FAMILIES OF POLAR CURVES

A Deep dive into the beautiful intricacies of polar curves

SP Jain School of Global Management
Advanced Calculus

- Pradeep Sandilya
- Tushar Sonthalia
- Vedant Kabra

Table of Contents

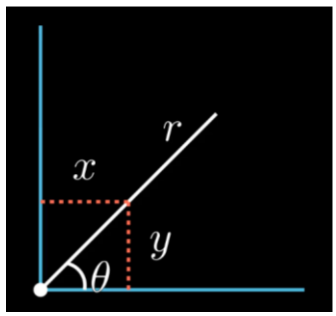
Introduction 2
Question 1 3
  Solution part (a) 3
  Solution part (b) 4
Question 2 5
  Solution 5
Question 3 8
  Solution 8
Question 4 10
  Solution 10
Conclusion 12
References 13

Introduction

The Cartesian coordinate system provides a means of mapping points to ordered pairs and ordered pairs to points. This is called a one-to-one mapping from points in the plane to ordered pairs. The polar coordinate system provides an alternative method of mapping points to ordered pairs.

Unlike Cartesian coordinate system, which uses an ordered pair of (x, y) as its coordinate basis, the Polar coordinate system used an ordered pair of (r, θ). r denotes the distance from a reference point (typically origin) and θ denotes the angle from a reference direction.

For a point located at (x, y) in the cartesian coordinate system, the length of the line joining the point and the origin is the r and the angle that line makes with the $x - axis$ is θ. The following graph lays out the difference between the two coordinate systems.



Using right-triangle trigonometry, the following equations are true for the point P:

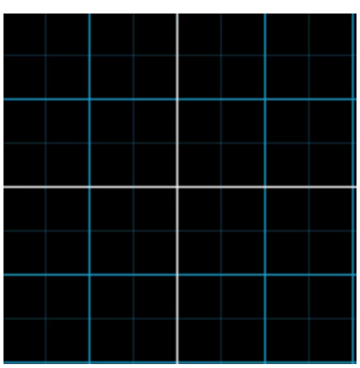
$$cos\,\theta = \frac{x}{r},\ \ so\ x = r\,cos\,\theta$$

$$sin\,\theta = \frac{y}{x},\ \ so\ x = r\,sin\,\theta$$

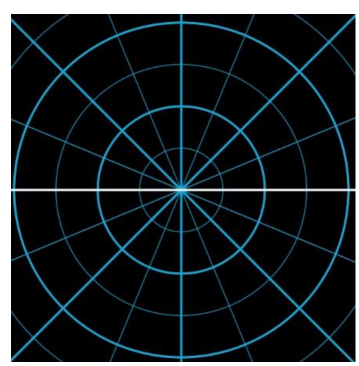
Furthermore,

$$r^2 = x^2 + y^2\ and\ \ tan\,\theta = \frac{y}{x}$$

The following two images illustrate the difference between the two coordinate systems.



Cartesian Coordinate System



Polar Coordinate System

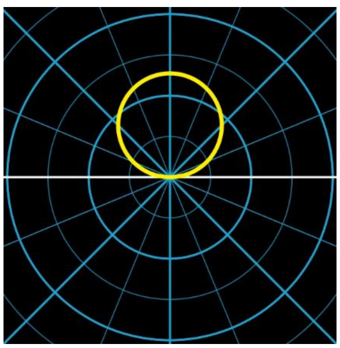
Question 1

(a) Investigate the family of curves defined by the polar equations $r = sin\, n\theta$ where n is a positive integer. How is the number of loops related to n?

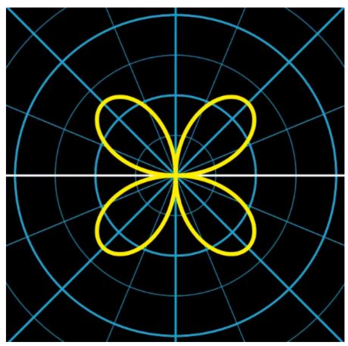
(b) What happens if the equation in part (a) is replace by $r = |sin(n\theta)|$?

Solution part (a)

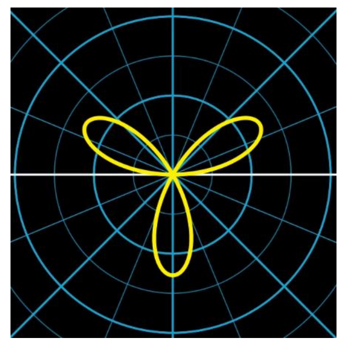
Plotting the curves for $r = sin\, n\theta$ on polar coordinate system:



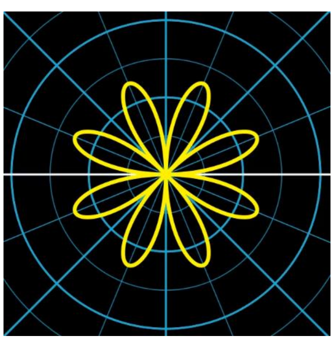
$\boldsymbol{r = sin\,\theta}$



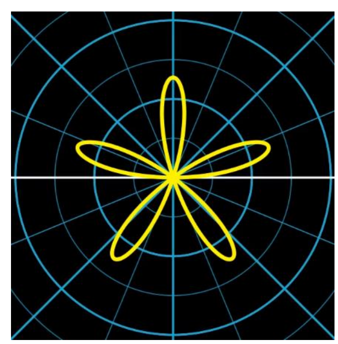
$\boldsymbol{r = sin\,2\theta}$



$\boldsymbol{r = sin\,3\theta}$



$\boldsymbol{r = sin\,4\theta}$



$\boldsymbol{r = sin\,5\theta}$

It is clearly observed that for odd values of n, the number of loops is n, whereas it is 2n for even values of n. The reason for this phenomenon is that every point on the graph is traversed just once for even values of n and twice for odd values of n. This happens due to the reason that:

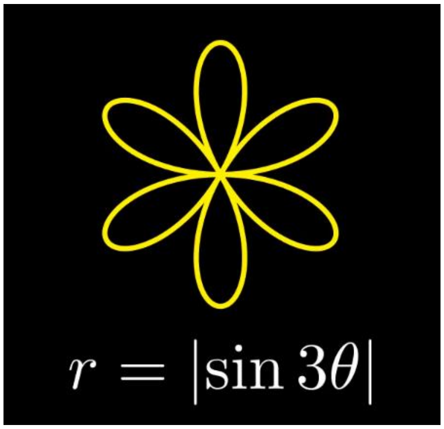
$$r(\theta + \pi) = sin[n(\theta + \pi)]$$

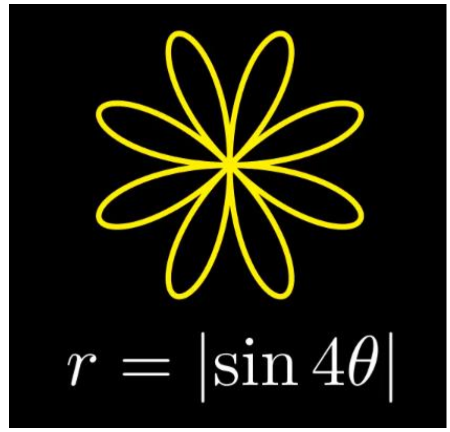
$$= sin\, n\theta\, cos\, n\pi + \cos n\theta \sin n\pi$$

$$= \begin{cases} \sin(n\theta), & if\ n\ is\ even \\ -\sin(n\theta), & if\ n\ is\ odd \end{cases}$$

Solution part (b)

Graphing $r = |sin(n\theta)|$





From the above graphs, it is visible that irrespective of whether n is even or odd, the number of loops in the curve is always 2n. This happens due to the fact that:

$$r(\theta + \pi) = r(\theta)$$

Question 2

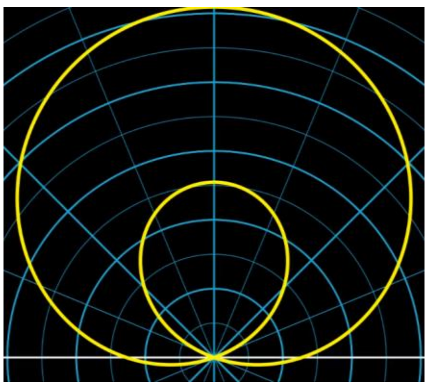
A family of curves is given by the equation:

$$r = 1 + c\sin(n\theta)$$

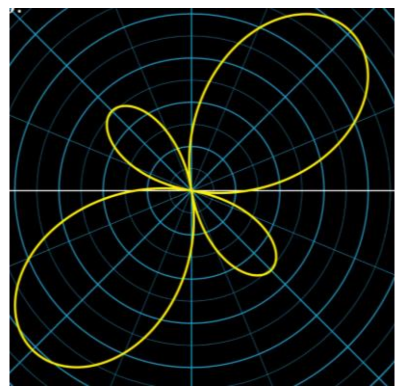
where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

Solution

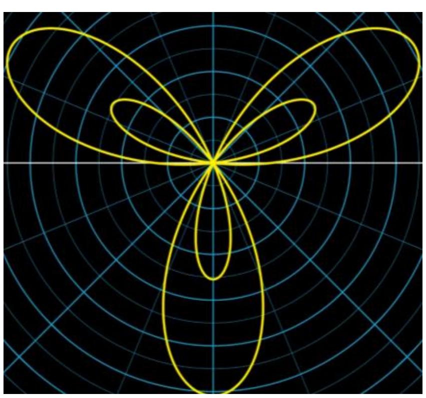
Firstly graphing $r = 1 + c\sin(n\theta)$ with c fixed as 3 with varying n:



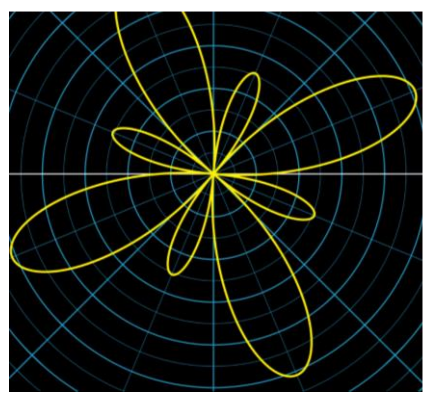
$\boldsymbol{(n, c) = (1, 3)}$



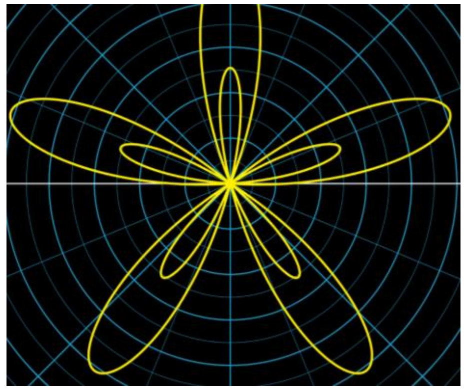
$\boldsymbol{(n, c) = (2, 3)}$



$\boldsymbol{(n, c) = (4, 3)}$



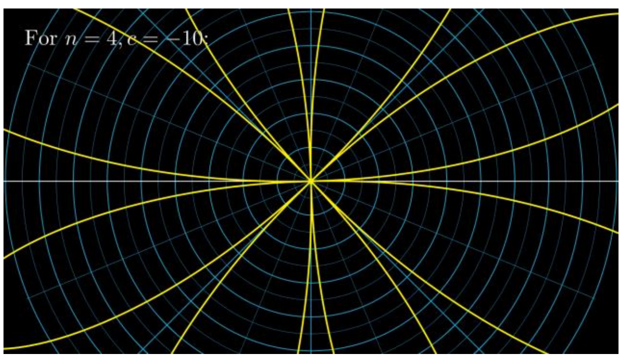
$\boldsymbol{(n, c) = (4, 3)}$



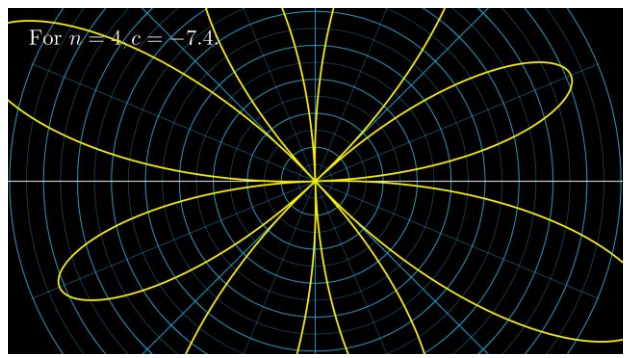
$(\boldsymbol{n}, \boldsymbol{c}) = (\mathbf{5}, \mathbf{3})$

From the plotted graphs, it is observed that, as the value of n increases, the number of loops of the curve also increases. More importantly, when n is even the smaller loops are outside the bigger loops, while they are inside the bigger loops when the value of n is odd.

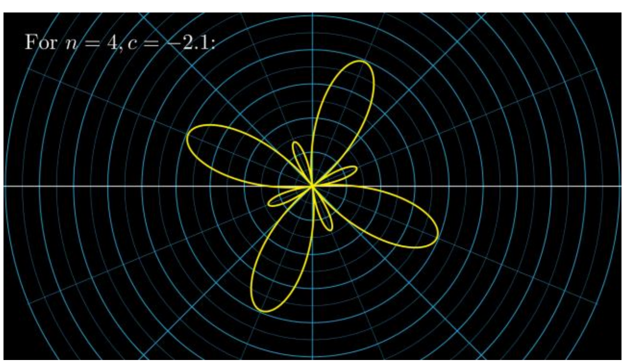
Now, graphing $r = 1 + c\sin(n\theta)$ with n fixed and varying c:



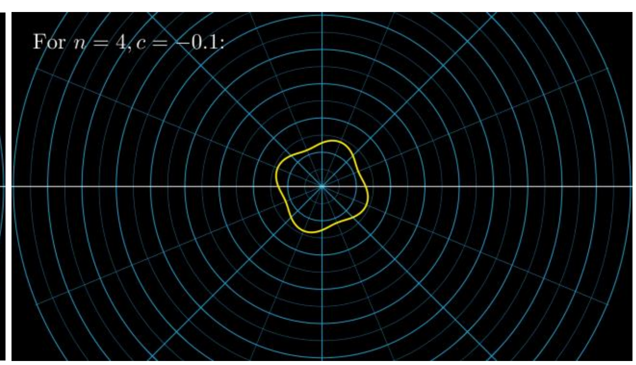
$(\boldsymbol{n}, \boldsymbol{c}) = (\mathbf{4}, \mathbf{-10})$



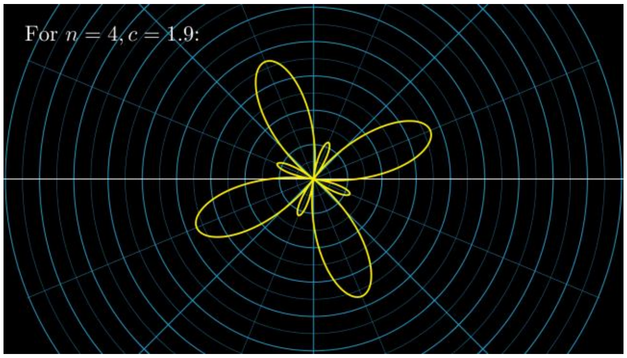
$(\boldsymbol{n}, \boldsymbol{c}) = (\mathbf{4}, \mathbf{-10})$



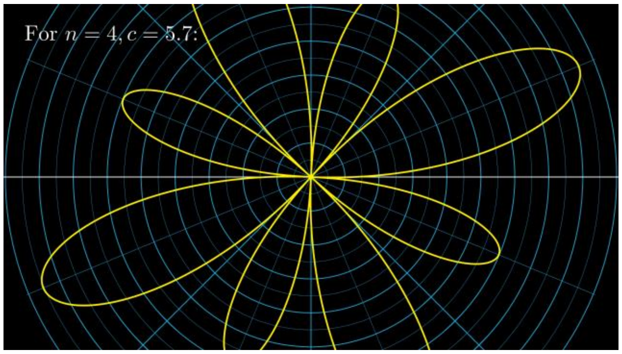
$(\boldsymbol{n}, \boldsymbol{c}) = (\mathbf{4}, \mathbf{-2.1})$



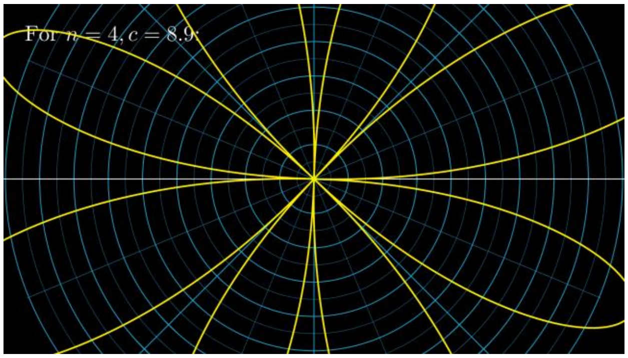
$(\boldsymbol{n}, \boldsymbol{c}) = (\mathbf{4}, \mathbf{-0.1})$



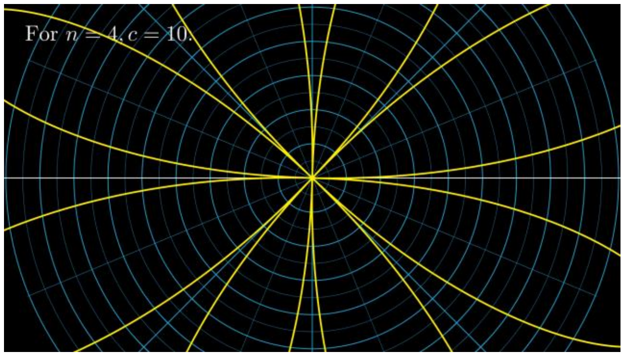
$(\boldsymbol{n}, \boldsymbol{c}) = (\mathbf{4}, \mathbf{1.9})$



$(\boldsymbol{n}, \boldsymbol{c}) = (\mathbf{4}, \mathbf{5.7})$



$(\boldsymbol{n}, \boldsymbol{c}) = (\mathbf{4}, \mathbf{8.9})$



$(\boldsymbol{n}, \boldsymbol{c}) = (\mathbf{4}, \mathbf{10})$

In this second set of graphs, it is observed that as c increased from -10, the curve seemed to get smaller, and as it reached close to 0, the inner loops completely disappeared. Furthermore, as the value of c increased to 10, the curve started growing and the inner loops start to get bigger.

Question 3

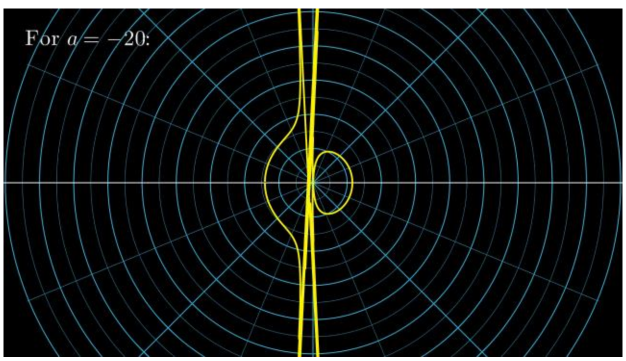
A family of curves has polar equations:

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

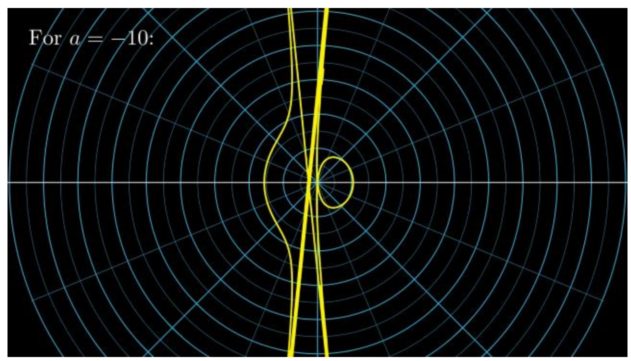
Investigate how the graph changes as the number a changes. In particular, you should identity the transitional values of a for which the basic shape of the curve changes.

Solution

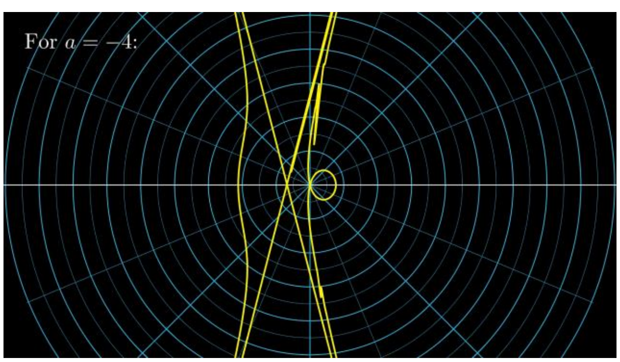
Plotting $r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$ for multiple values of a:



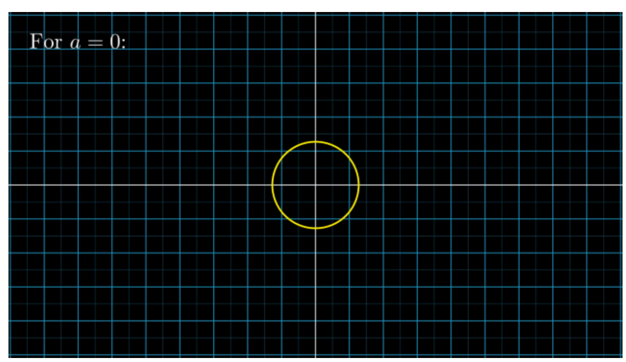
$\boldsymbol{a = -20}$



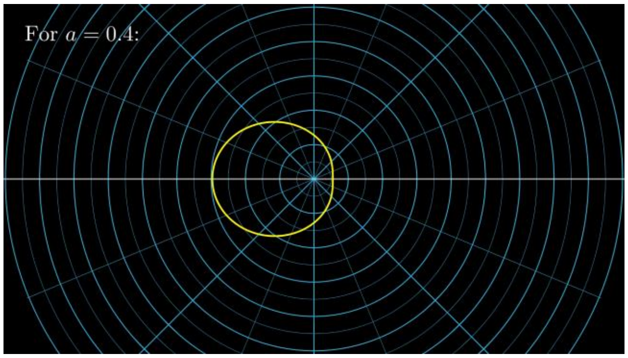
$\boldsymbol{a = 10}$



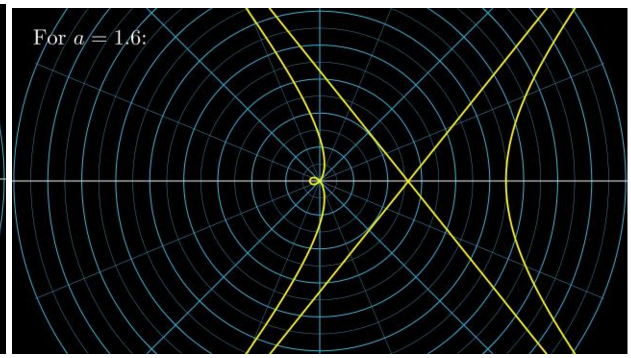
$\boldsymbol{a = -4}$



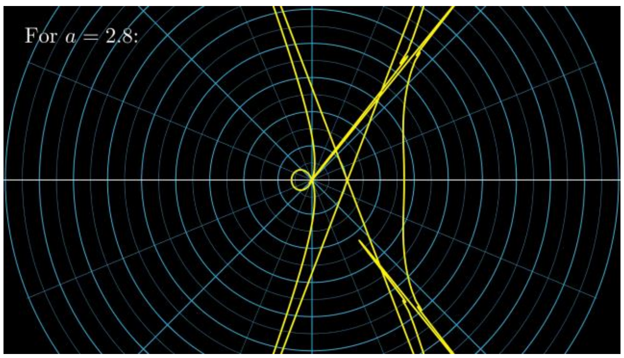
$\boldsymbol{a = 0}$



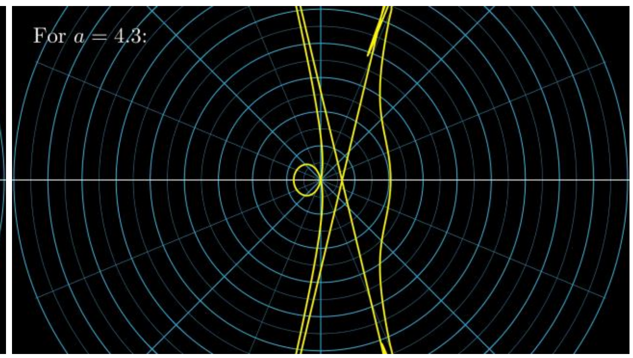
$\boldsymbol{a = 0.4}$



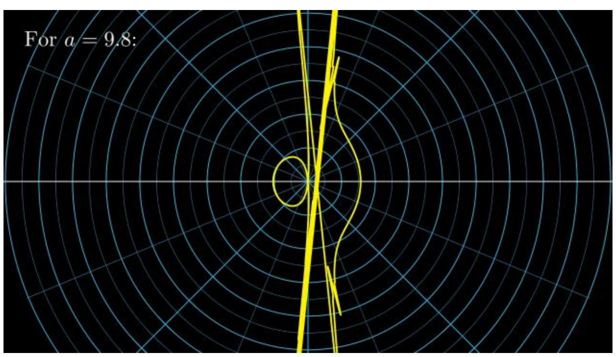
$\boldsymbol{a = 1.6}$



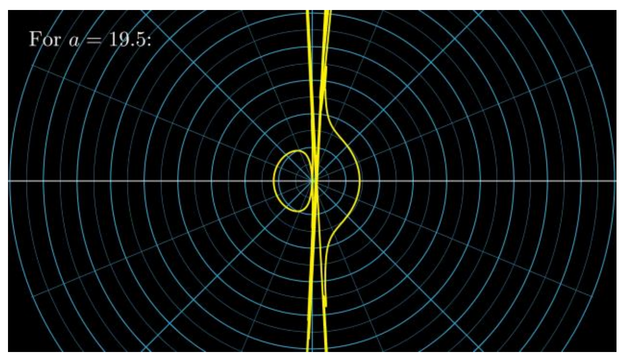
$\boldsymbol{a = 2.8}$



$\boldsymbol{a = 4.3}$



$\boldsymbol{a = 9.8}$



$\boldsymbol{a = 19.5}$

With the increase in the value of a, the graph moves to the left and its right side flattens. As a reaches 0.4, the curves seems to grow a dimple which gets more and more pronounced as a continues to increase. The curves begins to stretch out horizontally until a reaches 1, where the graph disappears at $\theta = \pi$. Further than that, the curves splits into two parts. The left part of the loop, which grows larger as a increases, and the right part grows broader vertically

Question 4

The astronomer Giovanni Cassini (1625-1712) studied the family of curves with polar equations:

$$r^4 - 2c^2r^2\cos\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the ovals of Cassini even though they are oval shaped only for certain values of a and c. Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into parts?

Solution

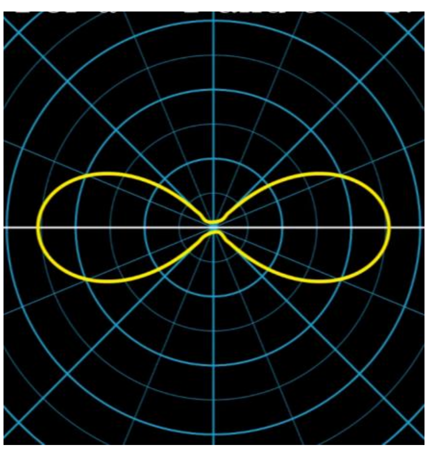
In order to plot the equation, the equation must be solved first:

$$r^2 = \frac{2\,c^2\cos(2\theta) \pm \sqrt{4c^4\cos^2(2\theta) - 4(c^4 - a\text{^}4)}}{2}$$

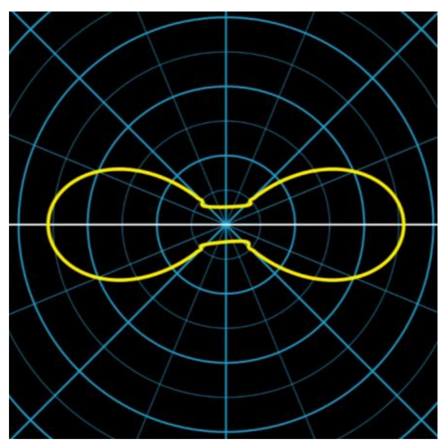
$$r^2 = c^2\cos(2\theta) \pm \sqrt{a^4 - c^4\sin^2(2\theta)}$$

$$r = \pm\sqrt{c^2\cos(2\theta) \pm \sqrt{a^4 - c^4\sin^2(2\theta)}}$$

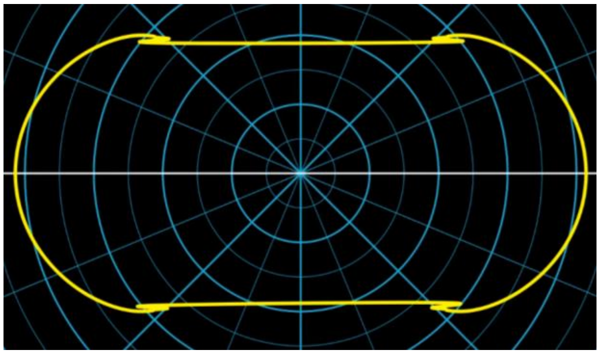
Plotting certain curves for $r = \pm\sqrt{c^2\cos(2\theta) \pm \sqrt{a^4 - c^4\sin^2(2\theta)}}$:



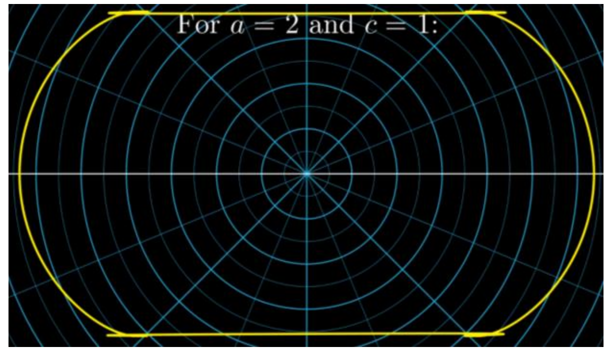
$(\boldsymbol{a}, \boldsymbol{c}) = (\mathbf{1}, \mathbf{1})$



$(\boldsymbol{a}, \boldsymbol{c}) = (\mathbf{1.1}, \mathbf{1})$



$(\boldsymbol{a}, \boldsymbol{c}) = (\mathbf{1.5}, \mathbf{1})$



$(\boldsymbol{a}, \boldsymbol{c}) = (\mathbf{2}, \mathbf{1})$

We start with the case a = c = 1, and the resulting curve resembles the symbol for infinity. If we let a decrease, the curve splits into two symmetric parts, and as a decreases further, the parts become smaller, further apart, and rounder. If instead we let a increase from 1, the two lobes of the curve join together, and as a increases further they continue to merge, until at a ≈ 1.4, the graph no longer has dimples, and has an oval shape. As a → ∞, the oval becomes larger and rounder, since the c^2 and c^4 terms lose their significance.

Conclusion

In this project we have explored the various breathtakingly beautiful polar curves and monitored how they change with respect to tuning certain arbitrary variables in the equations.

Polar coordinates are used often in navigation as the destination or direction of travel can be given as an angle and distance from the object being considered. For instance, aircraft use a slightly modified version of the polar coordinates for navigation.

Systems displaying radial symmetry provide natural settings for the polar coordinate system, with the central point acting as the pole. A prime example of this usage is the groundwater flow equation when applied to radially symmetric wells. Systems with a radial force are also good candidates for the use of the polar coordinate system. These systems include gravitational fields, which obey the inverse-square law, as well as systems with point sources, such as radio antennas.

References

The animations and graphs used in this project have been generated with python.

All the resources and the codes used in this project are available at: https://github.com/tusharsonthalia/Calculus-Project.