Chapter 7

Probability Theory

7.1 INTRODUCTION

Probability theory is a mathematical modeling of the phenomenon of chance or randomness. If a coin is tossed in a random manner, it can land heads or tails, but we do not know which of these will occur in a single toss. However, suppose we let s be the number of times heads appears when the coin is tossed n times. As n increases, the ratio $f = s/n$, called the *relative frequency* of the outcome, becomes more stable. If the coin is perfectly balanced, then we expect that the coin will land heads approximately 50% of the time or, in other words, the relative frequency will approach $\frac{1}{2}$. Alternatively, assuming the coin is perfectly balanced, we can arrrive at the value $\frac{1}{2}$ deductively. That is, any side of the coin is as likely to occur as the other; hence the chance of getting a head is 1 in 2 which means the probability of getting a head is $\frac{1}{2}$. Although the specific outcome on any one toss is unknown, the behavior over the long run is determined. This stable long-run behavior of random phenomena forms the basis of probability theory.

A probabilistic mathematical model of random phenomena is defined by assigning "probabilities" to all the possible outcomes of an experiment. The reliability of our mathematical model for a given experiment depends upon the closeness of the assigned probabilities to the actual limiting relative frequencies. This then gives rise to problems of testing and reliability, which form the subject matter of statistics and which lie beyond the scope of this text.

7.2 SAMPLE SPACE AND EVENTS

The set S of all possible outcomes of a given experiment is called the *sample space*. A particular outcome, i.e., an element in S, is called a *sample point*. An *event* A is a set of outcomes or, in other words, a subset of the sample space S. In particular, the set $\{a\}$ consisting of a single sample point $a \in S$ is called an *elementary event*. Furthermore, the empty set $\varnothing$ and S itself are subsets of S and so are events; $\varnothing$ is sometimes called the *impossible event* or the *null event*.

Since an event is a set, we can combine events to form new events using the various set operations:

(i) $A \cup B$ is the event that occurs iff A occurs *or* B occurs (or both).

(ii) $A \cap B$ is the event that occurs iff A occurs and B occurs.

(iii) A^c, the complement of A, also written $\bar{A}$, is the event that occurs iff A does *not* occur.

Two events A and B are called *mutually exclusive* if they are disjoint, that is, if $A \cap B = \varnothing$. In other words, A and B are mutually exclusive iff they cannot occur simultaneously. Three or more events are mutually exclusive if every two of them are mutually exclusive.

EXAMPLE 7.1

(*a*) ***Experiment***: Toss a die and observe the number (of dots) that appears on top.

The sample space S consists of the six possible numbers; that is,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let A be the event that an even number occurs, B that an odd number occurs, and C that a prime number occurs; that is, let

$$A = \{2, 4, 6\}, \quad B = \{1, 3, 5\}, \quad C = \{2, 3, 5\}$$

Then

$A \cup C = \{2, 3, 4, 5, 6\}$ is the event that an even or a prime number occurs.
$B \cap C = \{3, 5\}$ is the event that an odd prime number occurs.
$C^c = \{1, 4, 6\}$ is the event that a prime number does not occur.

Note that A and B are mutually exclusive: $A \cap B = \varnothing$. In other words, an even number and an odd number cannot occur simultaneously.

(*b*) ***Experiment***: Toss a coin three times and observe the sequence of heads (T) and tails (T) that appears.

The sample space S consists of the following eight elements:

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

Let A be the event that two or more heads appear consecutively, and B that all the tosses are the same; that is, let

$$A = \{\text{HHH, HHT, THH}\}. \qquad \text{and} \qquad B = \{\text{HHH, TTT}\}$$

Then $A \cap B = \{\text{HHH}\}$ is the elementary event in which only heads appear. The event that five heads appear is the empty set $\varnothing$.

(*c*) ***Experiment***: Toss a coin until a head appears, and then count the number of times the coin is tossed.

The sample space of this experiment is $S = \{1, 2, 3, \ldots\}$. Since every positive integer is an element of S, the sample space is infinite.

Remark: The sample space S in Example 7.1(*c*), as noted, is not finite. The theory concerning such samples space lies beyond the scope of this text. Thus, unless otherwise stated, all our sample spaces S shall be finite.

7.3 FINITE PROBABILITY SPACES

The following definition applies.

Definition: Let S be a finite sample space, say $S = \{a_1, a_2, \ldots, a_n\}$. A *finite probability space*, or *probability model*, is obtained by assigning to each point a_i in S a real number p_i, called the *probability* of a_i, satisfying the following properties:

(i) Each p_i is nonnegative, that is, $p_i \geq 0$.
(ii) The sum of the p_i is 1, that is, $p_1 + p_2 + \cdots + p_n = 1$.

The *probability* of an event A written $P(A)$, is then defined to be the sum of the probabilities of the points in A.

The singleton set $\{a_i\}$ is called an *elementary* event and, for notational convenience, we write $P(a_i)$ for $P(\{a_i\})$.

EXAMPLE 7.2 ***Experiment***: Let three coins be tossed and the number of heads observed. [Compare with the above Example 7.1(*b*).]

The sample space is $S = \{0, 1, 2, 3\}$. The following assignments on the elements of S defines a probability space:

$$P(0) = \tfrac{1}{8}, \qquad P(1) = \tfrac{3}{8}, \qquad P(2) = \tfrac{3}{8}, \qquad P(3) = \tfrac{1}{8}$$

That is, each probability is nonnegative, and the sum of the probabilities is 1. Let A be the event that at least one head appears, and let B be the event that all heads or all tails appear; that is, let $A = \{1, 2, 3\}$ and $B = \{0, 3\}$. Then, by definition,

$$P(A) = P(1) + P(2) + P(3) = \tfrac{3}{8} + \tfrac{3}{8} + \tfrac{1}{8} = \tfrac{7}{8} \qquad \text{and} \qquad P(B) = P(0) + P(3) = \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{4}$$

Equiprobable Spaces

Frequently, the physical characteristics of an experiment suggest that the various outcomes of the sample space be assigned equal probabilities. Such a finite probability space S, where each sample point has the same probability, will be called an *equiprobable space*. In particular, if S contains n points, then the probability of each point is $1/n$. Furthermore, if an event A contains r points, then its probability is $r(1/n) = r/n$. In other words,

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S} = \frac{n(A)}{n(S)} \quad \text{or} \quad P(A) = \frac{\text{number of outcomes favorable to } A}{\text{total number of possible outcomes}}$$

where $n(A)$ denotes the number of elements in a set A.

We emphasize that the above formula for $P(A)$ can only be used with respect to an equiprobable space, and cannot be used in general.

The expression *at random* will be used only with respect to an equiprobable space; the statement "choose a point at random from a set S" shall mean that every sample point in S has the same probability of being chosen.

EXAMPLE 7.3 Let a card be selected from an ordinary deck of 52 playing cards. Let

$$A = \{\text{the card is a spade}\} \quad \text{and} \quad B = \{\text{the card is a face card}\}$$

(A face card is a jack, queen, or king.) We compute $P(A)$, $P(B)$, and $P(A \cap B)$. Since we have an equiprobable space,

$$P(A) = \frac{\text{number of spades}}{\text{number of cards}} = \frac{13}{52} = \frac{1}{4}, \qquad P(B) = \frac{\text{number of face cards}}{\text{number of cards}} = \frac{12}{52} = \frac{3}{13}$$

$$P(A \cap B) = \frac{\text{number of spade face cards}}{\text{number of cards}} = \frac{3}{32}$$

Theorems on Finite Probability Spaces

The following theorem follows directly from the fact that the probability of an event is the sum of the probabilities of its points.

Theorem 7.1: The probability function P defined on the class of all events in a finite probability space has the following properties:

[P₁] For every event A, $0 \le P(A) \le 1$.
[P₂] $P(S) = 1$.
[P₃] If events A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.

The next theorem formalizes our intuition that if p is the probability that an event E occurs, then $1 - p$ is the probability that E does not occur. (That is, if we hit a target $p = 1/3$ of the times, then we miss the target $1 - p = 2/3$ of the times.)

Theorem 7.2: Let A be any event. Then $P(A^c) = 1 - P(A)$.

The following theorem (proved in Problem 7.16) follows directly from Theorem 7.1.

Theorem 7.3: Let $\varnothing$ be the empty set, and suppose A and B are any events. Then:

(i) $P(\varnothing) = 0$.
(ii) $P(A \setminus B) = P(A) - P(A \cap B)$.
(iii) If $A \subseteq B$, then $P(A) \le P(B)$.

Observe that Property **[P₃]** in Theorem 7.1 gives the probability of the union of events in the case that the events are disjoint. The general formula (proved in Problem 7.17) is called the Addition Principle. Specifically:

Theorem 7.4 (Addition Principle): For any events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

EXAMPLE 7.4 Suppose a student is selected at random from 100 students where 30 are taking mathematics, 20 are taking chemistry, and 10 taking mathematics and chemistry. Find the probability p that the student is taking mathematics or chemistry.

Let $M = \{\text{students taking mathematics}\}$ and $C = \{\text{students taking chemistry}\}$. Since the space is equiprobable,

$$P(M) = \frac{30}{100} = \frac{3}{10} \qquad P(C) = \frac{20}{100} = \frac{1}{5}, \qquad P(M \text{ and } C) = P(M \cap C) = \frac{10}{100} = \frac{1}{10}$$

Thus, by the Addition Principle (Theorem 7.4),

$$p = P(M \text{ or } C) = P(M \cup C) = P(M) + P(C) - P(M \cap C) = \frac{3}{10} + \frac{1}{5} = \frac{1}{10} = \frac{2}{5}$$

7.4 CONDITIONAL PROBABILITY

Suppose E is an event in a sample space S with $P(E) > 0$. The probability that an event A occurs once E has occurred or, specifically, the *conditional probability of A given E*, written $P(A \mid E)$, is defined as follows:

$$P(A \mid E) = \frac{P(A \cap E)}{P(E)}$$

As pictured in the Venn diagram in Fig. 7-1, $P(A \mid E)$ measures, in a certain sense, the relative probability of A with respect to the reduced space E.

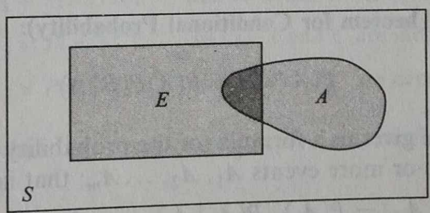


Fig. 7-1

Now suppose S is an equiprobable space, and we let $n(A)$ denote the number of elements in the event A. Then

$$P(A \cap E) = \frac{n(A \cap E)}{n(S)}, \qquad P(E) = \frac{n(E)}{n(S)}, \qquad \text{and so} \qquad P(A \mid E) = \frac{P(A \cap E)}{P(E)} = \frac{n(A \cap E)}{n(E)}$$

We state this result formally.

Theorem 7.5: Suppose S is an equiprobable space and A and B are events. Then

$$P(A \mid E) = \frac{\text{number of elements in } A \cap E}{\text{number of elements in } E} = \frac{n(A \cap E)}{n(E)}$$

EXAMPLE 7.5

(*a*) A pair of fair dice is tossed. The sample space S consists of the 36 ordered pairs (a, b) where a and b can be any of the integers from 1 to 6. (See Problem 7.3.) Thus the probability of any point is 1/36. Find the probability that one of the dice is 2 if the sum is 6. That is, find $P(A \mid E)$ where

$$E = \{\text{ sum is } 6\} \qquad \text{and} \qquad A = \{2 \text{ appears on at least one die}\}$$

Also find $P(A)$.

Now E consists of five elements, specifically

$$E = \{(1,5),\ (2,4),\ (3,3),\ (4,2),\ (5,1)\}$$

Two of them, (2, 4) and (4, 2), belong to A; that is,

$$A \cap E = \{(2,4),\ (4,2)\}$$

By Theorem 7.5, $P(A \mid E) = 2/5$.

On the other hand, A consists of 11 elements, specifically,

$$A = \{(2,1),\ (2,2),\ (2,3),\ (2,4),\ (2,5),\ (2,6),\ (1,2),\ 3,2),\ (4,2),\ (5,2),\ (6,2)\}$$

and S consists of 36 elements. Hence $P(A) = 11/36$.

(b) A couple has two children; the sample space is $S = \{bb, bg, gb, gg\}$ with probability 1/4 for each point. Find the probability p that both children are boys if it is known that: (i) at least one of the children is a boy, (ii) the older child is a boy.

(i) Here the reduced space consists of three elements, $\{bb, bg, gb\}$; hence $p = \frac{1}{3}$.

(ii) Here the reduced space consists of only two elements $\{bb, bg\}$; hence $p = \frac{1}{2}$.

Multiplication Theorem for Conditional Probability

Suppose A and B are events in a sample space S with $P(A) > 0$. By definition of conditional probability,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

Multiplying both sides by $P(A)$ gives us the following useful result:

Theorem 7.6 (Multiplication Theorem for Conditional Probability):

$$P(A \cap B) = P(A)P(B \mid A)$$

The multiplication theorem gives us a formula for the probability that events A and B both occur. It can easily be extended to three or more events $A_1, A_2, \ldots A_m$; that is,

$$P(A_1 \cap A_2 \cap \cdots \cap A_m) = P(A_1) \cdot P(A_2 \mid A_1) \cdot \cdots \cdot P(A_m \mid A_1 \cap A_2 \cap \cdots \cap A_{m-1})$$

EXAMPLE 7.6 A lot contains 12 items of which four are defective. Three items are drawn at random from the lot one after the other. Find the probability p that all three are nondefective.

The probability that the first item is nondefective is $\frac{8}{12}$ since eight of 12 items are nondefective. If the first item is nondefective, then the probability that the next item is nondefective is $\frac{7}{11}$ since only seven of the remaining 11 items are nondefective. If the first two items are nondefective, then the probability that the last item is nondefective is $\frac{6}{10}$ since only 6 of the remaining 10 items are now nondefective. Thus by the multiplication theorem,

$$p = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55} \approx 0.25$$

7.5 INDEPENDENT EVENTS

Events A and B in a probability space S are said to be *independent* if the occurrence of one of them does not influence the occurrence of the other. More specifically, B is independent of A if $P(B)$ is the same as $P(B \mid A)$. Now substituting $P(B)$ for $P(B \mid A)$ in the Multiplication Theorem $P(A \cap B) = P(A)P(B \mid A)$ yields

$$P(A \cap B) = P(A)P(B).$$

We formally use the above equation as our definition of independence.

Definition: Events A and B are *independent* if $P(A \cap B) = P(A)P(B)$; otherwise they are *dependent*.

We emphasize that independence is a symmetric relation. In particular, the equation

$$P(A \cap B) = P(A)P(B) \quad \text{implies both} \quad P(B \mid A) = P(B) \quad \text{and} \quad P(A \mid B) = P(A)$$

EXAMPLE 7.7 A fair coin is tossed three times yielding the equiprobable space

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

Consider the events:

$$A = \{\text{first toss is heads}\} = \{\text{HHH, HHT, HTH, HTT}\}$$
$$B = \{\text{second toss is heads}\} = \{\text{HHH, HHT, THH, THT}\}$$
$$C = \{\text{exactly two heads in a row}\} = \{\text{HHT, THH}\}$$

Clearly A and B are independent events; this fact is verified below. On the other hand, the relationship between A and C or B and C is not obvious. We claim that A and C are independent, but that B and C are dependent. We have

$$P(A) = \tfrac{4}{8} = \tfrac{1}{2}, \qquad P(B) = \tfrac{4}{8} = \tfrac{1}{2}, \qquad P(C) = \tfrac{2}{8} = \tfrac{1}{4}$$

Also,

$$P(A \cap B) = P(\{\text{HHH, HHT}\}) = \tfrac{1}{4}, \qquad P(A \cap C) = P(\{\text{HHT}\}) = \tfrac{1}{8}, \qquad P(B \cap C) = P(\{\text{HHT, THH}\}) = \tfrac{1}{4}$$

Accordingly,

$$P(A)P(B) = \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4} = P(A \cap B), \quad \text{and so } A \text{ and } B \text{ are independent}$$
$$P(A)P(C) = \tfrac{1}{2} \cdot \tfrac{1}{4} = \tfrac{1}{8} = P(A \cap C), \quad \text{and so } A \text{ and } C \text{ are independent}$$
$$P(B)P(C) = \tfrac{1}{2} \cdot \tfrac{1}{4} = \tfrac{1}{8} \neq P(B \cap C), \quad \text{and so } B \text{ and } C \text{ are dependent}$$

Frequently, we will postulate that two events are independent, or the experiment itself will imply that two events are independent.

EXAMPLE 7.8 The probability that A hits a target is $\frac{1}{4}$, and the probability that B hits the target is $\frac{2}{5}$. Both shoot at the target. Find the probability that at least one of them hits the target, i.e., that A or B (or both) hit the target.

We are given that $P(A) = \frac{1}{4}$ and $P(B) = \frac{2}{5}$, and we seek $P(A \cup B)$. Furthermore, the probability that A or B hits the target is not influenced by what the other does; that is, the event that A hits the target is independent of the event that B hits the target, that is, $P(A \cap B) = P(A)P(B)$. Thus

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A)P(B) = \tfrac{1}{4} + \tfrac{2}{5} - \left(\tfrac{1}{4}\right)\left(\tfrac{2}{5}\right) = \tfrac{11}{20}$$

7.6 INDEPENDENT REPEATED TRIALS, BINOMIAL DISTRIBUTION

We have previously discussed probability spaces which were associated with an experiment repeated a finite number of times, as the tossing of a coin three times. This concept of repetition is formalized as follows:

Definition: Let S be a finite probability space. By the space of n *independent repeated trials*, we mean the probability space S_n consisting of ordered n-tuples of elements of S, with the probability of an n-tuple defined to be the product of the probabilities of its components:

$$P((s_1, s_2, \ldots, s_n)) = P(s_1)P(s_2) \cdots P(s_n)$$

EXAMPLE 7.9 Whenever three horses a, b, and c race together, their respective probabilities of winning are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$. In other words, $S = \{a, b, c\}$ with $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{3}$, and $P(c) = \frac{1}{6}$. If the horses race twice, then the sample space of the two repeated trials is

$$S_2 = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$$

For notational convenience, we have written ac for the ordered pair (a, c). The probability of each point in S_2 is

$$\begin{array}{lll} P(aa) = P(a)P(a) = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}, & P(ba) = \frac{1}{6}, & P(ca) = \frac{1}{12} \\ P(ab) = P(a)P(b) = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6}, & P(bb) = \frac{1}{9}, & P(cb) = \frac{1}{18} \\ P(ac) = P(a)P(c) = \frac{1}{2}\left(\frac{1}{6}\right) = \frac{1}{12}, & P(bc) = \frac{1}{18}, & P(cc) = \frac{1}{36} \end{array}$$

Thus the probability of c winning the first race and a winning the second race is $P(ca) = \frac{1}{12}$.

Repeated Trials with Two Outcomes, Bernoulli Trials

Now consider an experiment with only two outcomes. Independent repeated trials of such an experiment are called Bernoulli trials, named after the Swiss mathematician Jacob Bernoulli (1654–1705). The term independent trials means that the outcome of any trial does not depend on the previous outcomes (such as, tossing a coin). We will call one of the outcomes *success* and the other outcome *failure*.

Let p denote the probability of success in a Bernoulli trial, and so $q = 1 - p$ is the probability of failure. A *binomial experiment* consists of a fixed number of Bernoulli trials. The notation

$$B(n, p)$$

will be used to denote a binomial experiment with n trials and probability p of success.

Frequently, we are interested in the number of successes in a binomial experiment and not in the order in which they occur. The following theorem (proved in Problem 7.38) applies.

Theorem 7.7: The probability of exactly k success in a binomial experiment $B(n, p)$ is given by

$$P(k) = P(k \text{ successes}) = \binom{n}{k} p^k q^{n-k}$$

The probability of one or more successes is $1 - q^n$.

Here $\binom{n}{k}$ is the binomial coefficient, which is defined and discussed in Chapter 6.

EXAMPLE 7.10 A fair coin is tossed 6 times; call heads a success. This is a binomial experiment with $n = 6$ and $p = q = \frac{1}{2}$.

(a) The probability that exactly two heads occurs (i.e., $k = 2$) is

$$P(2) = \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = \frac{15}{64} \approx 0.23$$

(b) The probability of getting at least four heads (i.e., $k = 4$, 5 or 6) is

$$\begin{aligned} P(4) + P(5) + P(6) &= \binom{6}{4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2 + \binom{6}{5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right) + \binom{6}{6}\left(\frac{1}{2}\right)^6 \\ &= \frac{15}{64} + \frac{6}{64} + \frac{1}{64} = \frac{11}{32} \approx 0.34 \end{aligned}$$

(c) The probability of getting no heads (i.e., all failures) is $q^6 = \left(\frac{1}{2}\right)^6 = \frac{1}{64}$, so the probability of one or more heads is $1 - q^n = 1 - \frac{1}{64} = \frac{63}{64} \approx 0.94$.

Remark: The function $P(k)$ for $k = 0, 1, 2, \ldots, n$, for a binomial experiment $B(n, p)$, is called the *binomial distribution* since it corresponds to the successive terms of the binomial expansion:

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \cdots + p^n$$

The use of the term *distribution* will be explained later in the chapter.

7.7 RANDOM VARIABLES

Let S be a sample of an experiment. As noted previously, the outcome of the experiment, or the points in S, need not be numbers. For example, in tossing a coin the outcomes are H (heads) or T (tails), and in tossing a pair of dice the outcomes are pairs of integers. However, we frequently wish to assign a specific number to each outcome of the experiment. For example, in coin tossing, it may be convenient to assign 1 to H and 0 to T; or, in the tossing of a pair of dice, we may want to assign the sum of the two integers to the outcome. Such an assignment of numerical values is called a *random variable*. More generally, we have the following definition.

Definition: A *random variable* X is a rule that assigns a numerical value to each outcome in a sample space S.

We shall let R_X denote the set of numbers assigned by a random variable X, and we shall refer to R_X as the *range space*.

Remark: In more formal terminology, X is a function from S to the real numbers **R**, and R_X is the range of X. Also, for some infinite sample spaces S, not all functions from S to **R** are considered to be random variables. However, the sample spaces here are finite, and every real-valued function defined on a finite sample space is a random variable.

EXAMPLE 7.11 A pair of fair dice is tossed. (See Problem 7.3.) The sample space S consists of the 36 ordered pairs (a, b) where a and b can be any integers between 1 and 6; that is,

$$S = \{(1,1),\ (1,2), \ldots, (6,6)\}$$

Let X assign to each point in S the sum of the numbers; then X is a random variable with range space

$$R_X = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

Let Y assign to each point the maximum of the two numbers; then Y is a random variable with range space

$$R_Y = \{1, 2, 3, 4, 5, 6\}$$

EXAMPLE 7.12 A box contains 12 items of which three are defective. A sample of three items is selected from the box. The sample space S consists of the $\binom{12}{3} = 220$ different samples of size 3. Let X denote the number of defective items in the sample; then X is a random variable with range space $R_X = \{0, 1, 2, 3\}$.

Probability Distribution of a Random Variable

Let $R_X = \{x_1, x_2, \ldots, x_t\}$ be the range space of a random variable X defined on a finite sample space S. Then X induces an assignment of probabilities on the range space R_X as follows:

$$p_i = P(x_i) = P(X = x_i) = \text{sum of probabilities of points in } S \text{ whose image is } x_i$$

The set of ordered pairs $(x_1, p_1), \ldots, (x_t, p_t)$, usually given by a table

x_1	x_2	$\cdots$	x_t
p_1	p_2	$\cdots$	p_t

is called the *distribution* of the random variable X.

In the case that S is an equiprobable space, we can easily obtain the distribution of a random variable from the following result.

Theorem 7.8: Let S be an equiprobable space, and let X be a random variable on S with range space

$$R_X = \{x_1, x_2, \ldots, x_t\}.$$

Then

$$p_i = P(x_i) = \frac{\text{number of points in } S \text{ whose image is } x_i}{\text{number of points in } S}$$

EXAMPLE 7.13 Consider the random variable X in Example 7.11 which assigns the sum to the toss of a pair of dice. We use Theorem 7.8 to obtain the distribution of X.

There is only one outcome $(1,1)$ whose sum is 2; hence $P(2) = \frac{1}{36}$. There are two outcomes, $(1,2)$ and $(2,1)$, whose sum is 3; hence $P(3) = \frac{2}{36}$. There are three outcomes, $(1,3)$, $(2,2)$, $(3,1)$, whose sum is 4; hence $P(4) = \frac{3}{36}$. Similarly $P(5) = \frac{4}{36}$, $P(6) = \frac{5}{36}, \ldots, P(12) = \frac{1}{36}$. The distribution of X consists of the points in R_X with their respective probabilities; that is,

x_i	2	3	4	5	6	7	8	9	10	11	12
p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

EXAMPLE 7.14 Let X be the random variable in Example 7.12. We use Theorem 7.8 to obtain the distribution of X.

There are $\binom{9}{3} = 84$ samples of size 3 with no defective items; hence $P(0) = \frac{84}{220}$. There are $3\binom{9}{2} = 108$ samples of size 3 containing one defective item; hence $P(1) = \frac{108}{220}$. There are $\binom{3}{2} \cdot 9 = 27$ samples of size 3 containing two defective items; hence $P(2) = \frac{27}{220}$. There is only one sample of size 3 containing the three defective items; hence $P(3) = \frac{1}{220}$. The distribution of X follows:

x_i	0	1	2	3
p_i	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$

Remark: Let X be a random variable on a probability space $S = \{a_1, a_2, \ldots, a_m\}$, and let $f(x)$ be any polynomial. Then $f(X)$ is the random variable which assigns $f(X(a_i))$ to the point a_i or, in other words, $f(X)(a_i) = f(X(a_i))$. Accordingly, if X takes on the values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$, then $f(X)$ takes on the values $f(x_1), f(x_2), \ldots, f(x_n)$ with the same corresponding probabilities.

Expectation of a Random Variable

Let X be a random variable. There are two important measurements (or parameters) associated with X, the *mean* of X, denoted by μ or μ_X, and the *standard deviation* of X denoted by σ or σ_X. The mean μ is also called the *expectation* of X, written $E(X)$. In a certain sense, the mean μ measures the "central tendency" of X, and the standard deviation σ measures the "spread" or "dispersion" of X. (The mean is sometimes called the *average* value since it corresponds to the average of a set of numbers where the probability of a number is defined by the relative frequency of the number in the set.) This subsection discusses the expectation $\mu = E(X)$ of X, and the next subsection discusses the standard deviation σ of X.

Let X be a random variable on a probability space $S = \{a_1, a_2, \cdots, a_m\}$. The *mean* or *expectation* of X is defined by

$$\mu = E(X) = X(a_1)P(a_1) + X(a_2)P(a_2) + \cdots + X(a_m)P(a_m) = \Sigma X(a_i)P(a_i)$$

In particular, if X is given by the distribution

x_1	x_2	$\cdots$	x_n
p_1	p_2	$\cdots$	p_n

then the *expectation* of X is

$$\mu = E(X) = x_1p_1 + x_2p_2 + \cdots + x_np_n = \Sigma x_ip_i$$

(For notational convenience, we have omitted the limits in the summation symbol Σ.)

EXAMPLE 7.15

(*a*) Suppose a fair coin is tossed six times. The number of heads which can occur with their respective probabilities are as follows:

x_i	0	1	2	3	4	5	6
p_i	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

Then the mean or expectation or expected number of heads is

$$\mu = E(X) = 0(\tfrac{1}{64}) + 1(\tfrac{6}{64}) + 2(\tfrac{15}{64}) + 3(\tfrac{20}{64}) + 4(\tfrac{15}{64}) + 5(\tfrac{6}{64}) + 6(\tfrac{1}{64}) = 3$$

(*b*) Consider the random variable X in Example 7.12 whose distribution appears in Example 7.14. It gives the possible numbers of defective items in a sample of size 3 with their respective probabilities. Then the expectation of X or, in other words, the expected number of defective items in a sample of size 3, is

$$\mu = E(X) = 0(\tfrac{84}{220}) + 1(\tfrac{108}{220}) + 2(\tfrac{27}{220}) + 3(\tfrac{1}{220}) = 0.75$$

(*c*) Three horses a, b, and c are in a race; suppose their respective probabilities of winning are $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$. Let X denote the payoff function for the winning horse, and suppose X pays \$2, \$6, or \$9 according as a, b, or c wins the race. The expected payoff for the race is

$$E(X) = X(a)P(a) + X(b)P(b) + X(c)P(c)$$
$$= 2(\tfrac{1}{2}) + 6(\tfrac{1}{3}) + 9(\tfrac{1}{6}) = 4.5$$

Variance and Standard Deviation of a Random Variable

Consider a random variable X with mean μ and probability distribution

x_1	x_2	x_3	$\cdots$	x_n
p_1	p_2	p_3	$\cdots$	p_n

The *variance* $\text{Var}(X)$ and *standard deviation* σ of X are defined by

$$\text{Var}(X) = (x_1 - \mu)^2p_1 + (x_2 - \mu)^2p_2 + \cdots + (x_n - \mu)^2p_n = \Sigma(x_i - \mu)^2p_i = E((X - \mu)^2)$$
$$\sigma = \sqrt{\text{Var}(X)}$$

The following formula is usually more convenient for computing $\text{Var}(X)$ than the above:

$$\text{Var}(X) = x_1^2p_1 + x_2^2p_2 + \cdots + x_n^2p_n - \mu^2 = \Sigma x_i^2p_i - \mu^2 = E(X^2) - \mu^2$$

Remark: According to the above formula, $\mathrm{Var}(X) = \sigma^2$. Both σ^2 and σ measure the weighted spread of the values x_i about the mean μ; however, σ has the same units as μ.

EXAMPLE 7.16

(*a*) Let X denote the number of times heads occurs when a fair coin is tossed six times. The distribution of X appears in Example 7.15(*a*), where its mean $\mu = 3$ is computed. The variance of X is computed as follows:

$$\mathrm{Var}(X) = (0-3)^2 \tfrac{1}{64} + (1-3)^2 \tfrac{6}{64} + (2-3)^2 \tfrac{15}{64} + \cdots + (6-3) \tfrac{1}{64} = 1.5$$

Alternatively:

$$\mathrm{Var}(X) = 0^2 \tfrac{1}{64} + 1^2 \tfrac{6}{64} + 2^2 \tfrac{15}{64} + 3^2 \tfrac{20}{64} + 4^2 \tfrac{15}{64} + 5^2 \tfrac{6}{64} + 6^2 \tfrac{1}{64} - 3^2 = 1.5$$

Thus the standard deviation is $\sigma = \sqrt{1.5} \approx 1.225$ (heads).

(*b*) Consider the random variable X in Example 7.15(*b*), where its mean $\mu = 0.75$ is computed. (Its distribution appears in Example 7.14.) The variance of X is computed as follows:

$$\mathrm{Var}(X) = 0^2 \tfrac{84}{220} + 1^2 \tfrac{108}{220} + 2^2 \tfrac{27}{220} + 3^2 \tfrac{1}{220} - (0.75)^2 = 0.46$$

Thus the standard deviation is

$$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{0.46} = 0.66$$

Binomial Distribution

Consider a binomial experiment $B(n, p)$. That is, $B(n, p)$ consists of n independent repeated trials with two outcomes, success or failure, and p is the probability of success. The number X of k successes is a random variable with distribution appearing in Fig. 7-2.

k	0	1	2	$\cdots$	n
$P(k)$	q^n	$\binom{n}{1} q^{n-1} p$	$\binom{n}{2} q^{n-2} p^2$	$\cdots$	p^n

Fig. 7-2

The following theorem applies.

Theorem 7.9: Consider the binomial distribution $B(n, p)$. Then:

(i) Expected value $E(X) = \mu = np$.

(ii) Variance $\mathrm{Var}(X) = \sigma^2 = npq$.

(iii) Standard deviation $\sigma = \sqrt{npq}$.

EXAMPLE 7.17

(*a*) The probability that a man hits a target is $p = 1/5$. He fires 100 times. Find the expected number μ of times he will hit the target and the standard deviation σ.

Here $p = \frac{1}{5}$ and so $q = \frac{4}{5}$. Hence

$$\mu = np = 100 \cdot \frac{1}{5} = 20 \quad \text{and} \quad \sigma = \sqrt{npq} = \sqrt{100 \cdot \frac{1}{5} \cdot \frac{4}{5}} = 4$$

(*b*) Find the expected number $E(X)$ of correct answers obtained by guessing in a five-question true–false test.

Here $p = \frac{1}{2}$. Hence $E(X) = np = 5 \cdot \frac{1}{2} = 2.5$.

Solved Problems

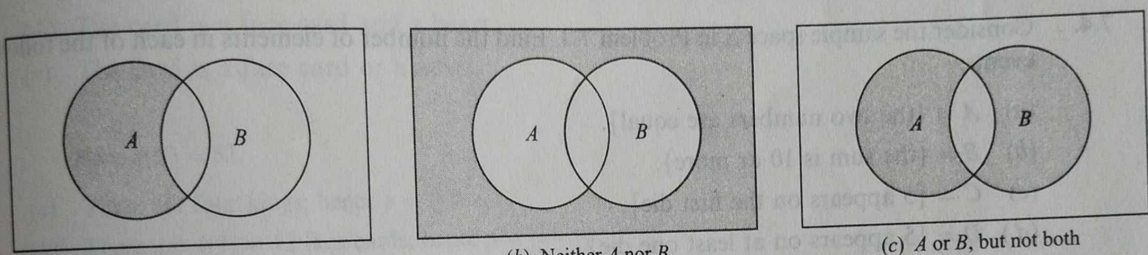
SAMPLE SPACES AND EVENTS

7.1. Let A and B be events. Find an expression and exhibit the Venn diagram for the events: (*a*) A but not B; (*b*) neither A nor B; (*c*) either A or B, but not both.

(*a*) Since A but not B occurs, shade the area of A outside of B, as in Fig. 7-3(*a*). Note that B^c, the complement of B, occurs, since B does not occur; hence A and B^c occur. In other words the event is $A \cap B^c$.

(*b*) "Neither A nor B" means "not A and not B", or $A^c \cap B^c$. By DeMorgan's law, this is also the set $(A \cup B)^c$; hence shade the area outside of A and B, i.e., outside $A \cup B$, as in Fig. 7-3(*b*).

(*c*) Since A or B, but not both, occurs, shade the area of A and B, except where they intersect, as in Fig. 7-3(*c*). The event is equivalent to the occurrence of A but not B or B but not A. Thus the event is $(A \cap B^c) \cup (B \cap A^c)$.



(*a*) A but not B (*b*) Neither A nor B (*c*) A or B, but not both

Fig. 7-3

7.2. Let a coin and a die be tossed; and let the sample space S consist of the 12 elements:

$$S = \text{ H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}\}$$

(*a*) Express explicitly the following events:

$$A = \text{ heads and an even number appears}\}$$

$$B = \{\text{a prime number appears}\}$$

$$C = \{\text{tails and an odd number appears}\}$$

(*b*) Express explicitly the events: (i) A or B occurs; (ii) B and C occur; (iii) only B occurs.

(*c*) Which pair of the events A, B, and C are mutually exclusive?

(*a*) The elements of A are those elements of S consisting of an H and an even number:

$$A = \{\text{H2, H4, H6}\}$$

The elements of B are those points in S whose second component is a prime number:

$$B = \{\text{H2, H3, H5, T2, T3, T5}\}$$

The elements of C are those points in S consisting of a T and an odd number: $C = \{\text{T1, T3, T5}\}$.

(*b*) (i) $A \cup B = \{\text{H2, H4, H6, H3, H5, T2, T3, T5}\}$.

(ii) $B \cap C = \{\text{T3, T5}\}$.

(iii) $B \cap A^c \cap C^c = \{\text{H3, H5, T2}\}$.

(*c*) A and C are mutually exclusive since $A \cap C = \varnothing$.

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7.3. A pair of dice is tossed and the two numbers appearing on the top are recorded. Describe the sample space S, and find the number $n(S)$ of elements in S.

There are six possible numbers, 1, 2, ..., 6, on each die. Hence $n(S) = 6 \cdot 6 = 36$, and S consists of the 36 pairs of numbers from 1 to 6. Figure 7-4 shows these 36 pairs of numbers in an array where each row has the same first element and each column has the same second element.

$$\begin{array}{cccccc}
(1,1), & (1,2), & (1,3), & (1,4), & (1,5), & (1,6) \\
(2,1), & (2,2), & (2,3), & (2,4), & (2,5), & (2,6) \\
(3,1), & (3,2), & (3,3), & (3,4), & (3,5), & (3,6) \\
(4,1), & (4,2), & (4,3), & (4,4), & (4,5), & (4,6) \\
(5,1), & (5,2), & (5,3), & (5,4), & (5,5), & (5,6) \\
(6,1), & (6,2), & (6,3), & (6,4), & (6,5), & (6,6)
\end{array}$$

Fig. 7-4

7.4. Consider the sample space S in Problem 7.3. Find the number of elements in each of the following events:

(*a*) $A = \{\text{the two numbers are equal}\}$.
(*b*) $B = \{\text{the sum is 10 or more}\}$.
(*c*) $C = \{\text{5 appears on the first die}\}$.
(*d*) $D = \{\text{5 appears on at least one die}\}$.
(*e*) $E = \{\text{the sum is 7 or less}\}$.

Use Fig. 7-4 to help count the number of elements which are in the event:

(*a*) $A = \{(1,1), (2,2), \ldots, (6,6)\}$, so $n(A) = 6$.
(*b*) $B = \{(6,4), (5,5), (4,6), (6,5), (5,6), (6,6)\}$, so $n(B) = 6$.
(*c*) $C = \{(5,1), (5,2), \ldots, (5,6)$, so $n(C) = 6$.
(*d*) There are six pairs with 5 as the first element, and six pairs with 5 as the second element. However, $(5,5)$ appears in both places. Hence

$$n(D) = 6 + 6 - 1 = 11$$

Alternatively, count the pairs in Fig. 7-4 which are in D to get $n(D) = 11$.

(*e*) Let $n(s)$ denote the number of pairs in S where the sum is s. The sum 7 appear on the diagonal of the array in Fig. 7-4; hence $n(7) = 6$. The sum 6 appears directly below the diagonal, so $n(6) = 5$. Similarly, $n(5) = 4$, $n(4) = 3$, $n(3) = 2$, and $n(2) = 1$. Thus

$$n(S) = 6 + 5 + 4 + 3 + 2 + 1 = 21$$

Alternatively, $n(7) = 6$ and there are $36 - 6 = 30$ pairs remaining. Half of them have sums exceeding 7 and half of them have sums less than 7. Thus $n(s) = 6 + 15 = 21$.

FINITE EQUIPROBABLE SPACES

7.5. Determine the probability p of each event:

(*a*) An even number appears in the toss of a fair die.
(*b*) One or more heads appear in the toss of three fair coins.
(*c*) A red marble appears in random drawing of one marble from a box containing four white, three red, and five blue marbles.

Each sample space S is an equiprobable space. Hence, for each event E, use

$$P(E) = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{n(E)}{n(S)}$$

(a) The event can occur in three ways (2, 4 or 6) out of 6 cases; hence $p = \frac{3}{6} = \frac{1}{2}$.

(b) Assuming the coins are distinguished, there are 8 cases:

$$\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT,}$$

Only the last case is not favorable; hence $p = 7/8$.

(c) There are $4 + 3 + 5 = 12$ marbles of which three are red; hence $p = \frac{3}{12} = \frac{1}{4}$.

7.6. A single card is drawn from an ordinary deck S of 52 cards. Find the probability p that:

(a) The card is a king.

(b) The card is a face card (jack, queen or king).

(c) The card is a heart.

(d) The card is a face card and a heart.

(e) The card is a face card or a heart.

Here $n(S) = 52$.

(a) There are four kings; hence $p = \frac{4}{52} = \frac{1}{13}$.

(b) There are $4(3) = 12$ face cards; hence $p = \frac{12}{52} = \frac{3}{13}$.

(c) There are 13 hearts; hence $p = \frac{13}{52} = \frac{1}{4}$.

(d) There are three face cards which are hearts; hence $p = \frac{3}{52}$.

(e) Letting $F = \{\text{face cards}\}$ and $H = \{\text{hearts}\}$, we have

$$n(F \cup H) = n(F) + n(H) - n(F \cap H) = 12 + 13 - 3 = 22$$

Hence $p = \frac{22}{52} = \frac{11}{26}$.

7.7. Consider the sample space S in Problem 7.2. Assume the coin and die are fair; hence S is an equiprobable space. Find: (a) $P(A)$, $P(B)$, $P(C)$; (b) $P(A \cup$

Since S is an equiprobable space, use $P(E) = n(E)/n(S)$. Here $n(S) = 12$. So we need only count the number of elements in the given set.

(a) $P(A) = \frac{3}{12}$, $P(B) = \frac{6}{12}$, $P(C) = \frac{3}{12}$.

(b) $P(A \cup B) = \frac{8}{12}$, $P(B \cap C) = \frac{2}{12}$, $P(B \cap A^c \cap C^c) = \frac{3}{12}$.

7.8. Two cards are drawn at random from an ordinary deck of 52 cards. Find the probability p that: (a) both are spades; (b) one is a spade and one is a heart.

There are $\binom{52}{2} = 1326$ ways to draw 2 cards from 52 cards.

(a) There are $\binom{13}{2} = 78$ ways to draw 2 spades from 13 spades; hence

$$p = \frac{\text{number of ways 2 spades can be drawn}}{\text{number of ways 2 cards can be drawn}} = \frac{78}{1326} = \frac{3}{51}$$

(b) There are 13 spades and 13 hearts, so there are $13 \cdot 13 = 169$ ways to draw a spade and a heart. Thus $p = \frac{169}{1326} = \frac{13}{102}$.

7.9. A box contains two white socks and two blue socks. Two socks are drawn at random. Find the probability p they are a match (same color).

There are $\binom{4}{2} = 6$ ways to draw two of the socks. Only two pairs will yield a match. Thus $p = \frac{2}{6} = \frac{1}{3}$.

7.10. Five horses are in a race. Audrey picks two of the horses at random, and bets on them. Find the probability p that Audrey picked the winner.

There are $\binom{5}{2} = 10$ ways to pick two of the horses. Four of the pairs will contain the winner. Thus $p = \frac{4}{10} = \frac{2}{5}$.

FINITE PROBABILITY SPACES

7.11. A sample space S consists of four elements; that is, $S = \{a_1, a_2, a_3, a_4\}$. Under which of the following functions does S become a probability space?

(a) $P(a_1) = \frac{1}{2}$ $\quad P(a_2) = \frac{1}{3}$ $\quad P(a_3) = \frac{1}{4}$ $\quad P(a_4) = \frac{1}{5}$

(b) $P(a_1) = \frac{1}{2}$ $\quad P(a_2) = \frac{1}{4}$ $\quad P(a_3) = -\frac{1}{4}$ $\quad P(a_4) = \frac{1}{2}$

(c) $P(a_1) = \frac{1}{2}$ $\quad P(a_2) = \frac{1}{4}$ $\quad P(a_3) = \frac{1}{8}$ $\quad P(a_4) = \frac{1}{8}$

(d) $P(a_1) = \frac{1}{2}$ $\quad P(a_2) = \frac{1}{4}$ $\quad P(a_3) = \frac{1}{4}$ $\quad P(a_4) = 0$

(a) Since the sum of the values on the sample points is greater than one, the function does not define S as a probability space.

(b) Since $P(a_3)$ is negative, the function does not define S as a probability space.

(c) Since each value is nonnegative and the sum of the values is one, the function does define S as a probability space.

(d) The values are nonnegative and add up to one; hence the function does define S as a probability space.

7.12. A coin is weighted so that heads is twice as likely to appear as tails. Find $P(T)$ and $P(H)$.

Let $P(T) = p$; then $P(H) = 2p$. Now set the sum of the probabilities equal to one, that is, set $p + 2p = 1$. Then $p = \frac{1}{3}$. Thus $P(H) = \frac{1}{3}$ and $P(T) = \frac{2}{3}$.

7.13. A die is weighted so that the outcomes produce the following probability distribution:

Outcome	1	2	3	4	5	6
Probability	0.1	0.3	0.2	0.1	0.1	0.2

Consider the events:

$$A = \{\text{even number}\}, \quad B = \{2, 3, 4, 5\}, \quad C = \{x: x < 3\}, \quad D = \{x: x > 7\}$$

Find the following probabilities:

(a) (i) $P(A)$, (ii) $P(B)$, (iii) $P(C)$, (iv) $P(D)$.

(b) $P(A^c)$, $P(B^c)$, $P(C^c)$, $P(D^c)$.

(c) (i) $P(A \cap B)$, (ii) $P(A \cup C)$, (iii) $P(B \cap C)$.

(a) For any event, find $P(E)$ by summing the probabilities of the elements in E. Thus:

(i) $A = \{2,4,6\}$, so $P(A) = 0.3 + 0.1 + 0.2 = 0.6$.

(ii) $P(B) = 0.3 + 0.2 + 0.1 + 0.1 = 0.7$.

(iii) $C = \{1,2\}$, so $P(C) = 0.1 + 0.3 = 0.4$.

(iv) $D = \varnothing$, the empty set. Hence $P(D) = 0$.

(b) Use $P(E^c) = 1 - P(E)$ to get:

$P(A^c) = 1 = 0.6 = 0.4$, $\quad P(C^c) = 1 - 0.4 = 0.6$

$P(B^c) = 1 - 0.7 = 0.3$, $\quad P(D^c) = 1 - 0 = 1$

(c) (i) $A \cap B = \{2,4\}$, so $P(A \cap B) = 0.3 + 0.1 = 0.4$.

(ii) $A \cup C = \{1,2,3,4,5\} = \{6\}^c$, so $P(A \cup C) = 1 - 0.2 = 0.8$.

(iii) $B \cap C = \{2\}$, so $P(B \cap C) = 0.3$.

7.14. Suppose A and B are events with $P(A) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$. Find the probability that:

(a) A does not occur. (c) A or B occurs.

(b) B does not occur. (d) Neither A nor B occurs.

(a) $P(\text{not } A) = P(A^c) = 1 - P(A) = 0.4$.

(b) $P(\text{not } B) = P(B^c) = 1 - P(B) = 0.7$.

(c) By the Addition Principle,

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.6 + 0.3 - 0.2 = 0.7$$

(d) Recall [Fig. 7-3(b)] that neither A nor B is the complement of $A \cup B$. Therefore,

$$P(\text{neither } A \text{ nor } B) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$$

7.15. Prove Theorem 7.2: $P(A^c) = 1 - P(A)$.

$S = A \cup A^c$ where A and A^c are disjoint. Thus

$$1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$$

from which our result follows.

7.16. Prove Theorem 7.3: (i) $P(\varnothing) = 0$, (ii) $P(A \setminus B) = P(A) - P(A \cap B)$, (iii) If $A \subseteq B$, then $P(A) \le P(B)$.

(i) $\varnothing = S^c$ and $P(S) = 1$. Thus $P(\varnothing) = 1 - 1 = 0$.

(ii) As indicated by Fig. 7-5(a), $A = (A \setminus B) \cup (A \cap B)$ where $A \setminus B$ and $A \cap B$ are disjoint. Hence

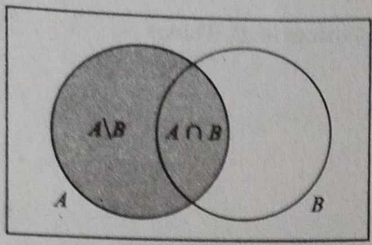
$$P(A) = P(A \setminus B) + P(A \cap B)$$

From which our result follows.

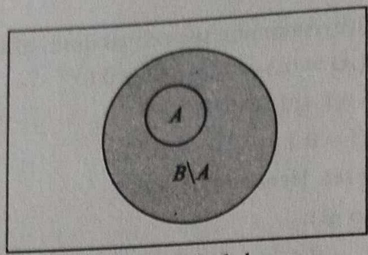
(iii) If $A \subseteq B$, then, as indicated by Fig. 7-5(b), $B = A \cup (B \setminus A)$ where A and $B \setminus A$ are disjoint. Hence

$$P(B) = P(A) + P(B \setminus A)$$

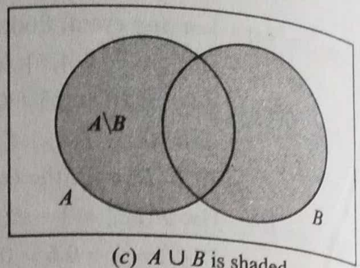
Since $P(B \setminus A) \ge 0$, we have $P(A) \le P(B)$.



(a) A is shaded



(b) B is shaded



(c) $A \cup B$ is shaded

Fig. 7-5

7.17. Prove Theorem (Addition Principle) 7.4: For any events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

As indicated by Fig. 7-5(c), $A \cup B = (A \setminus B) \cup B$ where $A \setminus B$ and B are disjoint sets. Thus, using Theorem 7.3(ii),

$$\begin{aligned} P(A \cup B) &= P(A \setminus B) + P(B) = P(A) - P(A \cap B) + P(B) \\ &= P(A) + P(B) - P(A \cap B) \end{aligned}$$

CONDITIONAL PROBABILITY

7.18. Three fair coins, a penny, a nickel, and a dime, are tossed. Find the probability p that they are all heads if: (a) the penny is heads; (b) at least one of the coins is heads.

The sample space has eight elements: $S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$.

(a) If the penny is heads, the reduced sample space is $A = \{\text{HHH, HHT, HTH, HTT}\}$. Since the coins are all heads in 1 of 4 cases, $p = \frac{1}{4}$.

(b) If one or more of the coins is heads, the reduced sample space is

$$B = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH}\}.$$

Since the coins are all heads in 1 of 7 cases, $p = \frac{1}{7}$.

7.19. A pair of fair dice is thrown. Find the probability p that the sum is 10 or greater if: (a) 5 appears on the first die; (b) 5 appears on at least one die.

(a) If a 5 appears on the first die, then the reduced sample space is

$$A = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)\}$$

The sum is 10 or greater on two of the six outcomes: $(5,5)$, $(5,6)$. Hence $p = \frac{2}{6} = \frac{1}{3}$.

(b) If a 5 appears on at least one of the dice, then the reduced sample space has eleven elements.

$$B = \{(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (1,5), (2,5), (3,5), (4,5), (6,5)\}$$

The sum is 10 or greater on three of the eleven outcomes: $(5,5)$, $(5,6)$, $(6,5)$. Hence $p = \frac{3}{11}$.

7.20. In a certain college town, 25% of the students failed mathematics, 15% failed chemistry, and 10% failed both mathematics and chemistry. A student is selected at random.

(a) If he failed chemistry, what is the probability that he failed mathematics?
(b) If he failed mathematics, what is the probability that he failed chemistry?
(c) What is the probability that he failed mathematics or chemistry?
(d) What is the probability that he failed neither mathematics nor chemistry?

(a) The probability that a student failed mathematics, given that he failed chemistry, is

$$P(M \mid C) = \frac{P(M \cap C)}{P(C)} = \frac{0.10}{0.15} = \frac{2}{3}$$

(b) The probability that a student failed chemistry, given that he failed mathematics is

$$P(C \mid M) = \frac{P(C \cap M)}{P(M)} = \frac{0.10}{0.25} = \frac{2}{5}$$

(c) By the Addition Principle (Theorem 7.4),

$$P(M \cup C) = P(M) + P(C) - P(M \cap C) = 0.25 + 0.15 - 0.10 = 0.30$$

(d) Students who failed neither mathematics nor chemistry form the complement of the set $M \cup C$, that is, they form the set $(M \cup C)^c$. Hence

$$P((M \cup C)^c) = 1 - P(M \cup C) = 1 - 0.30 = 0.70$$

7.21. A pair of fair dice is thrown. If the two numbers appearing are different, find the probability p that: (a) the sum is 6; (b) an ace appears; (c) the sum is 4 or less.

There are 36 ways the pair of dice can be thrown, and six of them, $(1, 1), (2, 2), \ldots, (6, 6)$, have the same numbers. Thus the reduced sample space will consist of $36 - 6 = 30$ elements.

(a) The sum 6 can appear in four ways: $(1,5), (2,4), (4,2), (5,1)$. (We cannot include $(3,3)$ since the numbers, are the same.) Hence $p = \frac{4}{30} = \frac{2}{15}$.
(b) An ace can appear in 10 ways: $(1,2), (1,3), \ldots, (1,6)$ and $(2,1), (3,1), \ldots, (6,1)$. Therefore $p = \frac{10}{30} = \frac{1}{3}$.
(c) The sum of 4 or less can occur in four ways: $(3,1), (1,3), (2,1), (1,2)$. Thus $p = \frac{4}{30} = \frac{2}{15}$.

7.22. A class has 12 boys and four girls. Suppose three students are selected at random from the class. Find the probability p that they are all boys.

The probability that the first student selected is a boy is 12/16 since there are 12 boys out of 16 students. If the first student is a boy, then the probability that the second is a boy is 11/15 since there are 11 boys left out of 15 students. Finally, if the first two students selected were boys, then the probability that the third student is a boy is 10/14 since there are 10 boys left out of 14 students. Thus, by the multiplication theorem, the probability that all three are boys is

$$p = \frac{12}{16} \cdot \frac{11}{15} \cdot \frac{10}{14} = \frac{11}{28}$$

Another Method

There are $C(16, 3) = 560$ ways to select three students out of the 16 students, and $C(12, 3) = 220$ ways to select three boys out of 12 boys; hence

$$p = \frac{220}{560} = \frac{11}{28}$$

Another Method

If the students are selected one after the other, then there are $16 \cdot 15 \cdot 14$ ways to select three students, and $12 \cdot 11 \cdot 10$ ways to select three boys; hence

$$p = \frac{12 \cdot 11 \cdot 10}{16 \cdot 15 \cdot 14} = \frac{11}{28}$$

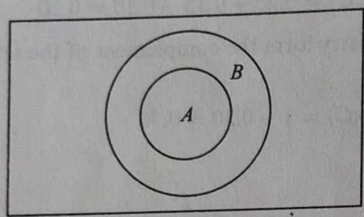
7.23. Find $P(B \mid A)$ if: (*a*) A is a subset of B; (*b*) A and B are mutually exclusive. (Assume $P(A) > 0$.)

(*a*) If A is a subset of B [as pictured in Fig. 7-6(*a*)], then whenever A occurs B must occur; hence $P(B \mid A) = 1$. Alternatively, if A is a subset of B, then $A \cap B = A$; hence

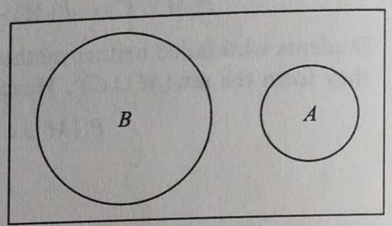
$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

(*b*) If A and B are mutually exclusive, i.e., disjoint [as pictured in Fig. 7-6(*b*)], then whenever A occurs B cannot occur; hence $P(B \mid A) = 0$. Alternatively, if A and B are disjoint, then $A \cap B = \varnothing$; hence

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{P(=\varnothing)}{P(A)} = \frac{0}{P(A)} = 0$$



(*a*) $A \subseteq B$



(*b*) $A \cap B = \varnothing$

Fig. 7-6

INDEPENDENCE

7.24. The probability that A hits a target is $\frac{1}{3}$ and the probability that B hits a target is $\frac{1}{5}$. They both fire at the target. Find the probability that: (*a*) A does not hit the target, (*b*) both hit the target; (*c*) one of them hits the target; (*d*) neither hits the target.

We are given $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{5}$ (and we assume the events are independent).

(*a*) $P(\text{not } A) = P(A^c) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$.

(*b*) Since the events are independent,

$$P(A \text{ and } B) = P(A \cap B) = P(A) \cdot P(B) = \tfrac{1}{3} \cdot \tfrac{1}{5} = \tfrac{1}{15}$$

(*c*) By the Addition Principle (Theorem 7.4),

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{15} = \tfrac{7}{15}$$

(*d*) We have

$$P(\text{neither } A \text{ nor } B) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - \tfrac{7}{15} = \tfrac{8}{15}$$

7.25. Consider the following events for a family with children:

$$A = \{\text{children of both sexes}\}, \qquad B = \{\text{at most one boy}\}$$

(*a*) Show that A and B are independent events if a family has three children.

(*b*) Show that A and B are dependent events if a family has only two children.

(a) We have the equiprobable space $S = \{bbb, bbg, bgb, bgg, gbb, gbg, ggb, ggg\}$. Here

$$A = \{bbg, bgb, bgg, gbb, gbg, ggb\} \quad \text{and so} \quad P(A) = \tfrac{6}{8} = \tfrac{3}{4}$$
$$B = \{bgg, gbg, ggb, ggg\} \quad \text{and so} \quad P(B) = \tfrac{4}{8} = \tfrac{1}{2}$$
$$A \cap B = \{bgg, gbg, ggb\} \quad \text{and so} \quad P(A \cap B) = \tfrac{3}{8}$$

Since $P(A)P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} = P(A \cap B)$, A and B are independent.

(b) We have the equiprobable space $S = \{bb, bg, gb, gg\}$. Here

$$A = \{bg, gb\} \quad \text{and so} \quad P(A) = \tfrac{1}{2}$$
$$B = \{bg, gb, gg\} \quad \text{and so} \quad P(B) = \tfrac{3}{4}$$
$$A \cap B = \{bg, gb\} \quad \text{and so} \quad P(A \cap B) = \tfrac{1}{2}$$

Since $P(A)P(B) \neq P(A \cap B)$, A and B are dependent.

7.26. Box A contains five red marbles and three blue marbles, and box B contains three red and two blue. A marble is drawn at random from each box.

(a) Find the probability p that both marbles are red.

(b) Find the probability p that one is red and one is blue.

(a) The probability of choosing a red marble from A is $\frac{5}{8}$ and from B is $\frac{3}{5}$. Since the events are independent, $p = \frac{5}{8} \cdot \frac{3}{5} = \frac{3}{8}$.

(b) The probability p_1 of choosing a red marble from A and a blue marble from B is $\frac{5}{8} \cdot \frac{2}{5} = \frac{1}{4}$. The probability p_2 of choosing a blue marble from A and a red marble from B is $\frac{3}{8} \cdot \frac{3}{5} = \frac{9}{40}$. Hence $p = p_1 + p_2 = \frac{1}{4} + \frac{9}{40} = \frac{19}{40}$.

7.27. Prove: If A and B are independent events, then A^c and B^c are independent events.

Let $P(A) = x$ and $P(B) = y$. Then $P(A^c) = 1 - x$ and $P(B^c) = 1 - y$. Since A and B are independent, $P(A \cap B) = P(A)P(B) = xy$. Furthermore,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = x + y - xy$$

By DeMorgan's law, $(A \cup B)^c = A^c \cap B^c$; hence

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - x - y + xy$$

On the other hand,

$$P(A^c)P(B^c) = (1 - x)(1 - y) = 1 - x - y + xy$$

Thus $P(A^c \cap B^c) = P(A^c)P(B^c)$, and so A^c and B^c are independent.

In similar fashion, we can show that A and B^c, as well as A^c and B, are independent.

REPEATED TRIALS, BINOMIAL DISTRIBUTION

7.28. Whenever horses a, b, c, d race together, their respective probabilities of winning are 0.2, 0.5, 0.1, 0.2. That is, $S = \{a, b, c, d\}$ where $P(a) = 0.2$, $P(b) = 0.5$, $P(c) = 0.1$, and $P(d) = 0.2$. They race three times.

(a) Describe and find the number of elements in the product probability space S_3.

(b) Find the probability that the same horse wins all three races.

(c) Find the probability that a, b, c each win one race.

(a) By definition, $S_3 = S \times S \times S = \{(x, y, z) : x, y, z \in S\}$ and

$$P((x, y, z)) = P(x)P(y)P(z)$$

Thus, in particular, S_3 contains $4^3 = 64$ elements.

(*b*) Writing xyz for (x, y, z), we seek the probability of the event

$$A = \{aaa, bbb, ccc, ddd\}$$

By definition

$$P(aaa) = (0.2)^3 = 0.008, \qquad P(ccc) = (0.1)^3 = 0.001$$
$$P(bbb) = (0.5)^3 = 0.125, \qquad P(ddd) = (0.2)^3 = 0.008$$

Thus $P(A) = 0.0008 + 0.125 + 0.001 + 0.008 = 0.142$.

(*c*) We seek the probability of the event

$$B = \{abc, acb, bac, bca, cab, cba\}$$

Every element in B has the same probability

$$(0.2)(0.5)(0.1) = 0.01. \text{ Hence } P(B) = 6(0.01) = 0.06.$$

7.29. A fair coin is tossed three times. Find the probability that there will appear: (*a*) three heads; (*b*) exactly two heads; (*c*) exactly one head; (*d*) no heads.

Let H denote a head and T a tail on any toss. The three tosses can be modeled as an equiprobable space in which there are eight possible outcomes:

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

However, since the result on any one toss does not depend on the result of any other toss, the three tosses may be modeled as three independent trials in which $P(H) = \frac{1}{2}$ and $P(T) = \frac{1}{2}$ on any one trial. Then:

(*a*) $P(\text{three heads}) = P(\text{HHH}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$.

(*b*) $P(\text{exactly two heads}) = P(\text{HHT or HTH or THH})$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{8}$$

(*c*) As in (*b*), $P(\text{exactly one head}) = P(\text{exactly two tails}) = \frac{3}{8}$.

(*d*) As in (*a*), $P(\text{no heads}) = P(\text{three tails}) = \frac{1}{8}$.

7.30. The probability that John hits a target is $p = \frac{1}{4}$. He fires $n = 6$ times. Find the probability that he hits the target: (*a*) exactly two times; (*b*) more than four times; (*c*) at least once.

This is a binomial experiment with $n = 6$, $p = \frac{1}{4}$, and $q = 1 - p = \frac{3}{4}$; that is, $B(6, \frac{1}{4})$. Accordingly, we use Theorem 7.7.

(*a*) $P(2) = \binom{6}{2} (\frac{1}{4})^2 (\frac{3}{4})^4 = 15(3^4)/(4^6) = \frac{1215}{4096} \approx 0.297$.

(*b*) $P(5) + P(6) = \binom{6}{5} (\frac{1}{4})^5 (\frac{3}{4})^1 + (\frac{1}{4})^6 = \frac{18^6}{4} + \frac{1^6}{4} = \frac{19^6}{4} = \frac{19}{4096} \approx 0.0046$.

(*c*) $P(0) = (\frac{3}{4})^6 = \frac{729}{4096}$, so $P(X > 0) = 1 - \frac{729}{4096} = \frac{3367}{4096} \approx 0.82$.

7.31. Suppose 20% of the items produced by a factory are defective. Suppose four items are chosen at random. Find the probability that: (*a*) two are defective; (*b*) three are defective; (*c*) none is defective.

This is a binomial experiment with $n = 4$, $p = 0.2$ and $q = 1 - p = 0.8$; that is, $B(4, 0.2)$. Hence use Theorem 7.7.

(*a*) $P(2) = \binom{4}{2} (0.2)^2 (0.8)^2 = 0.1536$.

(b) $P(3) = \binom{4}{3}(0.2)^3(0.8)^1 = 0.0256.$

(c) $P(0) = (0.8)^4 = 0.4095.$

7.32. Team A has probability $\frac{2}{3}$ of winning whenever it plays. Suppose A plays four games. Find the probability p that A wins more than half of its games.

Here $n = 4$, $p = \frac{2}{3}$ and $q = 1 - p = \frac{1}{3}$. A wins more than half its games if it wins three or four games. Hence

$$p = P(3) + P(4) = \binom{4}{3}\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^1 + \binom{4}{4}\left(\frac{2}{3}\right)^4 = \frac{32}{81} + \frac{16}{81} = \frac{16}{27} = 0.59$$

7.33. A family has six children. Find the probability p that there are: (a) three boys and three girls; (b) fewer boys than girls. Assume that the probability of any particular child being a boy is $\frac{1}{2}$.

Here $n = 6$ and $p = q = \frac{1}{2}$.

(a) $p = P(3 \text{ boys}) = \binom{6}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{20}{64} = \frac{5}{16}.$

(b) There are fewer boys than girls if there are zero, one, or two boys. Hence

$$p = P(0 \text{ boys}) + P(1 \text{ boy}) + P(2 \text{ boys}) = \left(\frac{1}{2}\right)^6 + \binom{6}{1}\left(\frac{1}{2}\right)^5 + \binom{6}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^4 = \frac{11}{32} = 0.34$$

7.34. A certain type of missile hits its target with probability $p = 0.3$. Find the number of missiles that should be fired so that there is at least an 80% probability of hitting the target.

The probability of missing the target is $q = 1 - p = 0.7$. Hence the probability that n missiles miss the target is $(0.7)^n$. Thus we seek the smallest n for which

$$1 - (0.7)^n > 0.8 \quad \text{or equivalently} \quad (0.7)^n < 0.2$$

Compute:

$$(0.7)^1 = 0.7, \quad (0.7)^2 = 0.49, \quad (0.7)^3 = 0.343, \quad (0.7)^4 = 0.2401, \quad (0.7)^5 = 0.16807$$

Thus at least five missiles should be fired.

7.35. How many dice should be thrown so that there is a better than an even chance of obtaining a six?

The probability of not obtaining a six on n dice is $(\frac{5}{6})^n$. Hence we seek the smallest n for which $(\frac{5}{6})^n$ is less than $\frac{1}{2}$. Compute as follows:

$$\left(\frac{5}{6}\right)^1 = \frac{5}{6}, \quad \left(\frac{5}{6}\right)^2 = \frac{25}{36}, \quad \left(\frac{5}{6}\right)^3 = \frac{125}{216}, \quad \text{but} \quad \left(\frac{5}{6}\right)^4 = \frac{625}{1296} < \frac{1}{2}$$

Thus four dice must be thrown.

7.36. A certain soccer team wins (W) with probability 0.6, losses (L) with probability 0.3, and ties (T) with probability 0.1. The team plays three games over the weekend. (a) Determine the elements of the event A that the team wins at least twice and does not lose; and find $P(A)$. (b) Determine the elements of the event B that the team wins, loses, and ties in some order; and find $P(B)$.

(a) A consists of all ordered triples with at least two Ws and no Ls. Thus

$$A = \{\text{WWW, WWT, WTW, TWW}\}$$

Furthermore,

$$\begin{aligned}P(A) &= P(\text{WWW}) + P(\text{WWT}) + P(\text{WTW}) + P(\text{TWW})\\ &= (0.6)(0.6)(0.6) + (0.6)(0.6)(0.1) + (0.6)(0.1)(0.6) + (0.1)(0.6)(0.6)\\ &= 0.216 + 0.036 + 0.036 + 0.036 = 0.324\end{aligned}$$

(*b*) Here $B = \{\text{WLT, WTL, LWT, LTW, TWL, TLW}\}$. Every element in B has probability $(0.6)(0.3)(0.1) = 0.018$; hence $P(B) = 6(0.018) = 0.108$.

7.37. A man fires at a target $n = 6$ times and hits it $k = 2$ times. (*a*) List the different ways that this can happen. (*b*) How many ways are there?

(*a*) List all sequences with two Ss (successes) and four Fs (failures):

SSFFFF, SFSFFF, SFFSFF, SFFFSF, SFFFFS, FSSFFF, FSFSFF, FSFFSF, FSFFFS, FFSSFF, FFSFSF, FFSFFS, FFFSSF, FFFSFS, FFFFSS

(*b*) There are 15 different ways as indicated by the list. Observe that this is equal to $\binom{6}{2}$ since we are distributing $k = 2$ letters S among the $n = 6$ positions in the sequence.

7.38. Prove Theorem 7.7: The probability of exactly k successes in a binomial experiment $B(n, p)$ is given by

$$P(k) = P(k \text{ successes}) = \binom{n}{k} p^k q^{n-k}$$

The probability of one or more successes is $1 - q^n$.

The sample space of the n repeated trials consists of all n-tuples (i.e., n-element sequences) whose components are either S (success) or F (failure). Let A be the event of exactly k successes. Then A consists of all n-tuples of which k components are S and $n - k$ components are F. The number of such n-tuples in the event A is equal to the number of ways that k letters S can be distributed among the n components of an n-tuple; hence A consists of $C(n, k) = \binom{n}{k}$ sample points. The probability of each point in A is $p^k q^{n-k}$; hence

$$P(A) = \binom{n}{k} p^k q^{n-k}$$

In particular, the probability of no successes is

$$P(0) = \binom{n}{0} p^0 q^n = q^n$$

Thus the probability of one or more successes is $1 - q^n$.

RANDOM VARIABLES, EXPECTATION

7.39. A player tosses two fair coins. He wins \$2 if two heads occur, and \$1 if one head occurs. On the other hand, he loses \$3 if no heads occur. Find the expected value E of the game. Is the game fair? (The game is fair, favorable, or unfavorable to the player according as $E = 0$, $E > 0$ or $E < 0$.)

The sample space is $S = \{\text{HH, HT, TH, TT}\}$ and each sample point has probability $\frac{1}{4}$. For the player's gain, we have

$$X(\text{HH}) = \$2, \qquad X(\text{HT}) = X(\text{TH}) = \$1, \qquad X(\text{TT}) = -\$3$$

and so the distribution of X is

x_i	2	1	-3
p_i	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

and
$$E = E(X) = 2(\tfrac{1}{4}) + 1(\tfrac{2}{4}) - 3(\tfrac{1}{4}) = \$0.25$$

Since $E(X) > 0$, the game is favourable to the player.

7.40. Two numbers from 1 to 3 are chosen at random with repetitions allowed. Let X denote the sum of the numbers. (*a*) Find the distribution of X. (*b*) Find the expectation $E(X)$.

(*a*) There are nine equiprobable pairs making up the sample space S. X assumes the values 2, 3, 4, 5, 6 with the following probabilities:

$$P(2) = P(1,1) = \tfrac{1}{9}, \qquad P(3) = P(\{(1,2),(2,1)\}) = \tfrac{2}{9}$$
$$P(4) = P(\{(1,3),(2,2),(3,1)\}) = \tfrac{3}{9}$$
$$P(5) = P(\{(2,3),(3,2)\}) = \tfrac{2}{9}, \qquad P(6) = P(3,3) = \tfrac{1}{9}$$

Hence, the distribution is

x_i	2	3	4	5	6
$P(x_i)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

(*b*) The expected value $E(X)$ is obtained by multiplying each value of x by its probability and taking the sum. Hence

$$E(X) = 2(\tfrac{1}{9}) + 3(\tfrac{2}{9}) + 4(\tfrac{3}{9}) + 5(\tfrac{2}{9}) + 6(\tfrac{1}{9}) = \tfrac{36}{9} = 4$$

7.41. A coin is weighted so that $P(H) = \frac{3}{4}$ and $P(T) = \frac{1}{4}$. The coin is tossed three times. Let X denote the number of heads that appear. (*a*) Find the distribution of X. (*b*) Find the expectation $E(X)$.

(*a*) The sample space is

$$S = \{\text{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}\}$$

X assumes the values 0, 1, 2, 3 with the following probabilities:

$$P(0) = P(\text{TTT}) = \tfrac{1}{4}\cdot\tfrac{1}{4}\cdot\tfrac{1}{4} = \tfrac{1}{64}$$
$$P(1) = P(\text{HTT, THT, TTH}) = \tfrac{3}{4}\cdot\tfrac{1}{4}\cdot\tfrac{1}{4} + \tfrac{1}{4}\cdot\tfrac{3}{4}\cdot\tfrac{1}{4} + \tfrac{1}{4}\cdot\tfrac{1}{4}\cdot\tfrac{3}{4} = \tfrac{9}{64}$$
$$P(2) = P(\text{HHT, HTH, THH}) = \tfrac{3}{4}\cdot\tfrac{3}{4}\cdot\tfrac{1}{4} + \tfrac{3}{4}\cdot\tfrac{1}{4}\cdot\tfrac{3}{4} + \tfrac{1}{4}\cdot\tfrac{3}{4}\cdot\tfrac{3}{4} = \tfrac{27}{64}$$
$$P(3) = P(\text{HHH}) = \tfrac{3}{4}\cdot\tfrac{3}{4}\cdot\tfrac{3}{4} = \tfrac{27}{64}$$

Hence, the distribution is as follows:

x_i	0	1	2	3
$P(x_i)$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{27}{64}$	$\frac{27}{64}$

(*b*) The expected value $E(X)$ is obtained by multiplying each value of x by its probability and taking the sum. Hence

$$E(X) = 0(\tfrac{1}{64}) + 1(\tfrac{9}{64}) + 2(\tfrac{27}{64}) + 3(\tfrac{27}{64}) = \tfrac{144}{64} = 2.25$$

7.42. You have won a contest. Your prize is to select one of three envelopes and keep what is in it. Each of two of the envelopes contains a check for \$30, but the third envelope contains a check for \$3000. What is the expectation E of your winnings (as a probability distribution)?

Let X denote your winnings. Then $X = 30$ or 3000, and $P(30) = \frac{2}{3}$ and $P(3000) = \frac{1}{3}$. Hence

$$E = E(X) = 30\cdot\tfrac{2}{3} + 3000\cdot\tfrac{1}{3} = 20 + 1000 = 1020$$

7.43. A fair coin is tossed until a head or five tails occurs. Find the expected number E of tosses of the coin.

The possible outcomes are

$$\text{H}, \quad \text{TH}, \quad \text{TTH}, \quad \text{TTTH}, \quad \text{TTTTH}, \quad \text{TTTTT}$$

with respective probabilities (independent trials)

$$\tfrac{1}{2}, \quad (\tfrac{1}{2})^2 = \tfrac{1}{4}, \quad (\tfrac{1}{2})^3 = \tfrac{1}{8}, \quad (\tfrac{1}{2})^4 = \tfrac{1}{16}, \quad (\tfrac{1}{2})^5 = \tfrac{1}{32}, \quad (\tfrac{1}{2})^5 = \tfrac{1}{32}$$

The random variable X of interest is the number of tosses in each outcome. Thus

$$X(\text{H}) = 1, \quad X(\text{TTH}) = 3, \quad X(\text{TTTTH}) = 5$$
$$X(\text{TH}) = 2, \quad X(\text{TTTH}) = 4, \quad X(\text{TTTTT}) = 5$$

and these X values are assigned the probabilities

$$P(1) = P(\text{H}) = \tfrac{1}{2}, \quad P(3) = P(\text{TTH}) = \tfrac{1}{8}, \quad P(5) = P(\text{TTTTH}) + P(\text{TTTTT})$$
$$P(2) = P(\text{TH}) = \tfrac{1}{4}, \quad P(4) = P(\text{TTTH}) = \tfrac{1}{16}, \quad = \tfrac{1}{32} + \tfrac{1}{32} = \tfrac{1}{16}$$

Accordingly,

$$E = E(X) = 1 \cdot \tfrac{1}{2} + 2 \cdot \tfrac{1}{4} + 3 \cdot \tfrac{1}{8} + 4 \cdot \tfrac{1}{16} + 5 \cdot \tfrac{1}{16} \approx 1.9$$

7.44. A linear array EMPLOYEE has n elements. Suppose NAME appears randomly in the array, and there is a linear search to find the location K of NAME, that is, to find K such that EMPLOYEE[K] = NAME. Let $f(n)$ denote the number of comparisons in the linear search.

(*a*) Find the expected value of $f(n)$.
(*b*) Find the maximum value (worst case) of $f(n)$.

(*a*) Let X denote the number of comparisons. Since NAME can appear in any position in the array with the same probability of $1/n$, we have $X = 1, 2, 3, \ldots, n$, each with probability $1/n$. Hence

$$f(n) = E(X) = 1 \cdot \frac{1}{n} + 2 \cdot \frac{1}{n} + 3 \cdot \frac{1}{n} + \cdots + n \cdot \frac{1}{n}$$
$$= (1 + 2 + \cdots + n) \cdot \frac{1}{n} = \frac{n(n+1)}{2} \cdot \frac{1}{n} = \frac{n+1}{2}$$

(*b*) If NAME appears at the end of the array, then $f(n) = n$.

MEAN, VARIANCE, AND STANDARD DEVIATION

7.45. Find the mean $\mu = E(X)$, variance $\sigma^2 = \text{Var}(X)$, and standard deviation $\sigma = \sigma_X$ of each distribution:

(*a*)

x_i	2	3	11
p_i	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

(*b*)

x_i	1	3	4	5
p_i	0.4	0.1	0.2	0.3

Use the formulas,

$$\mu = E(X) = x_1p_1 + x_2p_2 + \cdots + x_mp_m = \Sigma x_ip_i, \qquad \sigma^2 = \text{Var}(X) = E(X^2) - \mu^2$$
$$E(X^2) = x_1^2p_1 + x_2^2p_2 + \cdots + x_m^2p_m = \Sigma x_i^2p_i, \qquad \sigma = \sigma_X = \sqrt{\text{Var}(X)}$$
$$\sigma = \sigma_X = \sqrt{\text{Var}(X)}$$

(*a*) $\mu = \Sigma x_ip_i = 2(\tfrac{1}{3}) + 3(\tfrac{1}{2}) + 11(\tfrac{1}{6}) = 4.$
$E(X^2) = \Sigma x_i^2p_i = 2^2(\tfrac{1}{3}) + 3^2(\tfrac{1}{2}) + 11^2(\tfrac{1}{6}) = 26.$
$\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2 = 26 - 4^2 = 10.$
$\sigma = \sqrt{\text{Var}(X)} = \sqrt{10} = 3.2.$

(b) $\mu = \Sigma x_i p_i = 1(0.4) + 3(0.1) + 4(0.2) + 5(0.3) = 3.$
$E(X^2) = \Sigma x_i^2 p_i = 1(0.4) + 9(0.1) + 16(0.2) + 25(0.3) = 12.$
$\sigma^2 = \mathrm{Var}(X) = E(X^2) - \mu^2 = 12 - 9 = 3.$
$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{3} = 1.7.$

7.46. Five cards are numbered 1 to 5. Two cards are drawn at random. Let X denote the sum of the numbers drawn.

(a) Find the distribution of X.
(b) Find the mean μ, variance $\sigma^2 = \mathrm{Var}(X)$, and standard deviation $\sigma = \sigma_X$ of X.

(a) There are $C(5,2) = 10$ ways of drawing two cards at random. The 10 equiprobable sample points, with their corresponding X values, are shown below:

$$\{1,2\} \to 3 \quad \{1,3\} \to 4 \quad \{1,4\} \to 5 \quad \{1,5\} \to 6 \quad \{2,3\} \to 5$$
$$\{2,4\} \to 6 \quad \{2,5\} \to 7 \quad \{3,4\} \to 7 \quad \{3,5\} \to 8 \quad \{4,5\} \to 9$$

Observe that the values of X are the seven numbers 3, 4, 5, 6, 7, 8, and 9; of these 3, 4, 8, and 9 are each assumed at one sample point, while 5, 6, and 7 are each assumed at two sample points. Hence the distribution of X is

x_i	3	4	5	6	7	8	9
p_i	0.1	0.1	0.2	0.2	0.2	0.1	0.1

(b) $\mu = E(X) = \Sigma x_i p_i = 3(0.1) + 4(0.1) + 5(0.2) + 6(0.2) + 7(0.2) + 8(0.1) + 9(0.1) = 6.$
$E(X^2) - \Sigma x_i^2 p_i = 9(0.1) + 16(0.1) + 25(0.2) + 36(0.2) + 49(0.2) + 64(0.1) + 81(0.1) = 39.$
$\mathrm{Var}(X) = E(X^2) - \mu^2 = 39 - 6^2 = 3.$
$\sigma = \sqrt{\mathrm{Var}(X)} = \sqrt{3} \approx 1.7.$

7.47. A pair of fair dice is thrown. Let X denote the maximum of the two numbers which appear.

(a) Find the distribution of X.
(b) Find the mean μ, variance $\sigma^2 = \mathrm{Var}(X)$, and standard deviation $\sigma = \sigma_X$ of X.

(a) The sample space S is the equiprobable space consisting of the 36 pairs of integers (a, b) where a and b range from 1 to 6; that is,

$$S = \{(1,1), (1,2), \ldots, (6,6)\}$$

(See Problem 7.3.) Since X assigns to each pair in S the larger of the two integers, the values of X are the integers from 1 to 6. Observe:

(i) Only one pair $(1,1)$ give a maximum of 1; hence $P(1) = \frac{1}{36}$.
(ii) Three pairs, $(1,2), (2,2), (2,1)$, give a maximum of 2; hence $P(2) = \frac{3}{36}$.
(iii) Five pairs, $(1,3), (2,3), (3,3), (3,2), (3,1)$, give a maximum of 3; hence $P(3) = \frac{5}{36}$.

Similarly, $P(4) = \frac{7}{36}$, $P(5) = \frac{9}{36}$, $P(6) = \frac{11}{36}$.
Thus the distribution of X is as follows:

x_i	1	2	3	4	5	6
p_i	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

(b) We find the expectation (mean) of X by multiplying each x_i by its probability p_i and then summing:

$$\mu = E(X) = 1 \cdot \tfrac{1}{36} + 2 \cdot \tfrac{3}{36} + 3 \cdot \tfrac{5}{36} + 4 \cdot \tfrac{7}{36} + 5 \cdot \tfrac{9}{36} + 6 \cdot \tfrac{11}{36} = \tfrac{161}{36} \approx 4.5$$

We find $E(X^2)$ by multiplying x_i^2 by p_i and taking the sum:

$$E(X^2) = 1 \cdot \tfrac{1}{36} + 4 \cdot \tfrac{3}{36} + 9 \cdot \tfrac{5}{36} + 16 \cdot \tfrac{7}{36} + 25 \cdot \tfrac{9}{36} + 36 \cdot \tfrac{11}{36} = \tfrac{791}{36} \approx 22.0$$

Then

$$\operatorname{Var}(X) = E(X^2) - \mu^2 = 22.0 - (4.5)^2 = 1.75 \qquad \text{and} \qquad \sigma_X = \sqrt{1.75} \approx 1.3$$

7.48. A fair die is tossed. Let X denote twice the number appearing, and let Y be 1 or 3 according as an odd or even number appears. Find the distribution and expectation: (a) of X; (b) of Y.

The sample space is $S = \{1, 2, 3, 4, 5, 6\}$, with each sample point having probability $\frac{1}{6}$.

(a) The images of the sample points are:

$$X(1) = 2, \quad X(2) = 4, \quad X(3) = 6, \quad X(4) = 8, \quad X(5) = 10, \quad X(6) = 12$$

As these are distinct, the distribution of X is

x_i	2	4	6	8	10	12
$P(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Thus

$$E(X) = \sum x_i P(x_i) = \tfrac{2}{6} + \tfrac{4}{6} + \tfrac{6}{6} + \tfrac{8}{6} + \tfrac{10}{6} + \tfrac{12}{6} = 7$$

(b) The images of the sample points are:

$$Y(1) = 1, \quad Y(2) = 3, \quad Y(3) = 1, \quad Y(4) = 3, \quad Y(5) = 1, \quad Y(6) = 3$$

The two Y-values, 1 and 3, are each assumed at three sample points. Hence we have the distribution

y_i	1	3
$P(y_i)$	$\frac{3}{6}$	$\frac{3}{6}$

Thus

$$E(Y) = \sum y_i P(y_i) = \tfrac{3}{6} + \tfrac{9}{6} = 2$$

7.49. Let X and Y be random variables defined on the same sample space S. Then $Z = X + Y$ and $W = XY$ are also random variables on S defined by

$$Z(s) = (X + Y)(s) = X(s) + Y(s) \qquad \text{and} \qquad W(s) = (XY)(s) = X(s)Y(s)$$

Let X and Y be the random variable in Problem 7.48.

(a) Find the distribution and expectation of $Z = X + Y$. Verify that $E(X + Y) = E(X) + E(Y)$.

(b) Find the distribution and expectation of $W = XY$.

The sample space is still $S = \{1, 2, 3, 4, 5, 6\}$, and each sample point still has probability $\frac{1}{6}$.

(a) Using $(X + Y)(s) = X(s) + Y(s)$ and the values of X and Y from Problems 7.48, we obtain:

$$(X+Y)(1) = 2 + 1 = 3, \quad (X+Y)(3) = 6 + 1 = 7, \quad (X+Y)(5) = 10 + 1 = 11$$
$$(X+Y)(2) = 4 + 3 = 7, \quad (X+Y)(4) = 8 + 3 = 11, \quad (X+Y)(6) = 12 + 3 = 15$$

The image set is $\{3, 7, 11, 5\}$. The values 3 and 15 are each assumed at only one sample point and hence have probability $\frac{1}{6}$; the values 7 and 11 are each assumed at two sample points and hence have probability $\frac{2}{6}$. Thus the distribution of $Z = X + Y$ is:

z_i	3	7	11	15
$P(z_i)$	1/6	2/6	2/6	1/6

Thus

$$E(X+Y) = E(Z) = \sum z_i P(z_i) = \tfrac{3}{6} + \tfrac{14}{6} + \tfrac{22}{6} + \tfrac{15}{6} = 9$$

Moreover,

$$E(X+Y) = 9 = 7 + 2 = E(X) + E(Y)$$

(b) Using $(XY)(s) = X(s)Y(s)$, we obtain:

$$(XY)(1) = 2(1) = 2, \quad (XY)(3) = 6(1) = 6, \quad (XY)(5) = 10(1) = 10$$
$$(XY)(2) = 4(3) = 12, \quad (XY)(4) = 8(3) = 24, \quad (XY)(6) = 12(3) = 36$$

Each value of XY is assumed at just one sample point; hence the distribution of $W = XY$ is:

w_i	2	6	10	12	24	36
$P(w_i)$	1/6	1/6	1/6	1/6	1/6	1/6

Thus

$$E(XY) = E(W) = \sum w_i P(w_i) = \tfrac{2}{6} + \tfrac{6}{6} + \tfrac{10}{6} + \tfrac{12}{6} + \tfrac{24}{6} + \tfrac{36}{6} = 15$$

[Note $E(XY) = 15 \neq (7)(2) = E(X)E(Y)$.]

7.50. The probability that a man hits a target is $p = 0.1$. He fires $n = 100$ times. Find the expected number μ of times he will hit the target, and the standard deviation σ.

This is a binomial experiment $B(n, p)$ where $n = 100$, $p = 0.1$, and $q = 1 - p = 0.9$. Accordingly, we apply Theorem 7.9 to obtain

$$\mu = np = 100(0.1) = 10 \quad \text{and} \quad \sigma = \sqrt{npq} = \sqrt{100(0.1)(0.9)} = 3$$

7.51. A student takes an 18-question multiple-choice exam, with four choices per question. Suppose one of the choices is obviously incorrect, and the student makes an "educated" guess of the remaining choices. Find the expected number $E(X)$ of correct answers, and the standard deviation σ.

This is a binomial experiment $B(n, p)$ where $n = 18$, $p = \frac{1}{3}$, and $q = 1 - p = \frac{2}{3}$. Hence

$$E(X) = np = 18 \cdot \frac{1}{3} = 6 \quad \text{and} \quad \sigma = \sqrt{npq} = \sqrt{18 \cdot \frac{1}{3} \cdot \frac{2}{3}} = 2$$

7.52. The expectation function $E(X)$ on the space of random variables on a sample space S can be proved to be *linear*, that is,

$$E(X_1 + X_2 + \cdots + x_n) = E(X_1) + E(X_2) + \cdots + E(X_n)$$

Use this property to prove $\mu = np$ for a binomial experiment $B(n,p)$.

On the sample space of n Bernoulli trials, let X_i (for $i = 1, 2, \ldots, n$) be the random variable which has the value 1 or 0 according as the ith trial is a success or a failure. Then each X_i has the distribution

x	0	1
$P(x)$	q	p

Thus $E(X_i) = 0(q) + 1(p) = p$. The total number of successes in n trials is

$$X = X_1 + X_2 + \cdots + X_n$$

Using the linearity property of E, we have

$$\begin{aligned} E(X) &= E(X_1 + X_2 + \cdots + X_n) \\ &= E(X_1) + E(X_2) + \cdots + E(X_n) \\ &= p + p + \cdots + p = np \end{aligned}$$

Supplementary Problems

SAMPLE SPACES AND EVENTS

7.53. Let A and B be events. Rewrite each of the following events using set notation: (*a*) A or not B occurs; (*b*) only A occurs.

7.54. Let A, B, and C be events. Rewrite each of the following events using set notation: (*a*) A and B but not C occurs; (*b*) A or C, but not B occurs; (*c*) none of the events occurs; (*d*) at least two of the events occur.

7.55. A penny, a dime, and a die are tossed.

(*a*) Describe a suitable sample space S, and find $n(S)$.

(*b*) Express explicitly the following events:

$$A = \{\text{two heads and an even number}\}, \qquad B = \{2 \text{ appears}\}$$
$$C = \{\text{exactly one heads and an odd number}\}$$

(*c*) Express explicitly the events: (i) A and B; (ii) only B; (iii) B and C.

FINITE EQUIPROBABLE SPACES

7.56. Determine the probability of each event:

(*a*) An odd number appears in the toss of a fair die.

(*b*) One or more heads appear in the toss of four fair coins.

(*c*) One or both numbers exceed 4 in the toss of two fair dice.

7.57. A student is chosen at random to represent a class with five freshman, eight sophomores, three juniors, and two seniors. Find the probability that the student is: (*a*) a sophomore; (*b*) a junior; (*c*) a junior or a senior.