1 Question 1

Derive a generalized version of Bayes' Rule:
$P(X|Y,Z) = \frac{P(Y|X,Z)P(X|Z)}{P(Y|Z)}$
Solution:
$P(X|Y,Z) = \frac{P(X,Y,Z)}{P(Y,Z)} = \frac{P(Y|X,Z)P(X,Z)}{P(Y,Z)} = \frac{P(Y|X,Z)P(X|Z)P(Z)}{P(Y|Z)P(Z)} = \frac{P(Y|X,Z)P(X|Z)}{P(Y|Z)}$

2 Question 2

Prove the Symmetry and Decomposition properties, defined in equations (2.7) and (2.8):
Solution:

1. Symmetry:
   $(\mathbf{X} \perp \mathbf{Y}|\mathbf{Z}) \implies (\mathbf{Y} \perp \mathbf{X}|\mathbf{Z})$
   Since $\mathbf{X}$ and $\mathbf{Y}$ are independent given $\mathbf{Z}$, we have that: $P(\mathbf{X},\mathbf{Y}|\mathbf{Z}) = P(\mathbf{X}|\mathbf{Z})P(\mathbf{Y}|\mathbf{Z}) = P(\mathbf{Y}|\mathbf{Z})P(\mathbf{X}|\mathbf{Z}) = P(\mathbf{Y},\mathbf{X}|\mathbf{Z})$ by multiplicative symmetry and symmetry in probability. This directly implies $(\mathbf{Y} \perp \mathbf{X}|\mathbf{Z})$, which finishes the proof.

2. Decomposition:
   $(\mathbf{X} \perp \mathbf{Y},\mathbf{W}|\mathbf{Z}) \implies (\mathbf{X} \perp \mathbf{Y}|\mathbf{Z})$
   $(\mathbf{X} \perp \mathbf{Y},\mathbf{W}|\mathbf{Z})$ means that $P(\mathbf{X},\mathbf{Y},\mathbf{W}|\mathbf{Z}) = P(\mathbf{X}|\mathbf{Z})P(\mathbf{Y},\mathbf{W}|\mathbf{Z})$. Marginalizing over $\mathbf{W}$ gives us:
   $P(\mathbf{X},\mathbf{Y}|\mathbf{Z}) = \Sigma_{\mathbf{w}} P(\mathbf{X},\mathbf{Y},\mathbf{w}|\mathbf{Z}) = \Sigma_{\mathbf{w}} P(\mathbf{X}|\mathbf{Z})P(\mathbf{Y},\mathbf{w}|\mathbf{Z}) =$
   $P(\mathbf{X}|\mathbf{Z})\Sigma_{\mathbf{w}} P(\mathbf{Y},\mathbf{w}|\mathbf{Z}) = P(\mathbf{X}|\mathbf{Z})P(\mathbf{Y}|\mathbf{Z})$, which means $(\mathbf{X} \perp \mathbf{Y}|\mathbf{Z})$.

3 Question 3

Show that the variance of a random variable can be written as:
$Var[X] = E[X^2] - (E[X])^2$
Solution:
$Var[X] = E[(X - E[X])^2] = E[X^2 - 2XE[X] + (E[X])^2] =$
By linearity of expectation, and since the expectation of a constant is simply that constant, we have the following:
$E[X^2] - E[2XE[X]] + E[(E[X^2])] =$
$E[X^2] - 2E[X]E[X] + E[(E[X^2])] =$
$E[X^2] - 2E[X]^2 + E[X]^2 = E[X^2] - (E[X])^2$

4 Question 4

Let $X \sim \mathcal{N}(\mu; \sigma^2)$ and define a new variable $Y = aX + b$. Show that $Y \sim \mathcal{N}(a\mu + b; a^2\sigma^2)$

Solution:

$p_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is the pdf of X

We proceed by cases.

Case $a = 0$ is degenerate, as $Y = b$. We assume $a > 0$ or $a < 0$

Case $a > 0$:

$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(X \leq \frac{(y-b)}{a}) = \int_{-\infty}^{\frac{y-b}{a}} p_X(x)dx$

$p_Y(y)$ is easily calculated as the first derivative of $F_Y(y)$ with respect to y.

$p_Y(y) = F_Y'(y) = p_X(\frac{y-b}{a})\frac{1}{a}$, by chain rule of differentiation.

$p_Y(y) = \frac{1}{a} p_X(\frac{y-b}{a}) = \frac{1}{\sqrt{2\pi}\sigma a} e^{-\frac{(\frac{y-b}{a}-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma a} e^{-\frac{(\frac{y-b-a\mu}{a})^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi}\sigma a} e^{-\frac{(y-b-a\mu)^2}{2\sigma^2 a^2}} = \frac{1}{\sqrt{2\pi}\sigma a} e^{-\frac{(y-(a\mu+b))^2}{2\sigma^2 a^2}}$

Therefore Y is a Gaussian random variable with mean $a\mu + b$ and variance $a^2\sigma^2$. In other words, $Y \sim \mathcal{N}(a\mu + b; \sigma^2 a^2)$.

Case $a < 0$:

$F_Y(y) = P(Y \leq y) = P(aX + b \leq y) = P(X \geq \frac{(y-b)}{a}) = \int_{\frac{y-b}{a}}^{\infty} p_X(x)dx = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}(-\sigma)} e^{-\frac{(t-(a\mu+b))^2}{2\sigma^2 a^2}} dt$

Again, $p_Y(y)$ is first derivative of $F_Y(y)$ with respect to y.

$p_Y(y) = F_Y'(y) = p_X(\frac{y-b}{a})\frac{1}{a}$, by chain rule of differentiation.

Since $(-\sigma a) > 0$ ($a < 0$ and $\sigma > 0$), $p_Y(y)$ is once again the pdf of a Gaussian that satisfies $Y \sim \mathcal{N}(a\mu + b; \sigma^2 a^2)$. Notice that, in this case, the std. deviation is $(-a\sigma)$, rather than $(a\sigma)$, but both lead to the same distribution.