconditional statement. The statement $P \rightleftarrows Q$ has the truth value T whenver both P and Q have identical truth values. The biconditional $P \rightleftarrows Q$ is the conjunction of the conditionals $P \rightarrow Q$ and $Q \rightarrow P$. i.e., $(P \rightarrow Q) \wedge (Q \rightarrow P)$ is an alternate notation for $P \rightleftarrows Q$. The following table defines the biconditional:

Truth table for biconditional $P \rightleftarrows Q$

P	Q	$P \rightleftarrows Q$
T	T	T
T	F	F
F	T	F
F	F	T

Also Truth table for $(P \to Q) \wedge (Q \to P)$

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that both truth tables are identical.

Thus biconditional $P \rightleftarrows Q$ may be read by following way:

1. P if and only if Q.
2. P is equivalent to Q.
3. P is necessary and sufficient condition for Q.
4. Q is necessary and sufficient condition for P.

We also write '$P \rightleftarrows Q$' for '$P \leftrightarrow Q$'.

Example 3.5.1 1. $8 > 4$ if and only if $8 - 4$ is positive

2. $2 + 2 = 4$ if and only if it is raining
3. Two lines are parallel if and only if they have the same slope.

Example 3.5.2 Write the following statement in symbolic form.

If either Mr. Srinu takes calculus or Mr. Swamy takes Graph theory then Mr. Mahesh will take computer programming.

Solution: Denoting the statements as

S : Mr. Srinu takes calculus

W : Mr. Swamy takes Graph theory

M : Mr. Mahesh takes computer programming

the above statement can be symbolized as

$$(S \vee W) \to M$$

4. Statement Formulas

Statements which do not contain any connectives are called atomic or simple statements. On the other hand, the statements which contain one or more primary statements and at least one connective are called molecular or composite or compound statements.

For example, let P and Q be any two simple statements. Some of the compound statements formed by P and Q are

$\neg P \quad P \vee Q \quad (P \wedge Q) \vee (\neg P) \quad P \wedge (\neg Q) \quad (P \vee \neg Q) \wedge P$

The above compound statements are called *Statement formulas* derived from the *Statement variables* P and Q. Therefore P and Q are called as *Components* of the statement formulas.

A statement formula alone has no truth value. It has truth value only when the statement variables in the formula are replaced by definite statements and it depends on the truth values of the statements used in replacing the variables.

5. The Truth Table of a Statement Formula

A table showing all the possible truth values of a statement formula for each possible combination of the truth values of the component statements is called the *truth table* of the formula.

Truth tables have already been introduced in the definitions of the connectives.

In general, if there are 'n' distinct components in a statement formula, we need to consider 2^n possible combinations of truth values in order to obtain the truth table.

For example, if any statement formula have two component statements namely P and Q, $\because$ 2^2 possible combinations of truth values must be considered.

Example 5.1 Construct the truth table for $P \wedge \neg P$

Solution:

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

Example 5.2 Construct the truth table for $P \vee \neg P$

Solution:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Example 5.3 Construct the truth table for $P \to (Q \to R)$.

Solution: P, Q and R are the three statement variables that occur in this formula $P \rightarrow (Q \rightarrow R)$. There are $2^3 = 8$ different sets of truth value assignments for the variables P, Q and R. They are

The following table is the truth table fo $P \rightarrow (Q \rightarrow R)$

P	Q	R	$Q \rightarrow R$	$P \rightarrow (Q \rightarrow R)$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

□

Example 5.4 Construct the truth table for the formula

$$(P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \quad \text{(I)}$$

Solution:

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge \neg Q$	(I)
T	T	F	F	T	F	F	F	T
T	F	F	T	F	F	T	F	T
F	T	T	F	F	T	F	F	T
F	F	T	T	F	F	F	T	T

□

Exercise 1:'

(I) Write the following statements in symbolic form with statements

P : Pavan is rich

Q : Raghav is happy

(a) Pavan is rich and Raghav is not happy

(b) Pavan is not rich and Raghav is happy

(II) Write the following statements in symbolic form with statements

R : Naveen is rich

H : Naveen is happy

(a) Naveen is poor but happy

(b) Naveen is rich or unhappy

(c) Naveen is neither rich nor happy

(d) Naveen is poor or he is both rich and unhappy

(III) Write the following statements in symbolic form with statements.

P : Naveen is smart

Q : Amal is smart

(a) Naveen is smart and Amal is not smart

(b) Naveen and Amal are both smart

(c) Neither Naveen nor Amal are smart

(d) It is not true that Naveen and Amal are both smart.

(IV) Let P, Q, R denote the following statements:

P : Triangle ABC is isosceles

Q : Triangle ABC is Equilateral

R : Triangle ABC is Equiangular

Translate each of the following into a statement of English

(a) $Q \to P$

(b) $\neg P \to \neg Q$

(c) $Q \rightleftarrows \mathrm{R}$

(d) $P \wedge \neg Q$

(e) $\mathrm{R} \to P$

(V) If P, Q, R are three statements with truth values 'true', 'true' and 'false' respectively, Find the truth values of the following

(i) $P \vee Q$ (ii) $P \wedge R$ (iii) $(P \vee Q) \wedge R$ (iv) $P \wedge (\neg R)$

(v) $(P \wedge \neg Q) \wedge (\neg R)$ (vi) $P \rightarrow R$ (vii) $P \rightarrow Q$ (viii) $R \rightarrow P$

(ix) $(R \wedge P) \rightarrow Q$ (x) $(P \wedge \neg Q) \rightarrow R$ (xi) $(P \vee Q) \longleftrightarrow (P \rightarrow \neg R)$

(xii) $(P \longleftrightarrow R) \rightarrow R$

(VI) If P, Q are statements with truth values 'true' and R and S are statements with truth value 'false'. Find the truth value of the following

(a) $(R \wedge P) \rightarrow S$
(b) $(P \wedge Q) \wedge R$
(c) $(P \rightleftarrows Q) \rightarrow (S \longleftrightarrow R)$
(d) $P \vee (Q \wedge S)$
(e) $(P \rightarrow \neg Q) \rightarrow (S \rightleftarrows R)$
(f) $(P \rightarrow \neg Q) \rightarrow (P \vee Q)$
(g) $P \rightarrow (Q \longleftrightarrow (R \rightarrow S))$
(h) $S \rightarrow P$

(VII) Construct the truth tables for the following formulas

(a) $\neg(\neg P \wedge \neg Q)$
(b) $(\neg P \vee Q) \wedge (\neg Q \vee P)$
(c) $(P \wedge Q) \rightarrow (P \vee Q)$

(VIII) Given the truth values of P and Q as T and those of R and S as F, find the truth values of the following

(a) $P \vee (Q \wedge R)$
(b) $(P \wedge (Q \wedge R)) \wedge \neg((P \vee Q) \wedge (R \vee S))$

Answers 1

(I) (a) $P \wedge \neg Q$
(b) $\neg P \wedge Q$

(II) (a) $\neg R \wedge H$
(b) $R \vee \neg H$
(c) $\neg R \wedge \neg H$
(d) $\neg R \vee (R \wedge \neg H)$

(III) (a) $P \wedge \neg Q$

(b) $P \wedge Q$

(c) $\neg P \wedge \neg Q$

(d) $\neg(P \wedge Q)$

(IV) (a) If triangle ABC is equilateral, then it is isosceles

(b) If the triangle ABC is not isosceles then it is not equilateral

(c) The triangle ABC is equilateral if and only if it is equiangular

(d) If the triangle ABC is equiangular, then it is isosceles.

(V) (i) T (ii) F (iii) F (iv) T

(v) F (vi) F (vii) T (viii) T

(ix) T (x) T (xi) T (xii) T

(VI) (a) T (b) F (c) T (d) T

(e) T (f) T (g) T (b) T

(VII) (a) The variables that occur in the formula are P and Q so we have to consider $2^2 = 4$ possible combinations of truthvalues of two statements P and Q

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

The entries in the last column are the truth values of the formula $\neg(\neg P \wedge \neg Q)$.

(b) The variables are P and Q, clearly there are 2^2 rows in the truth table of this formula

P	Q	$\neg P$	$\neg Q$	$\neg P \vee Q$	$\neg Q \vee P$	$(\neg P \vee Q) \wedge (\neg Q \vee P)$
T	T	F	F	T	T	T
T	F	F	T	F	T	F
F	T	T	F	T	F	F
F	F	T	T	T	T	T

(c)

P	Q	$P \wedge Q$	$P \vee Q$	$(P \wedge Q \rightarrow (P \vee Q))$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

(VIII) (a)

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	F	F	T

(b) Check that

$$(P \wedge (Q \wedge R)) \wedge \neg((P \vee Q) \wedge (R \vee S)) \text{ is true}$$

6. Well-formed Formulae

Definition 6.1 A statement formula is an expression which is a string consisting of variables (capital letters with or without subscripts), parentheses and connective symbols ($\wedge$, $\vee$, $\rightarrow$ $\rightleftarrows$, $\neg$), which produces a statement when the variables are replaced by statements.

But, every string of these symbols is not a formula. We now give a recursive definition of a statement formula, called a well-formed formula (wff).

Definition 6.2 A Well-formed formula (wff) can be generated by the following rules:

1. A statement variable standing alone (ie., a string of length one, consisting of a statement variable) is a well-formed formula.
2. If A is a well-formed formula, then $\neg A$ is a well-formed formula
3. If A and B are well-formed formulas, then $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \rightleftarrows B)$ are well-formed formulas.
4. A string of symbols containing the statement variables, connectives and parentheses is a well-formed formula, if and only if it can be obtained by finitely many applications of rules (1), (2) and (3).

Similarly the set of all well-formed formulas can be defined as follows:

Definition 6.3 The set of all well-formed formulas is the smallest set of strings such that

i) Every statement variable is in the set
ii) If A and B are in the set, then so are $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \rightleftarrows B)$

Note 6.1 Now-onwards, formula means well-formed formula.

Example 6.1 *Formulae:* $\neg(P \wedge Q)$, $\neg(P \vee Q)$

$(P \wedge Q)$; $(\neg(P \wedge Q)) \rightarrow (R \wedge (\neg S))$;

$(\neg((\neg P) \wedge (\neg Q)))$; $(P \rightarrow (P \vee Q))$;

$(((P \vee Q) \wedge (P \vee \neg S)) \rightarrow (P \rightleftarrows R))$;

$((P \wedge Q) \rightarrow Q)$; $(P \rightarrow (Q \rightarrow R))$

Not formulae :

1. $\neg P \vee Q$, obviously P and Q are wffs, A wff would be either $(\neg P \vee Q)$ or $\neg(P \vee Q)$.
2. $((P \rightarrow Q) \rightarrow (\wedge Q))$ is not a formula, as $(\wedge Q)$ is not a wff.
3. $(P \rightarrow Q$ is not a wff as ')' is omitted. Note that $(P \rightarrow Q)$ is a formula.
4. $(P \wedge Q \rightarrow Q)$. The cause for this not being a wff is that one of the parentheses in the beginning is missing. $((P \wedge Q) \rightarrow Q)$ is a wff, while $(P \wedge Q) \rightarrow Q$ is still not a wff.

It is possible to introduce some conventions so that the number of parentheses used can be reduced.

1. For the sake of convenience, we shall omit the outer parentheses. Thus we write $P \vee Q$ for $(P \vee Q)$, $(P \vee Q) \rightarrow Q$ in place of $((P \vee Q) \rightarrow Q)$, $((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightleftarrows (P \rightarrow R)$ instead of $(((P \rightarrow Q) \wedge (Q \rightarrow R)) \rightleftarrows (P \rightarrow R).)$.
2. Also $(\neg(P \wedge Q) \rightarrow ((\neg P) \wedge (\neg R)))$ can be abbreviated to $\neg(P \wedge Q) \rightarrow (\neg P \wedge \neg R)$.

Note 6.2 It should be remembered that the above are just conventions, and the precise definitions must include the parentheses.

7. Tautology

Definition 7.1 A statement formula which is true regardless of the truth values of the statements which replace the variables in it is called a universally valid formula or a *tautology* or a logical truth.

i.e., If each entry in the final columr. of the truthtable of a statement formula is T alone, then it is called as *tautology*.

Similarly,

Definition 7.2 A statement formula which is false regardless of the truth values of the statements which replaces the variables in it is called a *contradiction*.

i.e., If each entry in the final column of the truth table of a statement formula is F alone, then it is called as *contradiction*.

Clearly, the negation of a contradiction is a tautology and vice-versa.

We can call a statement formula which is a tautology as identically true and a formula which is a contradiction as identically false.

Determining whether a given formula is a Tautology:

I) BY TRUTH TABLE

The first, straight forward method to determine whether a given formula is a tautology is to construct its *truth table*.

Example 7.1 i) Verify whether $P \vee \neg P$ is a tautology.

Solution:

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

Since, the entries in the last column of the truth table are T, thereore the given is a tautology.

ii) Verify whether $P \wedge \neg P$ is a tautology

Solution: Check that the last column of the truth table of $P \wedge \neg P$ contains false,

$\because$ The formula is not a tautology in particular, it is a contradiction

iii) Verify whether $(P \vee Q) \rightarrow P$ is a tautology

Solution:

P	Q	$P \vee Q$	$(P \vee Q) \rightarrow P$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	T

Since the entries in the last column of the truth table $(P \vee Q) \rightarrow P$, contain one false, the formula is not a tautology. □

Example 7.2 Verify whether $(P \wedge (P \rightleftarrows Q)) \rightarrow Q$ is a tautology

Solution:

P	Q	$(P \rightleftarrows Q)$	$P \wedge (P \rightleftarrows Q)$	$(P \wedge (P \rightleftarrows Q)) \rightarrow Q$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

As the entries in the last column are T, the given formula is a tautology.

Exercise 2:

I. Prove the following are tautologies (using truth tables):

(a) $\neg(P \vee Q) \vee (\neg P \wedge Q) \vee P$

(b) $((P \to Q) \wedge (R \to S) \wedge (P \vee R)) \to (Q \vee S)$

(c) $((P \to R) \wedge (Q \to R)) \to ((P \vee Q) \to R)$

(d) $((P \to (Q \vee R)) \wedge (\neg Q)) \to (P \to R)$

(e) $(((P \cup Q) \to R) \wedge (\neg P)) \to (Q \to R)$

(f) $(P \to Q) \rightleftarrows (\neg P \vee Q)$

(g) $Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

II. Show that the truth values of the following formula is independent of its components.

$(P \to Q) \rightleftarrows (\neg P \vee Q)$

Answers 2:

I. Construct truth tables for all the given formulae.

II. Since the given formulae has truth value T for any truth values of its statement variables (or) components. It is independent of its components.

(II) Alternative Method:

Recall that the numbers of rows in a truth table is 2^n, where n is the number of distinct variables in the formula. Therefore, this process of determining whether a given formula is a tautology is tedious, particularly when the number of distinct variables is large (or) when the formula is complicated.

Keeping the above point in view, we now consider alternative methods to determine whether a statement formula is a tautology without constructing its truth table.

(i) It is very clear, that the conjunction of two tautologies is also a tautology. Let us denote by A and B two statement formulas which are tautologies. If we assign any truth values to the variables of A and B, then the truth values of both A and B will be T. Thus the truth value of $A \wedge B$ will be T, so that $A \wedge B$ will be a tautology.

(ii) A formula A is called a *substitution instance* of another formula B if A can be obtained from B by substituting formulas for some variables of B, with the condition that the same formula is substituted for the same variable each time it occurs.

It should be noted that in constructing substitution instances of a formula, substitutions are made for the atomic formula and never for the molecular formula. Thus $P \to Q$ is not a substitution instance of $P \to \neg R$ because it is R which must be replaced and not $\neg R$.

Example 7.3 Substitution instances of $P \to \neg Q$ are:

1. $(R \wedge S) \to \neg(J \vee M)$
2. $Q \to \neg(P \wedge \neg Q)$
3. $(R \wedge \neg S) \to \neg P$
4. $(P \vee Q) \to \neg R$

It should also be noted that in constructing substitution instances of a formula, substitutions should be made simultaneously, not one after the other.

Example 7.4 Consider the following formulas from $P \to \neg Q$

(i) Substitute $P \vee Q$ for P and R for Q to get the substitution instance $(P \vee Q) \to \neg R$.

(ii) First substitute $P \vee Q$ for P to obtain the substitution instance $(P \vee Q) \to \neg Q$. Next, substitute R for Q in $(P \vee Q) \to \neg Q$, and we get $(P \vee R) \to \neg R$. This formula is a substitution instance of $(P \vee Q) \to \neg Q$, but it is not a substitution instance of $P \to \neg Q$ under the substitution $(P \vee Q)$ for P and R for Q, because we did not substitute simultaneously as we did in (i).

The importance of substitution instance lies in the fact that any substitution of a tautology is a tautology. For example, consider the tautology $P \vee \neg P$. Regardless of what is substituted for P, the truth value of $P \vee \neg P$ is always T. Therefore, if we substitute any statement formula for P, the resulting formula will be a tautology. Hence The following substitution instances of $P \vee \neg P$ are tautologies

$$((P \vee Q) \wedge R) \vee \neg((P \vee Q) \wedge R)$$

$$(((P \vee \neg S) \to R) \rightleftarrows S) \vee \neg(((P \vee \neg S) \to R) \rightleftarrows S)$$

Thus, if it is possible to detect whether a given formula is a substitution instance of a tautology, then it is immediately known that the given formula is also a tautology.

8. Equivalence of Formulae

Definition 8.1 Two formulas A and B are said to be *equivalent* to each other if and only if $A \rightleftarrows B$ is a tautology.

If $A \rightleftarrows B$ is a tautology, we write $A \Leftrightarrow B$.

Note 8.1 $A \Leftrightarrow B$ if and only if the truth tables of A and B are the same.

8.1 Truth table method

One method to determine whether any two statement formulas are *equivalent* is to construct their truth tables.

Example 8.1.1 Prove

$$P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q)$$

Solution:

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$	$P \vee Q \rightleftarrows \neg(\neg P \wedge \neg Q)$
T	T	T	F	F	F	T	T
T	F	T	F	T	F	T	T
F	T	T	T	F	F	T	T
F	F	F	T	T	T	F	T

As $P \vee Q \rightleftarrows \neg(\neg P \wedge \neg Q)$ is a tautology, then $P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q)$.

Example 8.1.2 Prove $(P \rightarrow Q) \Longleftrightarrow (\neg P \vee Q)$.

Solution:

P	Q	$P \rightarrow Q$	$\neg P$	$\neg P \vee Q$	$(P \rightarrow Q) \rightleftarrows (\neg P \vee Q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(P \rightarrow Q) \rightleftarrows (\neg P \vee Q)$ is a tautology.
then $(P \rightarrow Q) \Longleftrightarrow (\neg P \vee Q)$.

Equivalent formulas:

$P \vee P \Longleftrightarrow P$	$P \wedge P \Longleftrightarrow P$	Idempotent laws
$(P \vee Q) \vee R \Longleftrightarrow P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \Longleftrightarrow P \wedge (Q \wedge R)$	Associative laws
$P \vee Q \Longleftrightarrow Q \vee P$	$P \wedge Q \Leftarrow Q \wedge P$	Commutative laws
$P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$	$P \wedge (Q \vee R) \Longleftrightarrow (P \wedge Q) \vee (P \wedge R)$	Distributive laws
$P \vee \boldsymbol{F} \Longleftrightarrow P$	$P \wedge \boldsymbol{T} \Longleftrightarrow P$	
$P \vee \boldsymbol{T}, \boldsymbol{F} \Longleftrightarrow \boldsymbol{T}$	$P \wedge \boldsymbol{T}, \boldsymbol{F} \Longleftrightarrow \boldsymbol{F}$	
$P \vee \neg P \Longleftrightarrow \boldsymbol{T}$	$P \wedge \neg P \Longleftrightarrow \boldsymbol{F}$	
$P \vee (P \wedge Q) \Longleftrightarrow P$	$P \wedge (P \vee Q) \Longleftrightarrow P$	Absorption laws
$\neg(P \vee Q) \Longleftrightarrow \neg P \wedge \neg Q$	$\neg(P \wedge Q) \Longleftrightarrow \neg P \vee \neg Q$	Demorgan's laws

* Check the above formulas as an exercise by truth table technique

8.2 Replacement process

Consider the formula $A : P \to (Q \to R)$. The formula $Q \to R$ is a part of the formula A. If we replace $Q \to R$ by an *equivalent formula* $\neg Q \vee R$ in A, we get another formula $B : P \to (\neg Q \vee R)$. One can easily verify that the formulas A and B are equivalent to each other. This process of obtaining B from A is known as the replacement process.

Example 8.2.1 Prove that $P \to (Q \to R) \Longleftrightarrow P \to (\neg Q \vee R) \Longleftrightarrow (P \wedge Q) \to R$.

Solution: We know that $Q \to R \Longleftrightarrow \neg Q \vee R$
Replacing $Q \to R$ by $\neg Q \vee R$, we get $P \to (\neg Q \vee R)$, which is equivalent to $\neg P \vee (\neg Q \vee R)$ by the same rule,
Now

$$\neg P \vee (\neg Q \vee R) \Longleftrightarrow (\neg P \vee \neg Q) \vee R \Longleftrightarrow \neg(P \wedge Q) \vee R) \Longleftrightarrow (P \wedge Q) \to R$$

by associativity of $\vee$, Demorgan's law and the previously used rule.

Example 8.2.2 $(P \to Q) \wedge (R \to Q) \Longleftrightarrow (P \vee R) \to Q$

Solution: $(P \to Q) \wedge (R \to Q) \Longleftrightarrow (\neg P \vee Q) \vartriangle (\neg R \vee Q)$ replacing $P \to Q$ and $R \to Q$ by $\neg P \vee Q$ and $\neg R \vee Q$. respectively

$$\Longleftrightarrow (\neg P \wedge \neg R) \vee Q \left[\because (S_1 \vee S_2) \wedge (S_3 \vee S_2) \Leftrightarrow (S_1 \wedge S_3) \vee S_2\right]$$
$$\Longleftrightarrow \neg(P \vee R) \vee Q \left[\text{replacing } \neg P \wedge \neg R \text{ by } \neg(P \vee R)\right]$$
$$\Longleftrightarrow P \vee R \to Q \left[\because \neg A \vee B \Leftrightarrow (A \to B)\right]$$

Example 8.2.3 Prove that

$$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Longleftrightarrow R.$$

Solution:

$$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$$
$$\Longleftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R) \quad \text{(Associative Law \& Distributive Law)}$$
$$\Longleftrightarrow (\neg(P \vee Q) \wedge R) \vee ((Q \vee P) \wedge R) \quad \text{(Demorgan's Laws)}$$
$$\Longleftrightarrow (\neg(P \vee Q) \vee (P \vee Q)) \wedge R \quad \text{(Distributive Law)}$$
$$\Longleftrightarrow \mathbf{T} \wedge R \quad \text{since} \quad \neg S \vee S = \mathbf{T}$$
$$\Longleftrightarrow R \quad \text{as} \quad \mathbf{T} \wedge R \leftrightarrow R$$

Example 8.2.4 $P \to (Q \to P) \Longleftrightarrow \neg P \to (P \to Q)$

Solution: $P \to (Q \to P) \Longleftrightarrow \neg P \vee (Q \to P)$

$$\Leftrightarrow \neg P \vee (\neg Q \vee P)$$
$$\Leftrightarrow (\neg P \vee P) \vee \neg Q$$
$$\Leftrightarrow T \vee (\neg Q)$$
$$\Leftrightarrow T$$

$$\text{and } \neg P \to (P \to Q) \Leftrightarrow \neg(\neg P) \vee (P \to Q)$$
$$\Leftrightarrow P \vee (\neg P \vee Q)$$
$$\Leftrightarrow (P \vee \neg P) \vee Q$$
$$\Leftrightarrow T \vee Q$$
$$\Leftrightarrow T.$$

So $P \to (Q \to P) \Leftrightarrow T \Leftrightarrow \neg P \to (P \to Q)$

Example 8.2.5 $(P \to Q) \wedge (R \to Q) \Leftrightarrow (P \vee R) \to Q$

Solution: $(P \to Q) \wedge (R \to Q) \Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q)$

$$\Leftrightarrow (\neg P \wedge \neg R) \vee Q$$
$$\Leftrightarrow \neg(P \vee R) \vee Q$$
$$\Leftrightarrow (P \vee R) \to Q$$

Example 8.2.6

(i) $\neg(P \rightleftarrows Q) \Leftrightarrow (P \vee Q) \wedge \neg(P \wedge Q)$

(ii) $\neg(P \rightleftarrows Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Solution:

$$\neg(P \rightleftarrows Q) \Leftrightarrow \neg((P \to Q) \wedge (Q \to P))$$
$$\Leftrightarrow \neg((\neg P \vee Q) \wedge (\neg Q \vee P))$$
$$\Leftrightarrow \neg[((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)]$$
$$\Leftrightarrow \neg((\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) \vee (\neg P \wedge P) \vee (Q \wedge P))$$
$$\Leftrightarrow \neg(\neg(P \vee Q) \vee F \vee F \vee (Q \wedge P))$$
$$\Leftrightarrow \neg(\neg(P \vee Q) \vee (Q \wedge P))$$
$$\Leftrightarrow (P \vee Q) \wedge \neg(P \wedge Q) \qquad \text{(i)}$$
$$\Leftrightarrow (P \vee Q) \wedge (\neg P \vee \neg Q)$$
$$\Leftrightarrow (P \wedge (\neg P \vee \neg Q)) \vee (Q \wedge (\neg P \vee \neg Q))$$
$$\Leftrightarrow [(P \wedge \neg P) \vee (P \wedge \neg Q)] \vee [(Q \wedge \neg P) \vee (Q \wedge \neg Q)]$$

$$\Leftrightarrow F \vee (P \wedge \neg Q) \vee (Q \wedge \neg P) \vee \mathbf{F} \qquad \text{(By associative law)}$$
$$\Leftrightarrow (P \wedge \neg Q) \vee (Q \wedge \neg P)$$
$$\Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q) \qquad \text{(ii)}$$

Thus (i) and (ii) are proved.

Example 8.2.7 Show that $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology

Solution: By Demorgan's Laws, we have

$$\neg P \wedge \neg Q \Leftrightarrow \neg(P \vee Q)$$
$$\neg P \wedge \neg R \Leftrightarrow \neg(P \vee R)$$
$$\neg(P \wedge \neg Q) \vee (\neg P \wedge \neg R) \Leftrightarrow \neg(P \vee Q) \vee \neg(P \vee R)$$
$$\Leftrightarrow \neg((P \vee Q) \wedge (P \vee R))$$

Also

$$\neg(\neg P \wedge (\neg Q \vee \neg R)) \Leftrightarrow \neg(\neg P \wedge \neg(Q \wedge R))$$
$$\Leftrightarrow P \vee (Q \wedge R)$$
$$\Leftrightarrow (P \vee Q) \wedge (P \vee R)$$
$$(P \vee Q) \wedge ((P \vee Q) \wedge (P \vee R)) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

Consequently, the given formula is equivalent to

$$((P \vee Q) \wedge (P \vee R)) \vee \neg((P \vee Q) \wedge (P \vee R))$$

which is a substitution instance of $P \vee \neg P$.

Exercises (3)

(I) Show the following equivalences (without using truth table).

(a) $P \to (Q \vee R) \Longleftrightarrow (P \to Q) \vee (P \to R)$

(b) $\neg(P \to Q) \Longleftrightarrow P \wedge \neg Q$

(c) $(P \rightleftarrows Q) \Longleftrightarrow (P \to Q) \wedge (Q \to P)$

(d) $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Longleftrightarrow (\neg P \vee Q)$

(e) $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Longleftrightarrow (\neg P \wedge \neg Q)$

(f) $P \to Q \Longleftrightarrow \neg Q \to \neg P$

(II) Show that P is equivalent to the following formulas

$\neg\neg P$, $P \wedge P$, $P \vee P$, $P \vee (P \wedge Q)$, $P \wedge (P \vee Q)$, $(P \wedge Q) \vee (P \wedge \neg Q)$, $(P \vee Q) \wedge (P \vee \neg Q)$

9. Law of Duality

Definition 9.1 Two formulas A and A^* are said to be *duals* of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$. The connectives $\wedge$ and $\vee$ are also called duals of eath other. If the formula A contains the special variables T or F, the A^*, its dual is obtained by replacing T by F and F by T in addition to the above-mentioned interchanges

Example 9.1 If the formula A is given by

$$A : \neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg R))$$

The its dual A^* is given by $A^* : \neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg R))$

Result 9.1 Let A and A^* be dual formulas and let $P_1, P_2, \ldots, P_n$ be all the atomic variables that are in A and A^*.

i.e; $A : A(P_1, P_2, \ldots, P_n)$ and
$A^* : A^*(P_1, P_2, \ldots, P_n)$

Then using the Demorgan's Laws $P \wedge Q \Longleftrightarrow \neg(\neg P \vee \neg Q)$
$P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q)$

We can prove

$$\neg A(P_1, P_2, \ldots, P_n) \Longleftrightarrow A^*(\neg P_1, \neg P_2, \ldots, \neg P_n) \tag{1}$$

Thus the negation of a formula is equivalent to its dual in which every variable is replaced by its negation.

Similarly

$$A(\neg P_1, \neg P_2, \ldots \neg P_n) \Longleftrightarrow \neg A^*(P_1, P_2, \ldots P_n). \tag{2}$$

Example 9.2 Show that $\neg(\neg P \wedge \neg(Q \vee R)) \Longleftrightarrow P \vee (Q \vee R)$

Solution: Let $A(P, Q, R)$ be $\neg P \wedge \neg(Q \vee R)$

Then $A^*(P, Q, R)$ is $\neg P \vee \neg(Q \wedge R)$ and

$$A^*(\neg P, \neg Q, \neg R) : \neg\neg P \vee \neg(\neg Q \wedge \neg R) \Longleftrightarrow P \vee (Q \vee R)$$

On the other hand

$$\neg A(P, Q, R) \text{ is } \neg(\neg P \wedge \neg(Q \vee R)) \Longleftrightarrow P \vee (Q \vee R)$$

Result 9.2 If any two formulas are equivalent, then their duals are also equivalent to each other.

i.e., If $A \Longleftrightarrow B$ then $A^* \Longleftrightarrow B^*$

(OR) Let $P_1, P_2, \ldots, P_n$ be all the atomic variables appearing in the formulas A and B. Given that $A \Longleftrightarrow B$ means " $A \rightleftarrows B$ is a tautology", then the following are also tautologies.

$$A(P_1, P_2, \ldots, P_n) \rightleftarrows B(P_1, P_2, \ldots, P_n)$$
$$A(\neg P_1, \neg P_2, \ldots, \neg P_n) \rightleftarrows B(\neg P_1, \neq P_2, \ldots, \neq P_n)$$

Using (2) we get $\neg A^*(P_1, P_2, \ldots, P_n) \rightleftarrows \neg B^*(P_1, P_2, \ldots, P_n)$

Hence $A^* \Longleftrightarrow B^*$.

Example 9.3 Prove that

(a) $\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q)) \Longleftrightarrow (\neg P \vee Q)$

(b) $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Longleftrightarrow (\neg P \wedge Q)$

Solution:

(a) $\neg(P \wedge Q) \rightarrow (\neg P \vee (\neg P \vee Q))$

$$\begin{aligned}
&\Longleftrightarrow (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \\
&\Longleftrightarrow (P \wedge Q) \vee (\neg P \vee Q) \\
&\Longleftrightarrow (P \wedge Q) \vee \neg P \vee Q \\
&\Longleftrightarrow ((P \vee \neg P) \wedge (Q \vee \neg P)) \vee Q \\
&\Longleftrightarrow (Q \vee \neg P) \vee Q \\
&\Longleftrightarrow Q \vee \neg P \\
&\Longleftrightarrow \neg P \vee Q
\end{aligned}$$

(b) From (I) $(P \wedge Q) \vee (\neg P \vee (\neg P \vee Q)) \Longleftrightarrow \neg P \vee Q$
Writing dual,

$$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Longleftrightarrow \neg P \wedge Q.$$

10. Tautological Implications

Definition 10.1 A statement A is said to *tautologically imply* a statement B if and only if $A \rightarrow B$ is a tautology. In this case, we write $A \Longrightarrow B$, read as "A implies B"

Note 10.1 $\Longrightarrow$ is not a connective, $A \Longrightarrow B$ is not a statement formula.

i) Thus $A \Longrightarrow B$ states that $A \rightarrow B$ is a tautology or A tautologically implies B.
ii) Clearly $A \Longrightarrow B$ guarantees that B has the truthvalue T whenever A has the truth value T.
iii) By constructing the truth tables of A and B, we can determine whether $A \Longrightarrow B$.

Example 10.1 Prove that $(P \rightarrow Q) \Longrightarrow (\neg Q \rightarrow \neg P)$

Solution: We prove this by Using the truth table for $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$	$(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since all the entries in the last column are true, $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$ is a tautology
Hence $(P \rightarrow Q) \Longrightarrow (\neg Q \rightarrow \neg P)$.

(iv) In order to show any of the given implications, it is sufficient to show that an assignment of the truth value T to the antecedent of the corresponding conditional leads to the truth value T for the consequent. This procedure ensures that the conditional becomes a tautology, thereby proving the implication.

Example 10.2 Prove that $\neg Q \wedge (P \rightarrow Q) \Longrightarrow \neg P$

Solution: Assume that the antecedent $\neg Q \wedge (P \rightarrow Q)$ has the true value T, then both $\neg Q$ and $P \rightarrow Q$ have the truth value T, which means that Q has the value F. $P \rightarrow Q$ has the truth value T, and hence P must have the value F.
Therefore the consequent $\neg P$ must have the value T.

(v) Another method to show $A \Longrightarrow B$ is to assume that the consequent B has the value F and then show that this assumption leads to A's having the value F. Then $A \rightarrow B$ must have the value T.

Example 10.3 Show that $\neg(P \rightarrow Q) \Longrightarrow P$

Solution: Assume that P is false (F), when P is false $P \rightarrow Q$ has T, then $\neg(P \rightarrow Q)$ has F.
Then $\neg(P \rightarrow Q) \rightarrow P$ has T.
$\therefore \quad \neg(P \rightarrow Q) \Longrightarrow P$.

Note 10.2 $A \Longleftrightarrow B$ if and only if $A \Longrightarrow B$ and $B \Longrightarrow A$ i.e., If each of two formulas A and B implies the other, then A and B are equivalent.

Observations 10.1 If a formula is equivalent to a tautology then it must be a tautology

Observations 10.2 If a formula is implied by a tautology then it is a tautology

Observations 10.3 Both Implication and equivalence are transitive.

i.e., if $A \Longleftrightarrow B$ and $B \Longleftrightarrow C$, then $A \Longleftrightarrow C$.

It follows from the definition of equivalence.

To show that the implication is transitive:

Assume that $A \Longrightarrow B$ and $B \Longrightarrow C$.

The $A \rightarrow B$ and $B \rightarrow C$ are tautologies.

Hence $(A \rightarrow B) \wedge (B \rightarrow C)$ is also a tautology.

But from $(P \rightarrow Q) \wedge (Q \rightarrow R) \Longrightarrow P \rightarrow R$

$$(A \rightarrow B) \wedge (B \rightarrow C) \Longrightarrow (A \rightarrow C)$$

Hence $A \rightarrow C$ is a tautology.

Hence Implication is transitive. □

(vi) In order to show that $A \Longrightarrow C$, it is convenient to introduce a series of formulas $B_1, B_2, \ldots, B_m$ such that $A \Longrightarrow B_1, B_1 \Longrightarrow B_2, \ldots, B_{m-1} \Longrightarrow B_m$ and $B_m \Longrightarrow C$.

(vii) Important property of implication:
If $A \Longrightarrow B$ and $A \Longrightarrow C$, then $A \Longrightarrow (B \wedge C)$.

Proof: Assume if A is true, then B and C are both true. Thus $B \wedge C$ is true and hence $A \rightarrow (B \wedge C)$ is true. □

Result 10.1 If $H_1, H_2, \ldots, H_m$ and P imply Q, then $H_1, H_2, \ldots, H_m$ imply $P \rightarrow Q$.

Proof: We have $(H_1 \wedge H_2 \wedge \ldots \wedge H_m \wedge P) \Longrightarrow Q$

This means $(H_1 \wedge H_2 \wedge \ldots \wedge H_m \wedge P) \rightarrow Q$ is a tautology.
Now from the equivalence $P_1 \rightarrow (P_2 \rightarrow P_3) \Longleftrightarrow (P_1 \wedge P_2) \rightarrow P_3$
We can conclude that

$$(H_1 \wedge H_2 \wedge \ldots \wedge H_m) \rightarrow (P \rightarrow Q)$$

is a tautology.
Hence the theorem. □

Implications

$P \wedge Q \Longrightarrow P$

$P \wedge Q \Longrightarrow Q$

$P \Longrightarrow P \vee Q$

$\neg P \Longrightarrow P \rightarrow Q$

$Q \Longrightarrow P \rightarrow Q$

$\neg(P \rightarrow Q) \Longrightarrow P$

$\neg(P \rightarrow Q) \Longrightarrow \neg Q$

$P \wedge (P \rightarrow Q) \Longrightarrow Q$

$\neg Q \wedge (P \rightarrow Q) \Longrightarrow \neg P$

$\neg P \wedge (P \vee Q) \Longrightarrow Q$

$(P \rightarrow Q) \wedge (Q \rightarrow R) \Longrightarrow P \rightarrow R$

$(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \Longrightarrow R$

Check the above implications.

Exercise 4

(I) Show the following implications

(a) $(P \to (Q \to R)) \Longrightarrow (P \to Q) \to (P \to R)$
(b) $Q \Longrightarrow P \to R$
(c) $(P \wedge Q) \Longrightarrow P \to Q$

(II) show the following implications without Constructing the truth tables

(a) $\neg Q \wedge (P \to Q) \Longrightarrow \neg P$

(b) $(P \vee Q) \wedge (\neg P) \Longrightarrow Q$

(c) $P \to Q \Longrightarrow P \to (P \wedge Q)$

(d) $(P \to Q) \to Q \Longrightarrow P \vee Q$

(e) $((P \vee \neg P) \to Q) \to ((P \vee \neg P) \to R) \Longrightarrow (Q \to R)$

(f) $(Q \to (P \wedge \neg P)) \to (R \to (P \wedge \neg P)) \Longrightarrow (R \to Q)$

Answers (4)

(I) (a) we prove this by Using the truth table for

$$(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$$

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$	$P \to (Q \to R)$	$(P \to Q) \to (P \to R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

as the columns of $P \to (Q \to R)$ and $(P \to Q) \to (P \to R)$ are identical $(P \to (Q \to R)) \to ((P \to) \to (P \to R))$ is a tautology

Therefore $(P \to (Q \to R)) \Longrightarrow ((P \to Q) \to (P \to R))$.

(II) (a). To prove that $\neg Q \wedge (P \to Q) \Longrightarrow \neg P$, it is enough to show that the assumption that $\neg Q \wedge (P \to Q)$ has the truth value T guarentees the truth value T for $\neg P$.

Now assume that $\neg Q \wedge (P \to Q)$ has the truth value T. Then both $\neg Q$ and $P \to Q$ have the truth value T. Since $\neg Q$ has truth value T, Q has the truth value F. As Q has the truth value F and $P \to Q$ has the truth value T, it follows that the truth value of P is F and the truth value of $\neg P$ is T. Thus we have proved that $\neg Q \wedge (P \to Q) \Longrightarrow \neg P$.

11. Formulas containing n Variables

A statement formula containing n variables must have as its truth table one of the 2^{2^n} possible truth tables each of them having 2^n rows. This fact suggests that there are many formulas which may look very different from one another but are equivalent.

For the case $n = 1$, any formula involving only one variable will have one of these four truth tables:

	1	2	3	4
P	P	$\neg P$	$P \vee \neg P$	$P \wedge \neg P$
T	T	F	T	F
F	F	T	T	F

Every other formula depending upon P alone would then be equivalent to one of these four formulas.

In case of $n = 2$, the number of distinct truth tables for formulas involving two variables is $2^{2^2} = 2^4 = 16$. Clearly there are 2^2 rows in the truth table and since each row will have any of the two entries T or F, we have 16 possible tables.

P	Q	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
T	T	T	T	T	T	T	T	T	T	F	F	F	F	F	F	F	F
T	F	T	T	T	T	F	F	F	F	T	T	T	T	F	F	F	F
F	T	T	T	F	F	T	T	F	F	T	T	F	F	T	T	F	F
F	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F	T	F

Distinct Formulae with two variables

12. Functionally Complete Sets of Connectives

1. We have already defined the connectives $\wedge, \vee, \neg, \to \rightleftarrows$. Now we introduce some *other connectives* namely NAND and NOR which have useful applications in the design of computers. The word 'NAND' is a combination of 'NOT' and 'AND', while the word 'NOR' is a combination of 'NOT' and 'OR', where

'NOT' stands for negation, 'AND' stands for conjunction and 'OR' stands for the disjunction. The connective 'NAND' is denoted by the symbol $\uparrow$.
For any two formulas P and Q

$$P \uparrow Q \Longleftrightarrow \neg(P \wedge Q)$$

The connective 'NOR' is denoted by the symbol $\downarrow$. For any two formulae

$$P \downarrow Q \Longleftrightarrow \neg(P \vee Q)$$

The connectives $\uparrow$ and $\downarrow$ have been defined interms of the connectives $\wedge$, $\vee$ and $\neg$. Therefore, for any formula containing the connectives $\uparrow$ or $\downarrow$, one can obtain an equivalent formula containing the connectives $\wedge$, $\vee$ and $\neg$ only. Note that $\uparrow$ and $\downarrow$ are duals of each other. Therefore in order to obtain the dual of a formula which includes $\uparrow$ or $\downarrow$, we should interchange $\uparrow$ and $\downarrow$ in addition to the other interchanges mentioned earlier.

Definition 12.1 A set of connectives is said to be *functionally complete set of connectives* if every formula can be expressed interms of an equivalent formula containing the connectives only from this set.

Note 12.1 Functionally complete set should not contain any redundant connectives i.e., a connective which can be expressed interms of the other connectives.

Already, we have

(1) $P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q)$

(2) $P \wedge Q \Longleftrightarrow \neg(\neg P \vee \neg Q)$

(3) $P \rightarrow Q \Longleftrightarrow \neg P \vee Q$

(4) $P \rightleftarrows Q \Longleftrightarrow (\neg P \vee Q) \wedge (P \vee \neg Q)$

Hence by first replacing all bi conditionals, then the conditionals and finally all the conjuctions or all the disjunctions in any formula, we can obtain an equivalent formula which contains either the negation and disjunction only or the negation and conjunction only. In other words, for every formula, we can find an equivalent formula containing the connectives $\vee$ and $\neg$ only or $\wedge$ and $\neg$ only. From the definition of functionally complete set of connectives, the sets of connectives $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ are functionally complete sets.

Note 12.2 The set $\{\wedge, \vee\}$ is not functionally complete, as for the formula $\neg P$, it is not possible to find an equivalent formula containing connectives only from the set $\{\wedge, \vee\}$.

Result 12.1 $\{\uparrow\}$, $\{\downarrow\}$ are functionally complete.

Proof: In order to prove, it is sufficient to show that the sets of connectives $\{\wedge, \neg\}$ and $\{\vee, \neg\}$ can be expressed either interms of $\uparrow$ alone or interms of $\downarrow$ alone.

To show that $\{\vee, \neg\}$ is functionally complete, it is enough to show that $\neg$ and $\vee$ can be expressed interms of $\downarrow$ alone.

We have

$$\neg P \Longleftrightarrow \neg P \wedge \neg P \Longleftrightarrow \neg(P \vee P) \Longleftrightarrow P \downarrow P \quad \text{and}$$

$$P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q) \Longleftrightarrow \neg P \downarrow \neg Q \Longleftrightarrow (P \downarrow P) \downarrow (Q \downarrow Q)$$

Then $\{\downarrow\}$ is a functionally complete set.

To show that $\{\neg, \wedge\}$ is functionally complete, we have to express $\neg$ and $\wedge$ in terms of $\uparrow$ alone. The following valid equalities help us in this direction.

$\neg P \Longleftrightarrow \neg P \vee \neg P \Longleftrightarrow \neg(P \wedge P) \Longleftrightarrow P \uparrow P$ and

$P \wedge Q \Longleftrightarrow \neg(P \uparrow Q) \Longleftrightarrow (P \uparrow Q) \uparrow (P \uparrow Q)$

Then $\{\uparrow\}$ is a functionally complete set.

Thus we proved that each of the sets $\{\uparrow\}$ and $\{\downarrow\}$ are functionally complete. □

Note 12.3

(a)

$$\neg P \Longleftrightarrow \neg P \vee \neg P \Longleftrightarrow \neg(P \wedge P) \Longleftrightarrow P \uparrow P$$

$$P \wedge Q \Longleftrightarrow \neg(P \uparrow Q) \Longleftrightarrow (P \uparrow Q) \uparrow (P \uparrow Q)$$

$$P \vee Q \Longleftrightarrow \neg(\neg P \wedge \neg Q) \Longleftrightarrow \neg P \uparrow \neg Q \Longleftrightarrow (P \uparrow P) \uparrow (Q \uparrow Q)$$

(b)

$$\neg P \Longleftrightarrow \neg(P \vee P) \Longleftrightarrow P \downarrow P$$

$$P \vee Q \Longleftrightarrow \neg(P \downarrow Q) \Longleftrightarrow (P \downarrow Q) \downarrow (P \downarrow Q)$$

$$P \wedge Q \Longleftrightarrow \neg P \downarrow \neg Q \Longleftrightarrow (P \downarrow P) \downarrow (Q \downarrow Q).$$

We call each of the sets $\{\uparrow\}$ and $\{\downarrow\}$ a *minimal functionally complete set* or inshort a *minimal set*.

Note (c):

(i) $P \uparrow Q \Longleftrightarrow Q \uparrow P$; $\quad P \downarrow Q \Longleftrightarrow Q \downarrow P$ (Commutative)

(ii) The connectives $\uparrow$ and $\downarrow$ are not associative

since

$$\begin{aligned} P \uparrow (Q \uparrow R) &\Longleftrightarrow P \uparrow \neg(Q \wedge R) \Longleftrightarrow (P \wedge \neg(Q \wedge R)) \\ &\Longleftrightarrow P \neg(Q \wedge R) \\ (P \uparrow Q) \uparrow R &\Longleftrightarrow (P \wedge Q) \wedge \neg R \end{aligned}$$

Similarly

$$P \downarrow (Q \downarrow R) \Longleftrightarrow \neg P \wedge (Q \vee R)$$

$$(P \downarrow Q) \downarrow R \Longleftrightarrow (P \vee Q) \wedge \neg R$$

(iii) $P \uparrow Q \uparrow R \iff \neg(P \wedge Q \wedge R)$

However $P \uparrow Q \uparrow R$ is not equivalent to any of

$$P \uparrow (Q \uparrow R), \quad (P \uparrow Q) \uparrow R, \quad Q \uparrow (P \uparrow R)$$

$$P \uparrow Q \iff \neg(P \wedge Q) \iff \neg P \vee \neg Q \iff (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$$

$$P \downarrow Q \iff \neg(P \vee Q) \iff \neg P \wedge \neg Q \iff (\neg P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee \neg Q)$$

13. Normal Forms

By constructing and comparing truth tables, we can determine whether two statement formulas A and B are equivalent. But this is very tedious and difficult to follow even on a computer because the number of entries increases very rapidly as n increases.

A better method is to transform the statement formulas A and B to some standards forms A' and B' such that a simple comparison of A' and B' shows whether $A \iff B$. The standard forms are called canonical forms or normal forms.

Let $A(P_1, P_2, \dots P_n)$ be a statement formula where $P_1, P_2, \dots P_n$ are the primitive variables.

If A has the truthvalue T for atleast one combination of truth values assigned to $P_1, P_2, \dots, P_n$ then A is said to be *satisfiable.*

The problem of determining, in a finite number of steps, whether a given statement formula is a tautology or a contradiction or at least satisfiable is known as a *decision problem*.

However, the solution of the decision problem may not be simple as we mentioned earlier, the construction of truth tables may not be practical, even with the help of a computer. Therefore we are in need of other procedures known as reduction to normal forms.

Note 13.1 It will be convenient to use the word "product" in place of "conjunction" and "sum" inplace of "disjunction" in our discussion throughout.

Definition 13.1 A product of (statement) variables and their negations is called an elementary product.

Similarly, a sum of the variables and their negations is called an elementary sum.

Example 13.1 The formulae P, $\neg P$, $\neg P \wedge Q$, $\neg Q \wedge P \wedge \neg P$, $P \wedge \neg P$, $Q \wedge \neg P \neg P \wedge \neg Q$ are some examples for elementary products. The formulae P, $\neg P$, $\neg P \vee Q$, $\neg Q \vee P \vee \neg P$, $P \vee \neg P$, $Q \vee \neg P$ are examples for elementary sums.

Definition 13.2 Any part of an elementary sum or product which is itself an elementary sum or product is called a *factor* of the original elementary sum or product.

Example 13.2 $\neg Q$, $P \wedge \neg P$ and $\neg Q \wedge P$ are some of the factors of $\neg Q \wedge P \wedge \neg P$.

Observations 13.1 A necessary and sufficient condition for an elementary product to be identically false is that it contain atleast one pair of factors in which one is the negation of the other.

Explanation 1 For any variable P, $P \wedge \neg P$ is identically false. Hence if $P \wedge \neg P$ appears in the elementary product, then the product is identically false.

Assume that if an elementary product is identically false and does not contain at least one factor of this type then we can assign truth values T and F to variables and negated variables respectively that appear in the product. But this assignment says that the elementary product has the truth value T. This is contrary to our assumption. Hence the observation follows.

Similarly,

Observations 13.2 A necessary and sufficient condition for an elementary sum to be identically true is that it contain at least one pair of factors in which one is the negation of the other.

The explanation of the observation 2 will follow along the similar lines of observation 1.

13.1 Disjunctive Normal Forms (d.n.f)

Definition 13.1.1 A formula which is equivalent to a given formula and which consists of a sum of elementary products is called a disjunctive normal form of the given formula.

Procedure to obtain a disjunctive normal form of a given formula

Step 1: If the connectives '$\rightarrow$' and $\rightleftarrows$ appear in the given formula, obtain an equivalent formula in which '$\rightarrow$' and '$\rightleftarrows$' do not appear. i.e., an equivalent formula can be obtained in which '$\rightarrow$' and '$\rightleftarrows$' do not appear. i.e., an equivalent formula can be obtained in which '$\rightarrow$' and '$\rightleftarrows$' are replaced by $\wedge$, $\vee$ and $\neg$.

Example 13.1.1 $P \rightarrow Q$ is replaced by $\neg P \vee Q$ and $P \rightleftarrows Q$ is replaced $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ or $(\neg P \vee Q) \wedge (\neg Q \vee P)$.

Therefore, there is no loss of generality in assuming that the given formula contains he connectives $\wedge$, $\vee$ and $\neg$ only.

Step 2: If the negation is applied to the formula or to a part of the formula and not to he variables appearing in it, (i.e., formula which is not a statement variable). Then by ısing DeMorgan's laws an equivalent formula can be obtained in which the negation is ıpplied to the statement variables only.

Step 3: Now apply the distributive law until a sum of elementary products is obtained. This will be a disjunctive normal form, after application of the Idempotent Law and suitable reordering. In this normal form, the elementary products which are equivalent to 'F' (false), if any, can be omitted.

Note 13.1.1 Extended Distributive Law

$$(P \vee Q) \wedge (R \vee S) \Longleftrightarrow (P \wedge R) \vee (P \wedge S) \vee (Q \wedge R) \vee (Q \wedge S)$$

This is as follows:

$$\begin{aligned}(P \vee Q) \wedge (R \vee S) &\Longleftrightarrow [(P \vee Q) \wedge R] \vee [(P \vee Q) \wedge S] \\ &\Longleftrightarrow (P \wedge R) \vee (Q \wedge R) \vee (P \wedge S) \vee (Q \wedge S)\end{aligned}$$

Example 13.1.2 obtain a disjuctive normal form of $P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$

Solution: $P \to ((P \to Q) \wedge \neg(\neg Q \vee \neg P))$

$$\begin{aligned}&\Longleftrightarrow \neg P \vee ((P \to Q) \wedge \neg(\neg Q \vee \neg P)) \\ &\Longleftrightarrow \neg P \vee ((\neg P \vee Q) \wedge \neg(\neg Q \vee \neg P)) \\ &\Longleftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P)) \\ &\Longleftrightarrow \neg P \vee [(\neg P \wedge (Q \wedge P)) \vee (Q \wedge (Q \wedge P))] \\ &\Longleftrightarrow \neg P \vee [(P \wedge \neg P \wedge Q)] \vee (Q \wedge P) \\ &\Longleftrightarrow \neg P \vee \boldsymbol{F} \vee (P \wedge Q) \\ &\Longleftrightarrow \neg P \vee (P \wedge Q)\end{aligned}$$

which is the required d.n.f

Example 13.1.3 obtain a d.n.f of $P \wedge (P \to Q)$

Solution:

$$P \wedge (P \to Q) \Longleftrightarrow P \wedge (\neg P \vee Q) \Longleftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$$

Example 13.1.4 obtain a d.n.f of $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

Solution:

$$\begin{aligned}&\neg(P \vee Q) \rightleftarrows (P \wedge Q) \\ &\Longleftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee ((P \vee Q) \wedge \neg(P \wedge Q)) \\ &\qquad \text{(Elimination of biconditional)} \\ &\Longleftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee ((P \vee Q) \wedge (\neg P \vee \neg Q)) \\ &\Longleftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \\ &\qquad \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q) \qquad \text{[by Extended distributive law]}\end{aligned}$$

Then the required d.n.f.: $(P \wedge \neg Q) \vee (\neg P \vee Q)$, by ingnoring '$\boldsymbol{F}$'

Example 13.1.5 Find a d.n.f of $(P \wedge \neg(Q \vee R)) \vee (((P \wedge Q) \vee \neg R) \wedge P)$.

Solution:

$$\begin{aligned}&(P \wedge \neg(Q \vee R)) \vee (((P \wedge Q) \wedge \neg R) \wedge P) \\ &\Longleftrightarrow (P \wedge (\neg Q \wedge \neg R)) \vee (((P \wedge Q) \vee \neg R) \wedge P) \\ &\Longleftrightarrow (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge P) \vee (\neg R \wedge P) \\ &\Longleftrightarrow (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q) \vee (\neg R \wedge P).\end{aligned}$$

Example 13.1.6 obtain a d.n.f of $(Q \vee (P \wedge R)) \wedge \neg((P \vee R) \wedge Q)$

Solution: $(Q \vee (P \wedge R)) \wedge \neg((P \vee R) \wedge Q)$

$$\Longleftrightarrow (Q \vee (P \wedge R)) \wedge (\neg(P \vee R) \vee \neg Q)$$
$$\Longleftrightarrow (Q \vee (P \wedge R)) \wedge ((\neg P \wedge \neg R) \vee \neg Q)$$

$$\Longleftrightarrow (Q \wedge (\neg P \wedge \neg R)) \vee (Q \wedge \neg Q) \vee [(P \wedge R) \wedge (\neg P \wedge \neg R)] \vee ((P \wedge R) \wedge \neg Q)$$

(By Extended distributive law)

$$\Longleftrightarrow (\neg P \wedge Q \wedge \neg R) \vee \mathbf{F} \vee (\mathbf{F} \wedge \mathbf{F}) \vee (P \wedge \neg Q \wedge R)$$
$$\Longleftrightarrow (\neg P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R)$$

Example 13.1.7 Find a d.n.f of $[(P \wedge Q) \vee (P \wedge \neg Q)] \wedge [(P \wedge \neg Q) \vee (\neg P \wedge \neg Q)]$

Solution:

$$\Longleftrightarrow [(P \wedge Q) \wedge (P \wedge \neg Q)] \vee [(P \wedge Q) \wedge (\neg P \wedge \neg Q]$$
$$\vee [(P \wedge \neg Q) \wedge (P \wedge \neg Q)] \vee [(P \wedge \neg Q) \wedge (\neg P \wedge \neg Q)]$$

[By Extended distributive Law]

$$\Longleftrightarrow \mathbf{F} \vee \mathbf{F} \vee (P \wedge \neg Q) \vee \mathbf{F}$$
$$\Longleftrightarrow P \wedge \neg Q.$$

Example 13.1.8 obtain a d.n.f of $\neg(P \rightarrow (Q \wedge R))$

Solution:

$$\neg(P \rightarrow (Q \wedge R)) \Longleftrightarrow \neg(\neg P \vee (Q \wedge R))$$
$$\Longleftrightarrow P \wedge (\neg Q \vee \neg R)$$
$$\Longleftrightarrow (P \wedge \neg Q) \vee (P \wedge \neg R).$$

Note 13.1.2 The d.n.f of a given formula is not unique. For example consider the formula $P \vee (Q \wedge R)$. This is already in the d.n.f. However, we may write

$$P \vee (Q \wedge R) \Longleftrightarrow (P \vee Q) \wedge (P \vee R)$$
$$\Longleftrightarrow (P \wedge P) \vee (P \wedge Q) \vee (P \wedge R) \vee (Q \wedge R)$$

the last equivalent formula is another equivalent d.n.f. Infact, different d.n.fs can be obtained for a given formula, of course, these different d.n.fs of the same formula are equivalent.

Exercise 5:

(I) obtain a d.n.f of the following:

(a) $(P \wedge \neg(Q \wedge R)) \vee (P \rightarrow Q)$

(b) $P \vee (\neg P \rightarrow (Q \vee (Q \rightarrow R)))$

(c) $(\neg P \vee \neg Q) \rightarrow (\neg P \wedge R)$

(d) $P \vee (\neg P \wedge \neg Q \vee R)$

(e) $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

Miscellneous Example (Truth table method to find d.n.f) 13.1.9

We know how to construct the truth table for a given formula. The truth table can also be used in opposite mode. That is given a table with truth values only we can determine from the table a formula for which the given table is the truth table.

We remark about d.n.f, that a given formula is identically false if every elementary product appearing in its d.n.f is identically false.

Q : Determine a formula having the truth values as shown in the table:

P	Q	R	f(P,Q,R)
T	T	T	T
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Solution:

Let $f(P, Q, R)$ be a formula whose truth table is the above table. The formula 'f' has the truth value T in first, third, fourth and seventh rows of the table and has the value F in all other rows. The assigned truth values for the variables P, Q, R in these rows are T, T, T; T, F, T; T, F, F and F, F, T respectively. The formulae $P \wedge Q \wedge R$, $P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$ and $\neg P \wedge \neg Q \wedge R$ have their truth values T only in the first, third, fourth and seventhrows respectively in their truth tables.

Thus the formula $f(P, Q, R)$ and

$$(P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R)$$

have the same truth table

Hence

$$f(P, Q, R) \Longleftrightarrow (P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge R) \\ \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R)$$

Clearly, the formula obtained is a disjunction of terms, each of which is a conjunction of statement variables and their negation. This particular form is known as a "disjunctive normal form" of the formula. Since the procedure used in the above problem is completely general, we conclude that every satisfiable formula can be expressed in a d.n.f.

Example 13.1.10 Find the d.n.f of the form $(\neg P \rightarrow R) \wedge (P \rightleftarrows Q)$

Solution: The truth of $(\neg P \rightarrow R) \wedge (P \rightleftarrows Q)$ is

P	Q	R	$(\neg P \rightarrow R) \wedge (P \rightleftarrows Q)$
T	T	T	T
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

The statement formula has the truthvalue T in first, second and seventh rows, and has the value F in all other rows. The assigned truth values for the variables P, Q, R in these rows are T, T, T; T, T, F; F, F, T; resphy.
Then the required d.n.f is

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R).$$

13.2 Conjuctive Normal Forms (c.n.f)

Definition 13.2.1 A formula which is equivalent to a given formula and which consists of a product of elementary sums is called a conjunctive normal form of the given formula.

The procedure of obtaining a c.n.f of a given formula is similar to the one given for d.n.f. Now, it will be illustrated by following examples.

Example 13.2.1 Obtain a c.n.f of each of the formulas

(a) $P \wedge (P \rightarrow Q)$

Solution:

$$P \wedge (P \rightarrow Q) \Longleftrightarrow P \wedge (\neg P \vee Q)$$

Hence $P \wedge (\neg P \vee Q)$ is a required c.n.f.

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

Solution:

$$\begin{aligned}
&\neg(P \vee Q) \rightleftarrows (P \wedge Q) \\
&\Longleftrightarrow (\neg(PVQ) \rightarrow (P \wedge Q)) \wedge ((P \wedge Q) \rightarrow \neg(PVQ)) \\
&\qquad\qquad (\because A \rightleftarrows B \Longleftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)) \\
&\Longleftrightarrow ((P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee (\neg P \wedge \neg Q)) \\
&\Longleftrightarrow [(P \vee Q \vee P) \wedge (P \vee Q \vee Q)] \wedge [(\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)] \\
&\Longleftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q).
\end{aligned}$$

which is the required c.n.f

(c) Find a c.n.f of $[Q \vee (P \wedge R)] \wedge \neg[(P \vee R) \wedge Q]$

Solution:

$$\begin{aligned}
&[Q \vee (P \wedge R)] \wedge \neg[(P \vee R) \wedge Q] \\
&\qquad \Longleftrightarrow [Q \vee (P \wedge R)] \wedge [\neg(P \vee R) \vee \neg Q] \\
&\qquad \Longleftrightarrow [Q \vee (P \wedge R)] \wedge [(\neg P \wedge \neg R) \vee \neg Q] \\
&\qquad \Longleftrightarrow (Q \vee P) \wedge (Q \vee R) \wedge (\neg P \vee \neg Q) \wedge (\neg R \wedge \neg Q).
\end{aligned}$$

Note 13.2.1 The c.n.f is also not unique. Furthermore, a given formula is tautology if every elementary sum in its c.n.f is tautology

Example 13.2.2 Show that the formula

$$Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) \text{ is a tautology}$$

Solution: First we obtain a c.n.f of the given formula

$$\begin{aligned}
Q \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q) &\Longleftrightarrow Q \vee ((P \vee \neg P) \wedge \neg Q) \\
&\Longleftrightarrow [Q \vee (P \vee \neg P)] \wedge (Q \vee \neg Q) \\
&\Longleftrightarrow (Q \vee P \vee \neg P) \wedge (Q \vee \neg Q)
\end{aligned}$$

Since each of the elementary sums is a tautology, the given formula is a tautology.

Example 13.2.3 The truthtable for a formula A is given in the following table. Determine its c.n.f.

P	Q	R	A
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	T

The formula A has the truthvalue F in first, second, fourth and seventh rows of the table and has the value T in all other rows. The assigned truthvalues for the P, Q, R in these rows are T, T, T; T, T, F; T, F, F; F, F, T respectively. The formulas $\neg P \vee \neg Q \vee \neg R$, $\neg P \vee \neg Q \vee R$, $\neg P \vee Q \vee R$, and $P \vee Q \vee \neg R$ have their truthvalues F only in first, second, fourth and seventh rows respectively in their truth tables.

Therefore

$$A \Longleftrightarrow (\neg P \vee \neg Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee R) \wedge (P \vee Q \vee \neg R)$$

Note 13.2.2 The d.n.f or c.n.f of a statement formula is not unique.

In order to arrive at a unique normal form of a given formula, we introduce the principal disjunctive (conjunctive) normal form.

Exercise 6

Obtain a c.n.f of the following formula:

(a) $P \to [(P \to Q) \wedge \neg(\neg Q \vee \neg P)]$

(b) $[P \wedge \neg(Q \vee R)] \vee [((P \wedge Q) \vee \neg R) \wedge P]$

13.3 Principal Disjunctive Normal Forms (PDNF)

Definition 13.3.1 A *minterm* consists of conjunctions in which each statement variable or its negation, but not both, appears only once.

For example, for two variables P and Q, there are 2^2 minterms given by

$$P \wedge Q,\ P \wedge \neg Q,\ \neg P \wedge Q \quad \text{and} \quad \neg P \wedge \neg Q$$

For example, Minterms for the three variables P, Q and R are

$$\begin{array}{llll} P \wedge Q \wedge R, & P \wedge Q \wedge \neg R & P \wedge \neg Q \wedge R & P \wedge \neg Q \wedge \neg R \\ \neg P \wedge Q \wedge R & \neg P \wedge Q \wedge \neg R & \neg P \wedge \neg Q \wedge R & \neg P \wedge \neg Q \wedge \neg R \end{array}$$

From the truth tables of these minterms of P and Q;

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

(i) It is clear that no two minterms are equivalent.

(ii) Each minterm has the truth value T for exactly one combination of the truth values of the variables P and Q.

The minterms of P, Q and R satisfy properties similar to those for two variables P and Q.

We conclude that if the truth table of any formula is known, then one can easily obtain an equivalent formula which consists of a disjunction of some of the minterms.

Note 13.3.1 For any formula containing n variables, an equivalent d.n.f can be obtained by selecting appropriate minterms out of its 2^n possible minterms.

Definition 13.3.2 An equivalent formula consisting of disjunctions of minterms only is known as its principal disjunctive normal form (sum-of-products canonical form), simply p.d.n.f.

Methods to obtain p.d.n.f of a given formula

(I) **By Truth table**: For every truth value T of the given formula, select the minterm which also has the value T for the same combination of the truth values of the statement variables.

Example 13.3.1 obtain the PDNF of $P \to Q$

Solution: From the truth table of $P \to Q$:

P	Q	$P \to Q$	Minterm
T	T	T	$P \wedge Q$
T	F	F	$P \wedge \neg Q$
F	T	T	$\neg P \wedge Q$
F	F	T	$\neg P \wedge \neg Q$

and from the previous truthtable of the minterms of P and Q:

The p.d.n.f of $P \to Q$ is

$$(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$

$$\therefore\ P \to Q \Longleftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$$

Example 13.3.2 obtain the PDNF for $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

Solution: Constructing the truth table:

P	Q	R	Minterm	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R)$ $\vee (Q \wedge R)$
T	T	T	$P \wedge Q \wedge R$	T	F	T	T
T	T	F	$P \wedge Q \wedge \neg R$	T	F	F	T
T	F	T	$P \wedge \neg Q \wedge R$	F	F	F	F
T	F	F	$P \wedge \neg Q \wedge \neg R$	F	F	F	F
F	T	T	$\neg P \wedge Q \wedge R$	F	T	T	T
F	T	F	$\neg P \wedge Q \wedge \neg R$	F	F	F	F
F	F	T	$\neg P \wedge \neg Q \wedge R$	F	T	F	T
F	F	F	$\neg P \wedge \neg Q \wedge \neg R$	F	F	F	F

The p.d.n.f of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ is

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R).$$

Note 13.3.2 The no. of minterms appearing in the normal form is the same as the number of entries with the truth value T in the truth table of the given formula. Thus every formula (which is not a contradiction) has an equivalent p.d.n.f.

Further, such a normal form is unique.

(II) without constructing truth table:

In order to obtain the p.d.n.f of a given formula without constructing its truthtable:

Step1 First replace the conditionals and biconditionals by their equivalent formulas containing only $\wedge$, $\vee$ and $\neg$.

Step2 Next, the negations are applied to the variables by using De Morgan's laws followed by the application of distributive laws (as that in obtaining d.n.f or c.n.f).

Step3 Any elementary product which is a contradiction is dropped. Minterms are obtained in the disjunctions by introducing the missing factors. Identical minterms appearing in the disjunctions are deleted.

Example 13.3.3 By not using the truth table, find PDNF for

(a) $P \rightleftarrows Q$ (b) $\neg P \vee Q$

Solution:

(a)
$$\begin{aligned} P \rightleftarrows Q &\Longleftrightarrow (\neg P \vee Q) \wedge (P \vee \neg Q) \\ &\Longleftrightarrow (\neg P \wedge P) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \vee (\neg Q \wedge Q) \\ &\Longleftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q) \end{aligned}$$

Hence $(P \wedge Q) \vee (\neg P \wedge \neg Q)$ is the PDNF of $P \rightleftarrows Q$.

(b)
$$\begin{aligned} \neg P \vee Q &\Longleftrightarrow (\neg P \wedge \mathbf{T}) \vee (\mathbf{T} \wedge Q) \\ &\Longleftrightarrow [\neg P \wedge (Q \vee \neg Q)] \vee [(P \vee \neg P) \wedge Q] \\ &\Longleftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q) \vee (\neg P \wedge Q) \\ &\Longleftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \end{aligned}$$

is p.d.n.f of $\neg P \vee Q$.

Observation 13.3.1 As we know that p.d.n.f is unique for a given statement formula. In that case, if two given formulas are equivalent, then both of them must have identical p.d.n.f. Therefore, by the second method, it can be determined whether two given formulas are equivalent.

Example 13.3.4 Show the following

(a) $P \vee (P \wedge Q) \Longleftrightarrow P$ (b) $P \vee (\neg P \wedge Q) \Longleftrightarrow P \vee Q$

Solution: We can show by comparing the p.d.n.fs of two formulas.

(a) $P \vee (P \wedge Q) \Longleftrightarrow [P \wedge (Q \vee \neg Q)] \vee (P \wedge Q) \Longleftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$

$$P \Longleftrightarrow P \wedge (Q \vee \neg Q) \Longleftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$$

(b) $P \vee (\neg P \wedge Q) \Longleftrightarrow [P \wedge (Q \vee \neg Q)] \vee [\neg P \wedge Q]$

$$\Longleftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

$$P \vee Q \Longleftrightarrow [P \wedge (Q \vee \neg Q)] \vee (Q \wedge (P \vee \neg P))$$

$$\Longleftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

Observation 13.3.2 If a formula is a tautology, then clearly all the minterms in its p.d.n.f. Therefore it is also possible to determine whether a givenformula is a tautology by determining its p.d.n.f.

Example 13.3.5 Find the p.d.n.f of $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$

Solution:

$$\Longleftrightarrow \neg P \vee ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$$

$$\Longleftrightarrow \neg P \vee ((\neg P \vee Q) \wedge (Q \wedge P))$$

$$\Longleftrightarrow \neg P \vee ((\neg P \wedge Q \wedge P) \vee (Q \wedge Q \wedge P))$$

$$\Longleftrightarrow \neg P \vee \mathbf{F} \vee (P \wedge Q)$$

$$\Longleftrightarrow \neg P \vee (P \wedge Q)$$

$$\Longleftrightarrow [\neg P \vee (Q \vee \neg Q)] \vee (P \wedge Q)$$

$$\Longleftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$$

Hence $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$ is the required p.d.n.f

Example 13.3.6 Find the minterm normal form of $[\neg((P \vee Q) \wedge R)] \wedge [(P \wedge R)]$

Solution:

$$\Longleftrightarrow [\neg(P \vee Q) \vee \neg R] \wedge (P \wedge R)$$

$$\Longleftrightarrow ((\neg P \wedge \neg Q) \vee \neg R) \wedge (P \wedge R)$$

$$\Longleftrightarrow (\neg P \wedge \neg Q \wedge P) \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P) \vee (\neg R \wedge R)$$

$$\Longleftrightarrow \mathbf{F} \vee (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P) \vee \mathbf{F}$$

$$\Longleftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (\neg R \wedge P)$$

$$\Longleftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)$$

$$(\because\ \neg R \wedge P \Longleftrightarrow (\neg R \wedge P \wedge Q) \vee (\neg R \wedge P \wedge \neg Q))$$

Hence $(\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge \neg R)$ is the minterm normal form of the given formula.

Exercise 7:

(1) obtain the p.d.n.f of the following formulas

(a) $P \vee (\neg P \wedge \neg Q \wedge R)$

(b) $(Q \wedge \neg R \wedge \neg S) \vee (R \wedge S)$

Answers 7

(*a*) $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee$
$\vee (\neg P \wedge \neg Q \wedge \neg R)$.

(*b*) $(Q \wedge \neg R \wedge \neg S) \vee (Q \wedge R \wedge S) \vee (\neg Q \wedge R \wedge S)$.

13.4 Principal Conjunctive Normal Forms

Definition 13.4.1 A *maxterm* consists of disjunctions in which each variable or its negation, but not both, appears only once.

For example, For two variables P and Q, there are 2^2 maxterms given by

$$P \vee Q,\ P \vee \neg Q,\ \neg P \vee Q \text{ and } \neg P \vee \neg Q$$

Maxterms for the three variables P, Q and R are

$$\begin{array}{llll} P \vee Q \vee R & P \vee Q \vee \neg R & P \vee \neg Q \vee R & P \vee \neg Q \vee \neg R \\ \neg P \vee Q \vee R & \neg P \vee Q \vee \neg R & \neg P \vee \neg Q \vee R & \neg P \vee \neg Q \vee \neg R \end{array}$$

clearly the maxterms are the duals of minterms.

Either from the duality principle or directly from the truth tables it can be concluded that each of the maxterms has the truth value F for exactly one combination of the truth values of the variables. Different maxterms have the truth value F for different combinations of the truth values of the variables.

Definition 13.4.2 An equivalent formula consisting of conjunctions of maxterms only is known as principal conjunctive normal form (product-of-sums cronical form), simply p.c.n.f.

- Every formula (which is not a tautology) has an equivalent p.c.n.f which is unique except for the rearrangement of the factors in the maxterms and conjunctions.
- By duality principle, All the assertions made for the pdnfs can also be made for the pcnfs.

Methods to obtain p.c.n.f of a given formula

(I) The method for obtaining the p.c.n.f for a given formula is similar to the one studied previously for the principal disjunctive normal form (p.d.n.f)

(II) If the PD(C)NF of a given formula A containing n variables is known, then the PD(C)NF of $\neg A$ will consist of the disjunction (conjunction) of the remaining minterms (maxterms) which do not appear in the PD(C)NF of A.
From $A \Longleftrightarrow \neg\neg A$ one can obtain the PC(D)NF of A by repeated applications of Demorgan's laws to the PD(C)NF of $\neg A$.

Example 13.4.1 obtain the PCNF of the formula $(\neg P \rightarrow R) \wedge (Q \rightleftarrows P)$

Solution:

$$\Longleftrightarrow [\neg(\neg P) \vee R] \wedge [(Q \rightarrow P) \wedge (P \rightarrow Q)]$$

$$\Longleftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$$

$$\Longleftrightarrow [(P \vee R) \vee (Q \wedge \neg Q) \wedge [(P \vee \neg Q) \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \wedge (R \wedge \neg R)]$$

$$\Longleftrightarrow (P \vee R \vee Q) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

$$\Longleftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$$

which is the p.c.n.f. of $(\neg P \rightarrow R) \wedge (Q \rightleftarrows P)$.

Example 13.4.2 Given the PDNF of A: $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$. Find PCNF

Solution: PDNF of A: $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

Now PDNF of $\neg A$ is the disjunction of the remaining minterms

i.e., PDNF of $\neg A$: $(P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$.

Therefore

$$\neg A \Longleftrightarrow (P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)$$

Now

$$\neg\neg A \Longleftrightarrow \neg[(P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge \neg R)]$$

$$\Longleftrightarrow (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R \wedge (P \vee \neg Q \vee R) \wedge (P \vee Q \vee R)$$

is the p.c.n.f of A.

Example 13.4.3 Find PDNF and PCNF for $A : (P \wedge Q) \vee (\neg P \wedge Q) \vee (Q \wedge R)$

Solution: From the problems of PDNF

PDNF of A : $(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$

Now PDNF of $\neg A$: $(\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$.

Now

$$\begin{aligned} \neg\neg A &\Longleftrightarrow \neg((\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)) \\ &\Longleftrightarrow (P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \end{aligned}$$

is the p.c.n.f of the formula A.

Example 13.4.4 Find PDNF from PCNF of $S : P \vee (\neg P \to (Q \vee (\neg Q \to R)))$

Solution:

$$\begin{aligned} S &\Longleftrightarrow P \vee (\neg P \to (Q \vee (\neg Q \to R))) \\ &\Longleftrightarrow P \vee (\neg(\neg P) \vee (Q \vee (\neg(\neg Q) \vee R))) \\ &\Longleftrightarrow P \vee (P \vee (Q \vee (Q \vee R))) \\ &\Longleftrightarrow P \vee (P \vee Q \vee R) \\ &\Longleftrightarrow P \vee Q \vee R \quad \text{is the p.c.n.f of S} \end{aligned}$$

Now PCNF of $\neg S$ is the conjunction of the remaining maxterms so

$$\begin{aligned} \text{PCNF of} \neg S : &(P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \\ &\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \\ &\wedge (\neg P \vee \neg Q \vee R) \end{aligned}$$

Hence the PDNF of S is $\neg$(PCNF of$\neg S$):

$$\begin{aligned} &(\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \\ &\vee (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R) \end{aligned}$$

Exercise 8 Show that the PCNF of $(\neg P \to R) \wedge (Q \rightleftarrows P)$ is $(P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$.

Observation 13.4.2 Any of the principal normal forms can be used to determine whether two given formulas A and B are equivalent. It is not necessary to assume that both formulas have the same variables. In fact, each formula can be assumed to depend upon all the variables that appear in both formulas, by introducing the missing variables and then reducing them to their principal normal forms.

14. Normal Forms: Order, Uniqueness

Normal forms: PCNF, PDNF are unique except for the rearrangements of the factors in the disjunctions / conjunctions as well as in each of the minterms / maxterms.

Now, we can get a unique normal form by imposing a certain order in which the variables appear in the minterms (maxterms) as well as a definite order in which minterms (maxterms) appear in the disjunction (conjunction).

(I) Let us assume that n variables are given and are arranged in a particular order. The 2^n minterms corresponding to the n variables can be designated by $m_0, m_1, \ldots m_{(2^n)-1}$.

Thus each of $m_0, m_1, \ldots, m_{(2^n)-1}$ corresponds to a unique minterm, which can be determined from the binary representation of its subscript (the number of digits in the subscript is exactly n). We can obtain minterm is the following manner:

If 1 appears in the i^{th} location from the left, then the i^{th} variable appears in the conjunction.

If 0 appears in the i^{th} location from the left, then the negation of the i^{th} variable appears in the conjunction forming the minterm.

Conversely, given any minterm, one can find which of $m_0, m_1, \ldots, m_{(2^n)-1}$ designates it.

Example 14.1 Let P, Q and R be three variables arranged in that order. The corresponding minterms are denoted by $m_0, m_1, \ldots, m_7 (2^3 - 1 = 7)$.

Consider m_5, the subscript 5 in binary as 101 and the minterm m_5 is $P \wedge \neg Q \wedge R$. Similarly m_0 corresponds to $\neg P \wedge \neg Q \wedge \neg R$. To obtain the minterm m_3, we write the subscript 3 in binary as 11 and append a zero on the left to get 011 and m_3 is $\neg P \wedge Q \wedge R$.

With the above notation for the representation of the minterms we designate the disjunction (Sum) of minterms by the compact notation Σ.

Using such a notation, the sum-of-products canonical form representing the disjunction of m_i, m_j and m_k can be written down as $\Sigma i, j, k$.

Example 14.2 The p.d.n.f of $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$ is $(\neg P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$, From the previous notation: we denote the p.d.n.f as $\sum 1, 3, 6, 7$.

(II) The 2^n *maxterms* corresponding to the n statement variables can be designated by $M_0, M_1 \ldots, M_{(2^n)-1}$. Here also the maxterm corresponding to M_j is obtained by writing j in binary and appending the required number of zeros to the left in order to get n digits.
If 0 appears in the i^{th} location from the left of this binary number, then the i^{th} variable appears in the disjunction. If 1 appears in the i^{th} location, then the negation of the i^{th} variable appears.
Thus the binary representation of the subscript uniquely determines the maxterm. Conversely every binary representation of numbers between 0 and $2^n - 1$ determines a maxterm.
Note that the convention regarding 1 and 0 here is the opposite of what was used for minterms.

Example 14.3 The maxterms $M_0, M_1, \ldots, M_7$ corresponding to three variables P, Q and R are

$$\begin{array}{llll} P \vee Q \vee R & P \vee Q \vee \neg R & P \vee \neg Q \vee R & P \vee \neg Q \vee \neg R \\ \neg P \vee Q \vee R & \neg P \vee Q \vee \neg R & \neg P \vee \neg Q \vee R & \neg P \vee \neg Q \vee \neg R \end{array}$$

with the above notation for the representation of the maxterms we designate the conjunction (product) of maxterms by the compact notation Π.

Thus $\Pi i, j, k$ represents the conjunction of maxterms M_i, M_j, M_k.

Example 14.4 The PCNF of $(P \wedge Q) \vee (\neg P \wedge R)$: $(\neg P \wedge Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R)$ it can be represented as $\prod 0, 2, 4, 5$.

15. Statement Calculus: Theory of Inference

Definition 15.1 The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with rules of inference is known as *inference theory*.

Definition 15.2 If a conclusion is derived from a set of premises by using the accepted rules of reasoning, then such a process of derivation is called a *deduction* or a *formal proof*, and the arguement or conclusion is called a *valid arguement or valid conclusion*.

Note 15.1 Premises $\approx$ assumptions, axioms, hypotheses.

(I) The method to determine whether the conclusion logically follows from the given premises by constructing the relevant truth table is called "**truth table technique**"

Definition 15.3 Let A and B be two statement formulas we say that "B logically follows from A" or "B is a valid conclusion (consequence) of the premise A" iff $A \to B$ tautology i.e., $A \Rightarrow B$.

By extending the above definition, we say that from a set of premises $\{H_1, H_2, \ldots, H_m\}$ a conclusion C follows logically iff

$$H_1 \wedge H_2 \wedge \cdots \wedge H_m \Rightarrow C. \tag{1}$$

(I) By Truthtable

(i) Let $P_1, P_2, \ldots, P_n$ be all the atomic variables appearing in the premises $H_1, H_2, \ldots, H_m$ and the conclusion C. If all possible combinations of truth values are assigned to $P_1, P_2, \ldots, P_n$ and if the truth values of $H_1, H_2, \ldots, H_m$ and C are entered in a table, then it is easy to see from such a table whether (1) is true.

(ii) Look for the rows in which all $H_1, H_2, \ldots, H_m$ have the value *T*. If for every such row, C also has the value **T**, *then (1) holds.*

(iii) Atternatively, look for the rows in which C has the value *F*. If in every such row, atleast one of the values of $H_1, H_2, \ldots, H_m$ is **F**, *then (1) also holds.*

Example 15.1 Determine whether the conclusion C follows logically from the hypotheses H_1 and H_2.

(i) $H_1 : P \to Q \quad H_2 : P \quad C : Q$
(ii) $H_1 : P \to Q \quad H_2 : Q \quad C : P$
(iii) $H_1 : \neg P \quad H_2 : P \rightleftarrows Q \quad C : (P \wedge Q)$

P	Q	$P \to Q$	$\neg P$	$\neg Q$	$\neg(P \wedge Q)$	$P \rightleftarrows Q$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	T	T	F	T	F
F	F	T	T	T	T	T

For

(i) we observe that the first row is the only row in which both the premises have the value T. The conclusion also has the value T in that row. Hence it is valid.
(ii) The conclusion does not follow logically from the premises $P \to Q$ and Q.
(iii) Similarly, we can show that the conclusions are valid in (d).

Exercise 9 Determine whether the conclusion C is valid in the following the premises (By Truth table technique).

(a) $H_1 : P \to Q \quad H_2 : \neg P \quad C : Q$
(b) $H_1 : \neg Q \quad H_2 : P \to Q \quad C : \neg P$
(c) $H_1 : R \quad H_2 : P \vee \neg P \quad C : R$

Answers 9 (a) not valid (b) valid (c) valid

(II) Without Using Truth Table

The truth table technique becomes tedious when the number of atomic variables present in all the formulae representing the premises and the conclusion is large. To overcome this disadvantage, we need to investigate other possible methods, without using the truth table.

Now, we discuss the process of derivation by which one demonstrate that a particular formula is a valid consequence of a given set of premises. Before we go to actual process of derivation, we give three rules of inference, which are called Rule P, Rule T and Rule CP respectively. For the moment we consider only two of these rules. One permits us to introduce premises when needed and the other permits piecemeal use of tautological implications.

Before we proceed with the actual process of derivation, we list some important implications and equivalences that will be referred to frequently.

Implications

I_1	$P \wedge Q \Rightarrow P$	Simplification
I_2	$P \wedge Q \Rightarrow Q$	
I_3	$P \Rightarrow P \vee Q$	addition
I_4	$Q \Rightarrow P \vee Q$	
I_5	$\neg P \Rightarrow P \to Q$	
I_6	$Q \Rightarrow P \to Q$	
I_7	$\neg(P \to Q) \Rightarrow P$	
I_8	$\neg(P \to Q) \Rightarrow \neg Q$	
I_9	$P, Q \Rightarrow P \wedge Q$	
I_{10}	$\neg P, P \vee Q \Rightarrow Q$	(disjunctive syllogism)
I_{11}	$P, P \to Q \Rightarrow Q$	(modus ponens)
I_{12}	$\neg Q, P \to Q \Rightarrow \neg P$	(modus tollens)
I_{13}	$P \to Q, Q \to R \Rightarrow P \to R$	hypothetical syllogism
I_{14}	$P \vee Q, P \to R, Q \to R \Rightarrow R$	dilemma

Equivalences

E_1	$\neg\neg P \Leftrightarrow P$	double negation
E_2	$P \wedge Q \Leftrightarrow Q \wedge P$	Communtative laws
E_3	$P \vee Q \Leftrightarrow Q \vee P$	
E_4	$(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$	Associative laws
E_5	$(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$	
E_6	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$	distributive laws
E_7	$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	
E_8	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$	Demorgan's laws
E_9	$\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	
E_{10}	$P \vee P \Leftrightarrow P$	
E_{11}	$P \wedge P \Leftrightarrow P$	
E_{12}	$R \vee (P \wedge \neg P) \Leftrightarrow R$	
E_{13}	$R \wedge (P \vee \neg P) \Leftrightarrow R$	
E_{14}	$R \vee (P \vee \neg P) \Leftrightarrow T$	
E_{15}	$R \wedge (P \wedge \neg P) \Leftrightarrow F$	
E_{16}	$P \rightarrow Q \Leftrightarrow \neg P \vee Q$	
E_{17}	$\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$	
E_{18}	$P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$	
E_{19}	$P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$	
E_{20}	$\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$	
E_{21}	$P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$	
E_{22}	$(P \rightleftarrows Q) \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$	

We now give the first two rules

Rule P:	We may introduce a premise at any point in the derivation.
Rule T:	We may introduce a formula S in a derivation if S is tautologically implied by any one or more of the preceding formulae in the derivation.

We now consider an example to show how these rules of inference are used. It is better to indicate the reason for each step of the derivation.

Example 15.2 Demonstrate that S is a valid inference from the premises: $P \longrightarrow \neg Q$, $Q \vee R$, $\neg S \longrightarrow P$ and $\neg R$.

Solution:

[1]	(1)	$Q \vee R$	Rule P
[2]	(2)	$\neg R$	Rule P
[1, 2]	(3)	Q	Rule T, (1), (2) and I_{10}
[4]	(4)	$P \rightarrow \neg Q$	Rule P
[1, 2, 4]	(5)	$\neg P$	Rule, (3), (4) and I_{12}
[6]	(6)	$\neg S \rightarrow P$	Rule P
[1, 2, 4, 6]	(7)	S	Rule T, (5), (6) and I_{12}

Hence S is a valid inference.

There are seven lines in this derivation. The second column of numbers designate the formula and the line of derivation in which it occurs. The introduction of each line is justified by one of two rules Rule P and Rule T. The lines (1), (2), (4) and (6) are just premises of the argument. The other three lines are obtained by showing that they are tautological implications of preceding lines.

For example, in the case of line (5), it is easy to see that the formula in that line ie., $\neg P$ is tautologically implied by the conjunction of the formula in lines (3) and (4). That is $(P \rightarrow \neg Q) \wedge Q \Rightarrow \neg P$. In case of line (7), the conjunction of (5) and (6) tautologically implies S.

All these reasons for each step are indicated in the last column of the derivation. The set of numbers in braces (the first column) in each line shows the premises on which the formula in that depends on the other hand, the numerals in the last column simply indicate the lines from which the statement in the 3rd column is inferred.

The argument is usually given in a condensed form. The letter 'P' and 'T' are used for Rule P (premise) and Rule T (tautology) respectively. Some important implications and equivalences are listed, which will be referred frequently. The tautology $(Q \vee R) \wedge (\neg R) \Rightarrow Q$ can be indicated by either I_{10} or disjunctive syllogism.

Example 15.3 Show that R$\vee$S follows logically from the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $\neg H \rightarrow (A \wedge \neg B)$ and $(A \wedge \neg B) \rightarrow (R \vee S)$.

Solution:

{1}	(1) $(C \vee D) \rightarrow \neg H$	Rule P
{2}	(2) $\neg H \rightarrow (A \wedge \neg B)$	Rule P
{1, 2}	(3) $(C \vee D) \rightarrow (A \wedge \neg B)$	Rule T, (1), (2) and I_{13}

$\{4\}$	(4)	$(A \wedge \neg B) \rightarrow (R \vee S)$	Rule P
$\{1, 2, 4\}$	(5)	$(C \vee D) \rightarrow (R \vee S)$	Rule T, (3), (4) and I_{13}
$\{6\}$	(6)	$C \vee D$	Rule P
$\{1, 2, 4, 6\}$	(7)	$R \vee S$	Rule T, (5), (6) and I_{11}

Note 15.2 I_{13}: hypothetical syllogism. I_{11} : modus ponens.

Example 15.4 Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \rightarrow R$, $P \rightarrow M$ and $\neg M$.

Solution:

$\{1\}$	(1)	$P \rightarrow M$	Rule P
$\{2\}$	(2)	$\neg M$	Rule P
$\{1, 2\}$	(3)	$\neg P$	Rule T, (1), (2) and I_{12}
$\{4\}$	(4)	$P \vee Q$	Rule P
$\{1, 2, 4\}$	(5)	Q	Rule T, (3), (4) and I_{10}
$\{6\}$	(6)	$Q \rightarrow R$	Rule P
$\{1, 2, 4,.6\}$	(7)	R	Rule T, (5), (6) and I_{11}
$\{1, 2, 4, 6\}$	(8)	$R \wedge (P \vee Q)$	Rule T, (4), (7) and I_9

Example 15.5 Show that

$$\{(P \rightarrow Q) \wedge (R \rightarrow S), (Q \rightarrow T) \wedge (S \rightarrow U), \neg(T \wedge U), P \rightarrow R\} \Rightarrow \neg P$$

Solution:

[1]	(1)	$(P \rightarrow Q) \wedge (R \rightarrow S)$	Rule P
[1]	(2)	$P \rightarrow Q$	Rule T, (1)
[1]	(3)	$R \rightarrow S$	Rule T, (1)
[4]	(4)	$(Q \rightarrow T) \wedge (S \rightarrow U)$	Rule P
[4]	(5)	$Q \rightarrow T$	Rule T, (4)
[4]	(6)	$S \rightarrow U$	Rule T, (4)
[1, 4]	(7)	$P \rightarrow T$	Rule T, (2), (5), hypothetical syllogism
[1, 4]	(8)	$\neg T \rightarrow \neg P$	Rule T, (7) and E_{16}
[1, 4]	(9)	$R \rightarrow U$	Rule T, (3), (6), hypothetical syllogism
[10]	(10)	$P \rightarrow R$	Rule P
[1, 10]	(11)	$P \rightarrow U$	Rule T, (10), (3) hypothetical syllogism
[1, 10]	(12)	$\neg U \rightarrow \neg P$	Rule T, (11) and E_{18}
[1, 4, 10]	(13)	$(\neg T \vee \neg U) \rightarrow \neg P$	Rule T, (8), (12) and I_{14}
[14, 10]	(14)	$\neg(T \wedge U) \rightarrow \neg P$	Rule T, (13) Demorgan's law
[15]	(15)	$\neg(T \wedge U)$	Rule P
[1, 4, 10, 15]	(16)	$\neg P$	Rule T (14), (15), modus ponens.

We shall now introduce the third and last rule of inference is the rule of conditional proof, which we call *Rule CP*.

The general idea of this rule is that we may introduce a new premise R conditionally and use it in conjunction with the original premises to derive a conclusion S, and then assert that the implication $R \to S$ follows from the original premises alone. If S is a valid inference from premises $P_1, P_2, \ldots, P_n$ and R, then $R \to S$ is a valid inference from premises $P_1, P_2, \ldots, P_n$.

> **Rule CP**: If we can derive S from R and *a* set of premises then we can derive $R \to S$ from the set of premises alone.

Simply, $(P \wedge R) \to S \Leftrightarrow P \to (R \to S)$

where P denote the conjunction of the set of premises say $P_1, P_2, \ldots, P_n$.

The above equivalence states that if R is included as an additional premise and S is derived from $P \wedge R$, then $R \to S$ can be derived from the premises P alone.

Rule CP is also called the deduction theorem.

Example 15.6 Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\neg R \vee P$ and Q

Solution: It is enough to include R as an additional premise and derive S.

[1]	(1)	$\neg R \vee P$	Rule P
[2]	(2)	R	Rule P (additional premise)
[1, 2]	(3)	P	Rule T, (1), (2) and I_{10}
[4]	(4)	$P \to (Q \to S)$	Rule P
[1, 2, 4]	(5)	$Q \to S$	Rule T, (3), (4) and I_{11}
[6]	(6)	Q	Rule P
[1, 2, 4, 6]	(7)	S	Rule T, (5), (6) and I_{11}
[1, 4, 6]	(8)	$R \to S$	Rule CP

Example 15.7 Show that $P \to S$ can be derived from the premises $\neg P \vee Q$, $\neg Q \vee R$, $R \to S$.

Solution: we include P as an additional premise and derive S.

[1]	(1)	$\neg P \vee Q$	Rule P
[2]	(2)	P	Rule P (additional premise)
[1, 2]	(3)	Q	Rule T, (1), (2) and I_{10}
[4]	(4)	$\neg Q \vee R$	Rule P

[1, 2, 4]	(5)	R	Rule T, (3), (4) and I_{10}
[6]	(6)	$R \to S$	Rule P
[1, 2, 4, 6]	(7)	S	Rule T, (5), (6) and I_{11}
[1, 4, 6]	(8)	$P \to S$	Rule CP

Example 15.8 Derive $P \to (Q \to S)$ using the rule CP if necessary from $P \to (Q \to R)$, $Q \to (R \to S)$

Solution:

[1]	(1)	$P \to (Q \to R)$	Rule P
[2]	(2)	P	Rule P (additional premise)
[1, 2]	(3)	$Q \to R$	Rule T, (1), (2) and I_{11}
[4]	(4)	$Q \to (R \to S)$	Rule P
[1, 2]	(5)	$\neg Q \vee R$	Rule T, (3), E_{16}
[4]	(6)	$\neg Q \vee (R \to S)$	Rule T, (4), E_{16}
[1, 2, 4]	(7)	$\neg Q \vee (R \vee (R \to S))$	Rule T, (5), (6) distributive law
[1, 2, 4]	(8)	$\neg Q \vee S$	Rule T, (7), I_{11}
[1, 2, 4]	(9)	$Q \to S$	Rule T, (8), E_{16}
[1, 4]	(10)	$P \to (Q \to S)$	Rule CP

Definition 15.4 A set of formulas $H_1, H_2, \ldots, H_m$ is said to be *consistent* if their conjunction has the truth value **T** for some assignment of the truth values to the atomic variables appearing in $H_1, H_2, \ldots, H_m$.

If for every assignment of the truth values to the atomic variables, atleast one of the formulas $H_1, H_2, \ldots, H_m$ is false, so that their conjunction is identically false, then the formulas $H_1, H_2, \ldots, H_m$ are called *inconsistent*.

In other words, a set of formulas $H_1, H_2, \ldots, H_m$ is inconsistent if their conjunction implies a contradiction, that is

$$H_1 \wedge H_2 \wedge \ldots \wedge H_m \Rightarrow R \wedge \neg R$$

where R is any formula.

Note 15.3 $R \wedge \neg R$ is a contradiction, and it is necessary and sufficient for the implication that $H_1 \wedge H_2 \wedge \ldots \wedge H_m$ be a contradiction.

16. Indirect Method of Proof

The method of using the rule of conditional proof and the notion of an inconsistent set of premises is called the indirect *method of proof* or *proof by contradiction* or *reduction ad absurdum*.

The technique of indirect method of proof is as follows:

i) Introduce the negation of the desired conclusion as a new premise.
That is, assume the conclusion C is false and consider $\neg C$ as an additional premise (or new premise)

ii) From the additional or new premise, together with the given premises, derive a contradiction.
That is, if the new set of premises is inconsistent, then they imply a contradiction. Therefore C is true whenever, $H_1 \wedge H_2 \wedge \ldots \wedge H_m$ is true.

iii) Assert the desired conclusion as a logical inference from the premises.
Thus C follows logically from the premises $H_1, H_2, \ldots, H_m$.

Example 16.1 Using indirect method of proof, derive $P \to \neg S$ from $P \to Q \vee R$, $Q \to \neg P$, $S \to \neg R$, P.

Solution: The desired result is $P \to \neg S$. Its negation is $P \wedge S$. Since $P \wedge S \rightleftarrows \neg(\neg P \vee \neg S) \rightleftarrows \neg(P \to \neg S)$ is a tautology, from the law of negation for implication. We include $P \wedge S$ as an additional premise

[1]	(1)	$P \to (Q \vee R)$	Rule P
[2]	(2)	P	Rule P
[1, 2]	(3)	$Q \vee R$	Rule T, (1), (2), modus ponens (I_{11})
[4]	(4)	$S \to \neg R$	Rule P
[5]	(5)	$P \wedge S$	Rule P (new premise)
[5]	(6)	S	Rule T, (5)
[4, 5]	(7)	$\neg R$	Rule T, (4), (6) modus ponens (I_{11})
[1, 2, 4, 5]	(8)	Q	Rule T, (3), (7), I_{10}
[9]	(9)	$Q \to \neg P$	Rule P
[1, 2, 4, 5, 9]	(10)	$\neg P$	Rule T, (8), (9), modus ponens
[1, 2, 4, 5, 9]	(11)	$P \wedge \neg P$	Rule T, (2), (10), contradiction

Thus additional premise $P \wedge S$ and the given premises together lead to a contradiction. So $\neg(P \wedge S)$ is derivable from $P \to Q, \vee R, Q \to \neg P, S \to \neg R, P$.

Example 16.2 Prove by indirect method that

$$(\neg Q),\ P \to Q,\ P \vee R \Rightarrow R$$

Solution: The desired result is R. Include its negation as a new premise.

[1]	(1)	$P \vee R$	Rule P
[2]	(2)	$\neg R$	Rule P (additional premise)

[1, 2]	(3)	P	Rule T, (1), (2)
[4]	(4)	$P \to Q$	Rule P
[1, 2, 4]	(5)	Q	Rule T, (3), (4), modus ponens
[6]	(6)	$\neg Q$	Rule P
[1, 2, 4, 6]	(7)	$Q \wedge \neg Q$	Rule T, (5), (6), Contradiction

The new premise, together with the given premises, leads to a contradiction. Thus

$$(\neg Q),\ P \to Q,\ P \vee R \Rightarrow R.$$

Example 16.3 By indirect proof, show that

$$P \to Q,\ Q \to R,\ \neg(P \wedge R),\ P \vee R \Rightarrow R$$

Solution: The desired result is R. Include $\neg R$ as a new premise.

[1]	(1)	$Q \to R$	Rule P
[2]	(2)	$\neg R$	Rule P (additional premise)
[1, 2]	(3)	$\neg Q$	Rule T, (1), (2)
[4]	(4)	$P \to Q$	Rule P
[1, 2, 4]	(5)	$\neg P$	Rule T, (3), (4), modus ponens
[6]	(6)	$P \vee R$	Rule P
[1, 2, 4, 6]	(7)	R	Rule T, (5), (6)
[1, 2, 4, 6]	(8)	$R \wedge \neg R$	Rule T, (2), (7), Contradiction

Thus we get a contradiction. Therefore we get

$$P \to Q,\ Q \to R,\ P \vee R,\ R.$$

But, the other premise $\neg(P \wedge R)$ will not yield a contradiction with R.

We shall now give some examples of derivation involving statements in English. In everyday life, we come across arguement expressed in (English) sentences. We can represent the premises in symbols and verity the validity as in earlier examples.

Example 16.4 Determine the validity of the following arguement: If 7 is a prime number, then 7 does not divide 35, 7 divides 35, implies 7 is not a prime number.

Solution: Let us indicate the statements as follows:

P : 7 is a prime number

Q : 7 divides 35

The given argument is of the form: $\{P \to \neg Q, Q\} \Longrightarrow \neg P$. Construct the truth table

P	Q	$\neg Q$	$P \to \neg Q$	Q	$\neg P$
T	T	F	F	T	F
T	F	T	T	F	F
F	T	F	T	T	T
F	F	T	T	F	T

$P \to \neg Q$ and Q are both true only in the third row and in that row $\neg P$ is also true. Hence $P \to \neg Q, Q \Longrightarrow \neg P$. Hence the arguement is true.

Note 16.1 The above arguement is valid by I_{12} i.e., modustollens.

Example 16.5 Determine the validity of the following arguement. If two sides of a triangle are equal, then two opposite angles are equal.

Two sides of a triangle are not equal

Therefore, the opposite angles are not equal.

Solution: Let us indicate the statements as follows:

P : Two sides of a triangle are equal

Q : The two opposite angle are equal

The given argument is of the form

$$P \to Q, \neg P \Longrightarrow \neg Q$$

Let us now construct the truth table for $[(P \to Q) \wedge (\neg P)] \to \neg Q$.

P	Q	$P \to Q$	$\neg P$	$(P \to Q) \wedge (\neg P)$	$\neg Q$	$((P \to Q) \wedge (\neg P)) \to \neg Q$
T	T	T	F	F	F	T
T	F	F	F	F	T	T
F	T	T	T	T	F	F
F	F	T	T	T	T	T

This shows that $(P \to Q) \wedge (\neg P) \to \neg Q$ is not a tautology. Hence the conclusion $\neg Q$ is not valid.

Example 16.6 Determine the validity of the following arguement. My father praises me only if I can be proud of myself Either I do well insports or I can't be pround of myself. If I study hard, then I can't do well in sports. Therefore, if father praises me, then I do not study well.

Solution: Let us indicate the statements as follows:

P: My father praises me

Q: I can be purd of myself

R: I do well in sports

S: I study hard

The given argument is of the form

$$P \rightarrow Q, R \vee \neg Q, S \rightarrow \neg R \Longrightarrow P \rightarrow \neg S$$

As the desired result is $P \rightarrow \neg S$, assume that P is true.

It is enough to verify the validity of

$$P, P \rightarrow Q, R \vee \neg Q, S \rightarrow \neg R \Longrightarrow \neg S.$$

From P and $P \rightarrow Q$, we have Q.

From Q and $R \vee \neg Q$, we have R.

From R and $S \rightarrow \neg R$, we have $\neg S$.

Thus the arguement is valid.

Example 16.7 Show that the following set of premises is in consistent.

If the contract is valid, then john is liable for penalty. If john is liable for penalty, he will go bankrupt. If the bank will loan him money, he will not go bankrupt. As a matter of fact, the contract is valid and the bank will loan him money.

Solution: We indicate the given statements as follows:

V: The contract is valid

L: John is liable for penalty.

M: Bank will loan him money

B: He will go bankrupt.

Then the given premises are

$$V \rightarrow L, L \rightarrow B, M \rightarrow \neg B, V \wedge M$$

[1]	(1)	$V \rightarrow L$	Rule P
[2]	(2)	$L \rightarrow B$	Rule P
[1, 2]	(3)	$V \rightarrow B$	Rule T, (1), (2) law of hypo. syllogism
[4]	(4)	$V \wedge M$	Rule P
[4]	(5)	V	Rule T, (4)
[4]	(6)	M	Rule T, (4)
[1, 2, 4]	(7)	B	Rule T, (3), (5), modus ponens
[8]	(8)	$M \rightarrow \neg B$	Rule P
[4, 8]	(9)	7B	Rule T, (6), (8)
[1, 2, 4, 8]	(10)	$B \wedge \neg B$	Rule T, (7), (9), contradiction

Thus the given set of premises leads to a contradiction and hence it is inconsistent.

Note 16.2 **on Notation:** Let A, B and C be three statement formulas.

The form $\boxed{\begin{array}{l} A \\ B \\ \hline \therefore C \end{array}}$ is equivalent to $A \wedge B \to C$ is a tautology.

The statement formulas above the horizontal line are called premises (or hypotheses). The statement formula below the line is called the conclusion.

The most common rules of inference

1. Modus ponens: $\begin{array}{r} P \\ P \to Q \\ \hline Q \end{array}$

2. Modus tollens: $\begin{array}{r} P \to Q \\ 7Q \\ \hline 7P \end{array}$

3. Disjunctive syllogism: $\begin{array}{r} P \vee Q \\ 7P \\ \hline Q \end{array}$

4. Chain rule: $\begin{array}{l} P \to Q \\ Q \to R \\ \hline P \to R \end{array}$

5. Resolution: $\begin{array}{l} P \vee R \\ Q \vee (7R) \\ \hline P \vee Q \end{array}$

Example 16.8 Show that the following argument is valid

$$\begin{array}{c} (P \vee Q) \to (S \wedge T) \\ [7((7S) \vee (7T))] \to [(7R) \vee Q] \\ \hline (P \vee Q) \to (R \to Q) \end{array}$$

Solution: Clearly

$$7((7S) \vee (7T)) \Longleftrightarrow (S \wedge T)$$
$$(7R) \vee Q \Longleftrightarrow R \to Q$$

Thus the given argument can be rewritten as

$$\begin{array}{c}(P \vee Q) \rightarrow (S \wedge T) \\ (S \wedge T) \rightarrow (R \rightarrow Q) \\ \hline (P \vee Q) \rightarrow (R \rightarrow Q)\end{array}$$

by The chain rule, the argument is valid.