3. Elementary Combinatorics

> Say mathematician, how many are the combinations in one composition with ingredients of six different tastes - sweet, pungent, astringent, sour, salt and bitter - taking them by ones, twos or threes, etc.,?
>
> -From Lilavathi by Bhaskara

1. Introduction

Combinatorics is the branch of mathematics meant to solve counting problems without enumerating all possible cases. At an elementary level, Combinatorics is usually considered as a part of discrete mathematics in which the main problem is that of counting the number of ways of arranging or choosing objects from a finite set according to some simple specified rules. At an advanced level, combinatorics deals with the enumeration analysis and optimization of discrete structures.

Combinational mathematics has a variety of applications. It is used in several physical and social sciences viz., computer science, operations research, statistics, probability chemistry.

The basic ideas, and concepts of combinatorics are necessary to make an assessment about the amount of storage in a computer. Also, they are useful to solve many problems of computer science.

2. Basics of counting

Two basic counting principles

Two elementary principles act as building blocks for all counting problems.

2.1 Disjunctive (or) Sum Rule

If an event can occur in m ways and another event can occur in n ways, and if these two events cannot occur simultaneously, then one of the two events can occur in $m+n$ ways.

More generally, If $E_1, E_2, \ldots, E_n$ are n events such that no two of them can occur at the same time, and E_1 can happen in n_1 ways, E_2 can happen in n_2 ways ..., E_n can happen in n_n ways, then one of the n events (E_1 or E_2 or E_n) can occur in $n_1+n_2+\ldots+n_n$ ways.

The sum rule can also be stated in terms of choices:

If an object can be selected from a collection in n_1 ways and an object can be selected from a seperate collection in n_2 ways, then the selection of one object from either one collection or the other can be made in $n_1 + n_2$ ways.

Example 2.1.1 If there are 9 boys and 10 girls in a class, there are $9 + 10 = 19$ ways of selecting one student (either a boy or a girl) as class representative.

Example 2.1.2 Suppose E is the event of selecting a prime number less than 10 and F is the event of selecting an even number less than 10, Then E can happey in 4 ways, and F can happen in 4 ways. But because 2 is an even prime, E or F can happen in only $4+4-1=7$ ways.

2.2 Sequential (or) product rule

If an event can occur in m ways and a second event can occur in n ways, and if the number of ways the second event occurs does not depend upon how the first event occurs, then the two events can occur simultaneously in mn ways.

Moregenerally, If events $E_1, E_2, \ldots, E_n$ can happen in $n_1, n_2, \ldots$, and n_n ways respectively, then the sequence of events E_1 first, followed by $E_2, \ldots$, followed by E_n can happen in $n_1 \cdot n_2 \cdot n_3 \ldots n_n$ ways.

The product rule can also be stated interms of choices:

If a first object can be chosen in n_1 ways, a second in n_2 ways, ..., and an n^{th} object can be chosen in e_n ways, then a choice of a first, second, ..., and an n^{th} object can be made in $n_1 \cdot n_2 \ldots n_n$ ways.

Example 2.2.1 A book shelf holds 5 different mathematics books, 6 different computer science books, and 10 different books of statistics. Therefore $(i) 5.6.10 = 300$ ways of selecting 3 books, 1 in each subject $(ii) 5+6+10 = 21$ ways of selecting 1 book in any one of the subject.

Example 2.2.2 From the previous example, A mathematics book and a computer science book can be selected in $(5)(6) = 30$ ways; A mathematics book and a statistics book can be selected in $5.10 = 50$ ways; a computer science book and a statistics book can be selected in $6.1 = 60$ ways. Thus there are $30 + 5 + 60 = 140$ ways of selecting 2 books in 2 subjects.

Example 2.2.3 If each of 12 Questions in a objective type examination has 4 answers (1 correct and 3 wrong), the number of ways of answering all Questions is 4^{12}.

We summarize the sum rule by saying that we add the numbers of elements in each subset when the elements being counted can be decomposed into disjoint subsets.

Also we summarize the product rule by saying that we multiply together the numbers of ways of doing each step when an activity is constructed in successive steps.

If we are counting objects that are constructed in successive steps, we use the product rule. If we have disjoint sets of objects and we want to know the total number of objects, we use the sum rule. It is important to recognize when to apply each principle. This will comes from practice and careful thinking about each problem. The first two above examples illustrate both counting principles.

Again, we consider some more examples that illustrate both counting rules.

Example 2.2.4 A computer password consists of a letter of the alphabet followed by 3 or 4 digits. Find (a) the total number of passwords that can be created and (b) the number of passwords in which no digit repeats.

(a) The number of 4-character passwords is $(26)\times 10^3$, and the number of 5-character passwords is $(26) \times 10^4$, by the product rule. So the total number of passwords is $(26) \cdot 10^3 + (26) \cdot 10^4 = 286,000$ by the sum rule.

(b) The number of 4-character passwords is $(26)(10)(9)(8) = 18,720$ and the number of 5-character passwords is $(26)(10)(9)(8)(7) = 131,040$ for a total of 149,760.

Example 2.2.5 A six person committee composed of Shiva, Brahma, Vishnu, Narada, Indra & Yama is to select a chairperson, secretary and treasurer.

(a) In how many ways can this be done?

(b) In how many ways can this be done if either Shiva or Brahma must be chairperson?

(c) In how many ways can this be done Indra must hold one of the offices?

(d) In how many ways can this be done if both Narada and Yama must hold office?

Solution:

(a) We use the product rule. The officers can be selected in three successive steps: Select the treasurer. The chairperson can be selected in sixways, once the chairperson has been selected, the secretary can be selected in five ways. After selection of the chair person and secretary, the treasuner can be selected in four ways. Therefore, the total number of possibilities is

$$6.5.4 = 120$$

(b) Arguing as in part(a), if shiva is chairperson, there are 5.4 = 20 ways to select the remaining officers. Similarly, if Brahma in chair person, there are 20 ways to select the remaining officers. Since these cases are disjonint, by the sum rule, there are

$$20 + 20 = 40 \text{ possibilities}$$

(c) Arguing as in purt (a), if Indra is chair person, there are 20 ways to select the remaining officers. Similarly, if Indra is secretary, there are 20 possibilities, and if Indra is treasurer, there are 20 possibilities. Since these three cases are pairwise disjoint, by the sum rule, there are

$$20 + 20 + 20 = 60 \text{ possibilities.}$$

[Alternate solution]: Let us consider the activity of assigning Indra and two others to offices to be made up of three successive steps: Assign Indra an office, fill the highest remaining office, fill the last office. Once Indra has been assigned, there are five ways to fill the highest remaining office. Once Indra has been assigned and the highest remaining office filled, there are four ways to fill the last office. By the product rule, there are 3.5.4 = 60 possibilities

(d) Let us consider the activity of Narada, Yama and one other person to offices to be made up of three successive steps Assign Narada, assign Yama, fill the remaining office. There are three ways to assign Narada. Once Narada has been assigned, there are two ways to assign Yama. Once Narada and Yama have been assigned, there are four ways to fill the remaining office. By the product rule, there are 3.2.4 = 24 possibilities.

3. Permutations and Combinations

Definition 3.1 A permutation of n distinct objects $x_1, \ldots x_n$ is an *ordering* of the n elements $x_1, \ldots x_n$.

Example 3.1 There are Six permutations of three objects A, B, and C. They are ABC, ACB, BAC, BCA, CAB, CBA

Definition 3.2 An r-permutation of n (distinct) elements $x_1, \ldots, x_n$ is an ordering of an r-element subset of $\{x_1, \ldots x_n\}$. The number of r-permutations of a set of n distinct elements is denoted by $P(n, r)$

Example 3.2 2-permutations of A, B, C are AB, BA, CA, AC, BC, CB.

Definition 3.3 A combination is a selection of objects *without regard to order*.

Example 3.3 ABC is the combination of three objects A, B and C.

Definition 3.4 An r-combination of n (distinct) elements $x_1, \ldots x_n$ is an *Unordered* selection of an r-element subset of $\{x_1, \ldots x_n\}$ The number of r-combinations of a set of n district elements is denoted by $C(n, r)$ or $\left(\begin{array}{c} n \\ r \end{array}\right)$.

Example 3.4 2-combinations of A, B, C are AB, AC, BC.

Note 3.1 In general, when order matters, we count the number of permutations, when order does not matter. We count the number of combinations.

4. Enumeration of permutations and combinations, (Enumeration without repetition)

4.1 Enumerating r-permutations without repetitions

$$P(n, r) = n(n-1) \ldots (n-r+1) = \frac{n!}{(n-r)!} \quad (0! \equiv 1)$$

Example 4.1 There are $P(10, 4) = 5040$ 4-digit numbers that contain no repeated digits, since each such number is just an arrangement of four of the digits 0, 1, 2, 3, ... , 9 (leading zeroes are allowed).

4.2 Enumerating r-combinations without repetitions

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

Example 4.2 A committee of 5 be chosen from 9 people in $C(9, 5)$ ways.

Observations 4.1 (i) $P(n, n) = n!$ $\quad n! = n(n-1) \ldots 2.1$

(ii) There are $n!$ permutations of n distinct objects

(iii) $C(n, r) = C(n, n-r)$

(iv) There are $(n-1)!$ permutations of n distinct objects in a circle.

Example 4.3 How many words of three distinct letters can be formed from the letters of the word MAST?

Solution: The number is $P(4, 3) = \frac{4!}{(4-3)!} = 24$.

Example 4.4 Compute the number of distinct five-card hands tht can be dealt from a deck of 52 cards.

Solution: The number is $C(52, 5) = \frac{52!}{5!47!} = 2598960$, because the order in which the cards were dealt is irrelevant.

5. The Pigeonhole Principle

Some of the most profound and complicated results in modern combinatorial theory flow from a very simple propostition:

If n pigenoholes shelter $n + 1$ or more pigeons, at least 1 pigeonhole shelters at least 2 pigeons.

Note that the pigeonhole principle tells us nothing about how to locate the pigeonhole that contains two or more pigeons. It only asserts the existence of a pigeonhole containing two or more pigeons.

To apply the pigenonhole principle, we must decide which objects will play the roles of the pigeons and which objects will play the roles of the pigeonholes.

Example 5.1 Among any group of 367 people, there must be atleast two with the same birthday, because there are only 366 possible birthdays.

Example 5.2 If eight people are chosen in any way from some group, at least two of them will have been born on the same day of the week. Here each person (pigeon) is assigned to the day of the week (pigeonhole) on which he or she was born. Since there are eight people and only seven days of the week, the pigeonhole principle tells us atleast two people must be assigned to the same day of the week.

A generalization of the pigeonhole principle is as follows:

If K pigeons are assigned to n pigeonholes, then one of the pigenonholes must contain at least $\left\lfloor \frac{k-1}{n} \right\rfloor + 1$ pigeons.

Note 5.1 If x is a real variable, the floor of x, denoted $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Example 5.3 Prove that if any 30 people are selected, then we may choose a subset of 5 so that all 5 were born on the same day of the week.

Solution: Assign each person to the day of the week on which she or he was born. Then 30 pigeons are being assigned to 7 pigeon holes. By the generalized pigeonhole principle with $K = 30$ and $n = 7$, at least $\left\lfloor \frac{30-1}{7} \right\rfloor + 1$ or 5 of the people must have been born on the same day of the week.

Example 5.4 Suppose there are 26 students and 7 cars to transport them. Then atleast one car must have 4 or more passengers.

Solution: Clearly, $K = 26, n = 7$

Now by generalized pigeonhole principle, at least $\left\lfloor \frac{26-1}{7} \right\rfloor + 1$ or 4 or more passengers should board atleast one car.

6. The Inclusion-Exclusion Principle:

We discussed the sum rule by which we can count the number of objects in the union of disjoint sets. However, if the sets are not disjoint we must refine the statement of the sum rule.

In other words, when two tasks can be done at the same time, we cannot use the sum rule to count the number of ways to do one of the two tasks. Adding the number of ways to do each task leads to an over count, since the ways to do both tasks are counted twice. To correctly count the number of ways to do one of the two tasks, we add the number of ways to do each of the two tasks and then subtract the number of ways to do both tasks. This technique is called the principle of inclusion-exclusion.

We can state the above principle in terms of sets. Let A_1 and A_2 be sets. Let T_1 be the task of choosing an element from A_1 and T_2 the task of choosing an element from A_2. There are $n(A_1)$ way to do T_1 and $n(A_2)$ ways to do T_2. The number of ways to do either T_1 or T_2 is the sum of the number of ways to do T_1 and the number of ways to do T_2, minus the number of ways to do both T_1 and T_2. Since there are $n(A_1 \cup A_2)$ ways to do either T_1 or T_2 and $n(A_1 \cap A_2)$ ways to do both T_1 and T_2, we have

$$n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$$

Example 6.1 From a group of 12 professors how many ways can a committee of 6 members be formed so that at least one of professor A and professor B will be included.

Solution: The total number of committees is $C(12, 6)$ among these committees, Let A_1 and A_2 be the set of committees that include Professor A and Professor B respectively. Since $n(A_1) = C(11, 5) = n(A_2)$ and $n(A_1 \cap A_2) = C(10, 4)$
Now $n(A_1 \cup A_2) = C(11, 5) + C(11, 5) - C(10, 4)$. □

The principle of inclusion-exclusion can be generalized to find the number of ways to do one of 'n' different tasks (or) equivalently, to find the number of elements in the union of n sets:
Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$n(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{1 \le i \le n} n(A_i) - \sum_{1 \le i < j \le n} n(A_i \cap A_j)$$

$$+ \sum_{1 \le < j < k \le n} n(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1} n(A_1 \cap A_2 \cap \ldots \cap A_k).$$

Example 6.2 Consider a set of integers from 1 to 250. Find how many of these numbers are divisible by 3 or 5 or 7. Also indicate how many are divisible by 3 or 7 but not by 5 and divisible by 3 or 5.

Let A_1, A_2, A_3 denote the set of integers 1 to 250 divisible by 3, 5 and 7

Then $n(A_1) = \frac{250}{3} = 83 \qquad n(A_1 \cap A_2) = \frac{250}{3 \times 5} = 16$

$n(A_2) = \frac{250}{5} = 50 \qquad n(A_1 \cap A_3) = \frac{250}{3 \times 7} = 11$

$n(A_3) = \frac{250}{7} = 35 \qquad n(A_2 \cap A_3) = \frac{250}{5 \times 7} = 7$

$$n(A_1 \cap A_2 \cap A_3) = \frac{250}{3 \times 5 \times 7} = 2.$$

Number of integers divisible by 3 or 5

$$\text{i.e.,} \quad n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$$
$$= 83 + 50 - 16 = 117.$$

Number of integers divisible by 3 or 5 or 7

$$n(A_1 \cup A_2 \cup A_3) = n(A_1) + n(A_2) + n(A_3) - n(A_1 \cap A_2) - n(A_1 \cap A_3)$$
$$- n(A_2 \cap A_3) + n(A_1 \cap A_2 \cap A_3)$$
$$= 83 + 50 + 35 - 16 - 11 - 7 + 2$$
$$= 136$$

Number of integers divisible by 3 or 7 but not by 5

$$= n(A_1 \cup A_2 \cup A_3) - n(A_2)$$
$$= 136 - 50 = 86$$

Exercise (I)

(1) A Label identifier for a computer program consists of one letter followed by three digits. If repetitions are allowed, how many distinct label identifiers are possible?

(2) How many different bit strings are there of length seven?

(3) A student can choose a computer project from one of three lists. The three lists contain 12, 18 and 24 possible projects respectively. How many possible projects are there to choose from?

(4) Suppose that either a member of mathematics faculty or a student who is a mathematics major is chosen as a representative to a university committee. How many different choices are there for this representative if there are 38 members of the mathematics faculty and 82 mathematics majors.

(5) A coin is tossed four times and the result of each toss is recorded. How many different sequences of heads and tails are possible?

(6) Currently, telephone area codes are three-digit numbers whose middle digit must be 0 or 1. Codes whose last two digits are 1's are being used for other purposes (Ex. 911). With these conditions, how many area codes are avaliable?

(7) In how many ways can a committee of three facutly members and two students be selected from seven faculty members and eight students?

(8) Suppose that there are eight runners in a race. the winner receives a gold medal, the second place finisher receives a silver medal, and the third place finisher receives a bronz medal. How many different ways are there to award these medals, if all possible outcomes of the race can occur?

(9) How many ways one there to distribute 12 different books among 15 people if no person is to receive more than one book?

(10) Show that if seven colors are used to paint 50 bicycles, at least 8 bicycles will be same color.

Answers (I)

(1) 26000
(2) 128
(3) 54
(4) 120
(5) 16
(6) 190
(7) 980
(8) 336
(9) P(15, 12)
(10) atleast 8