Lecture 19: April 1, 2015
cs 573: Probabilistic Reasoning
Professor Nevatia
Spring 2015

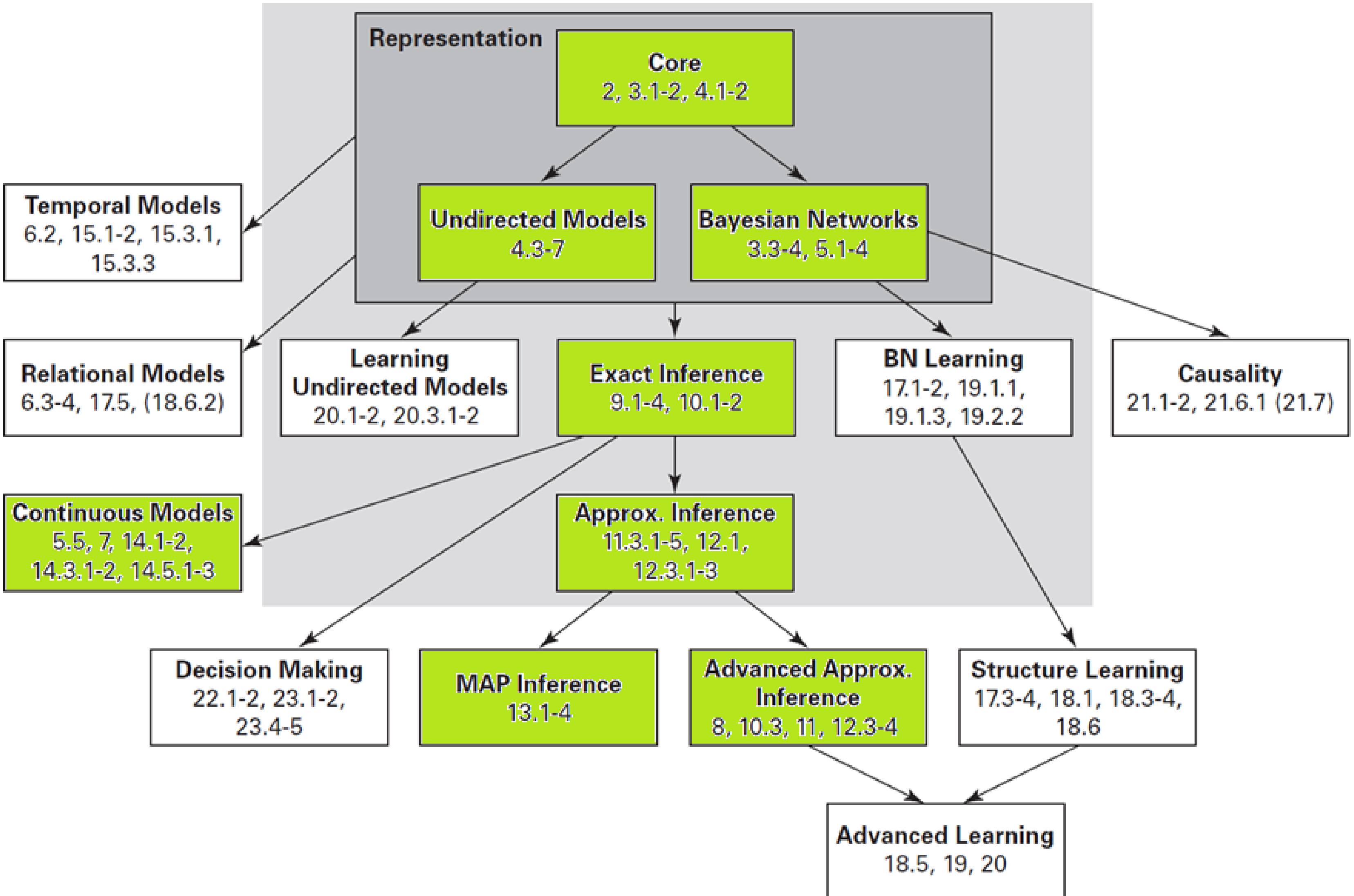
Review

- HW #6A posted, due 4/8/15
- Previous Lecture
  - MCMC
  - Metropolis-Hastings Algorithm
  - Mixing time
- Today's objective
  - Intro to temporal models

MCMC: Mixing

- How can we detect if a chain has “mixed”?
  - Distributions should be “stationary” but some variations can be expected from the distribution itself for a small number of samples
  - Distributions can also be stationary because the chain is in some high probability part of the space
  - We can initiate multiple chains, with different initialization, and test if they converge to the same distribution.
- Parallel MCMC
  - Inherently, MCMC is a sequential process.
  - We can parallelize, to some extent, by using multiple chains
  - Some parallel MCMC algorithms exist; out of scope of our course.

Book Plan



Representation
Core
2, 3.1-2, 4.1-2
Undirected Models
4.3-7
Bayesian Networks
3.3-4, 5.1-4
Temporal Models
6.2, 15.1-2, 15.3.1, 15.3.3
Relational Models
6.3-4, 17.5, (18.6.2)
Learning Undirected Models
20.1-2, 20.3.1-2
Exact Inference
9.1-4, 10.1-2
BN Learning
17.1-2, 19.1.1, 19.1.3, 19.2.2
Causality
21.1-2, 21.6.1 (21.7)
Continuous Models
5.5, 7, 14.1-2, 14.3.1-2, 14.5.1-3
Approx. Inference
11.3.1-5, 12.1, 12.3.1-3
Decision Making
22.1-2, 23.1-2, 23.4-5
MAP Inference
13.1-4
Advanced Approx. Inference
8, 10.3, 11, 12.3-4
Structure Learning
17.3-4, 18.1, 18.3-4, 18.6
Advanced Learning
18.5, 19, 20

Temporal Reasoning

- Temporal models are natural in a variety of domains
  - Estimate robot's position over time (based on some observations)
  - Monitor a patient in ICU (series of measurements such as blood pressure and heart rate)
  - Infer disease from temporal data (EKG)
  - Infer activity patterns in a video stream
  - Infer words from a speech (audio) stream
  - Predict weather, earth quakes, stock market…
- Number of variables can grow very large in temporal reasoning
  - In general, any variable at one time instance may have a direct influence on any other variable at any other time, and the influence itself could be a function of time.
  - Useful to have compact representations where possible. Also, we , make some simplifying assumptions for tractability.

Temporal Models (Chapter 6)

- System state at time *t* is an assignment to a the system variables (hidden or observed) at time *t*
- Notation: $X_i^{(t)}$ represents instantiation of X_i at time t
  - X_i is *not* a random variable that takes values, rather it is a *template variable*
  - Template is instantiated at different times; $X_i^{(t)}$ takes values in val (X_i)
- For a set of variables, **X**, we use notation $\mathbf{X}^{(t1:t2)}$ ($t_1 < t_2$) to denote the set of variables $\{\mathbf{X}^{(t)} : t \, \varepsilon \, [(t_1 , t_2)]\}$
  - $\boldsymbol{x}^{(t1:t2)}$ represents the assignment to this set of variables
- Trajectory: assignment to each variable $X_i^{(t)}$ for each relevant time
  - Space of trajectories is huge, need simplifying assumptions

Example

- Vehicle state: location, velocity, weather, failure (of the sensor), obs (observation of the sensor)
- One such set at each time t
- Joint probability distribution over all the sets (for some time interval) defines a probability distribution over a system trajectory (not in the same sense as the trajectory of car's position)
- Given a sequence of observations, we can answer questions about the distributions of other variables

Basic Assumptions

- Time is discretized (time slices)
- $P(\mathbf{X}^{(0:T)})$ short hand for $P(X^{(0)}, X^{(1)}, \ldots. X^{(T)})$
- Using chain rule:

  $P(\mathbf{X}^{(0:T)}) = (\prod_{t=0, T-1} P(\mathbf{X}^{(t+1)} | \mathbf{X}^{(o:t)})) P(\mathbf{X}^{(0)})$
- Markov assumption if $(\{\mathbf{X}^{(t+1)} \perp \mathbf{X}^{(o:(t-1))}\} | \mathbf{X}^{(t)})$
  - Next state is conditionally independent of previous states given the current state
- Distribution given Markov assumption

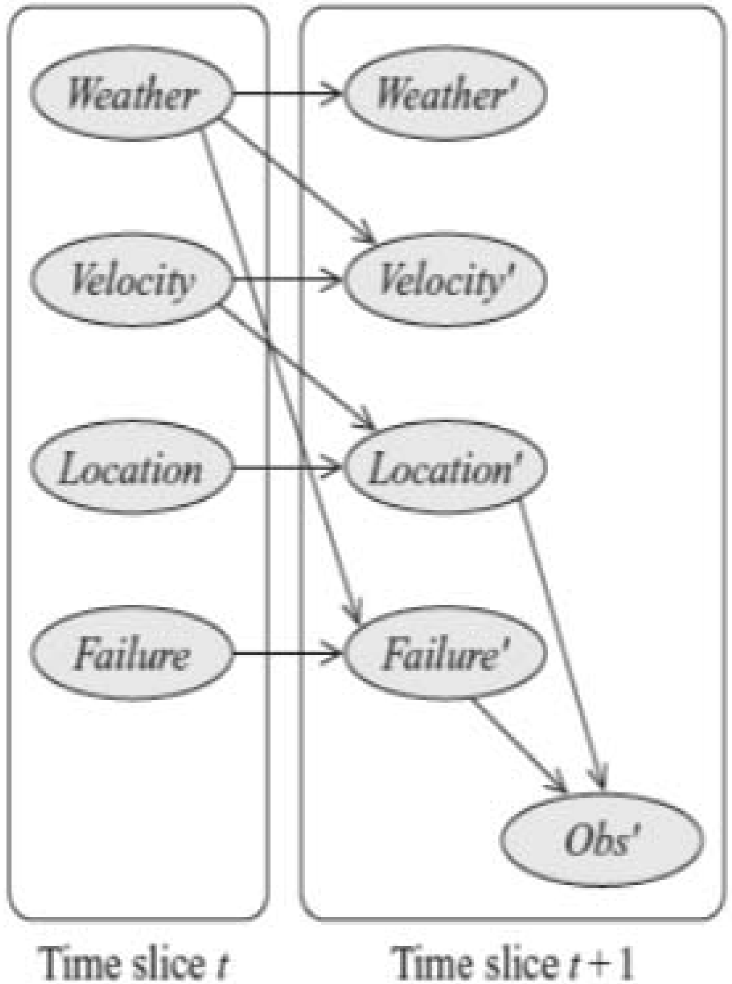
  $P(\mathbf{X}^{(0:T)}) = (\prod_{t=0, T-1} P(\mathbf{X}^{(t+1)} | \mathbf{X}^{(t)})) P(\mathbf{X}^{(0)})$ (eq 6.1)

Basic Assumptions

- Vehicle example
  - If variables are just the location l and observation of location o_l, Markov property is not accurate as it does not include info about speed
  - Change in speed may depend on weather, so also include weather
  - Adding more variables in state creates better approximation to Markovian properties
- *Stationary* dynamical system: $\mathbf{P} \; \mathbf{(} \; \boldsymbol{X}^{(t+1)} \mid \boldsymbol{X}^{(t)} \;)$ same for all t
- Transition model $P(\boldsymbol{X'} \mid \boldsymbol{X})$

$$P(\mathcal{X}^{(t+1)} = \xi' \mid \mathcal{X}^{(t)} = \xi) = P(\mathcal{X}' = \xi' \mid \mathcal{X} = \xi).$$

2-TBN Example



(a) $\mathcal{B}_{\rightarrow}$

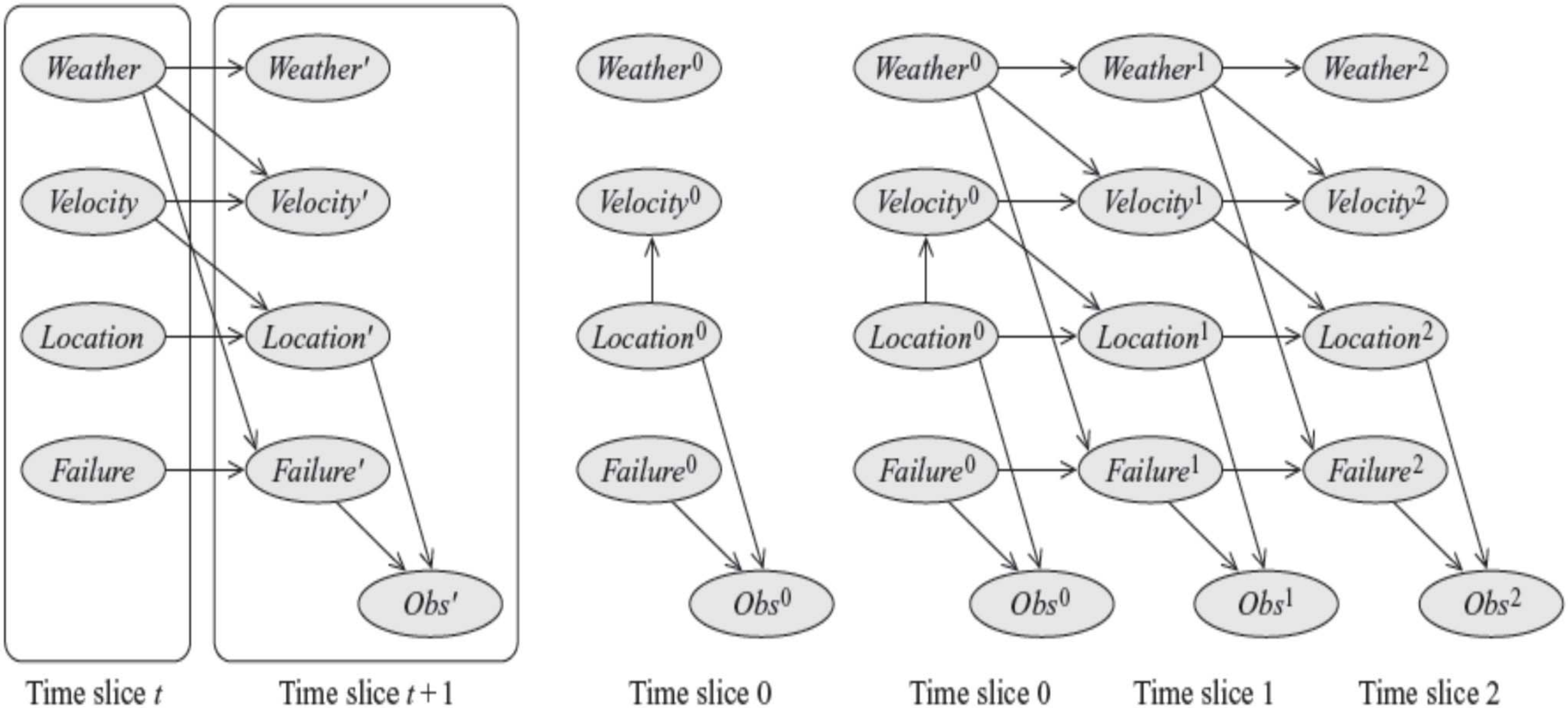
2-TBN

- 2-time slice Bayesian Network
  - Conditional Bayesian network over X' given X_I , X_I is a subset of X and is called the set of *interface* variables
- In the given example, all variables, except O, are in the interface
- Represents the following conditional distribution

$$P(\mathcal{X}' \mid \mathcal{X}) = P(\mathcal{X}' \mid \mathcal{X}_I) = \prod_{i=1}^{n} P(X_i' \mid \mathrm{Pa}_{X_i'}).$$

  - Term inside product is called a template factor; it is instantiated multiple times.

Dynamic Bayesian Networks (DBNs)



(a) $\mathcal{B}_{\rightarrow}$ (b) $\mathcal{B}_0$ (c) DBN unrolled over 3 steps

Dynamic Bayesian Network (DBN)

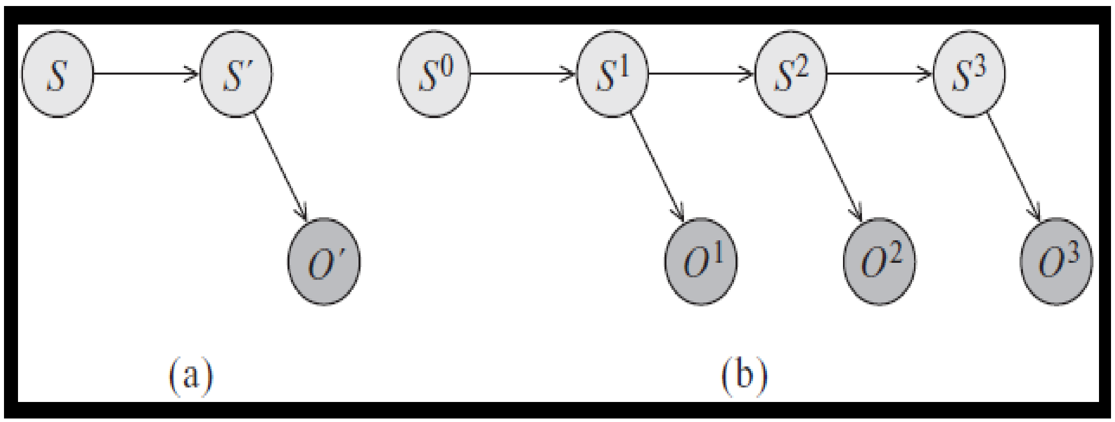
- A dynamic Bayesian network (DBN) is a pair (B_0 , $B_\rightarrow$)
- B_0 is a BN over $X^{(0)}$, representing initial state distribution
- $B_\rightarrow$ is a 2-TBN process
- For $T \geq 0$, the distribution over $X^{(0:T)}$ is defined as an unrolled BN where for any $i = 1,\ldots,n$:
  - Structure and CPDs of $X_i^{(0)}$ are same as for X_i in B_0
  - Structure and CPDs of $X_i^{(t)}$ for $t > 0$ are same as for X_i' in $B_\rightarrow$
- Generates an infinite set of BNs, one for every $T > 0$

Edges in DBN

- *Inter-time-slice* edges: edges across time slices
- *Intra-time-slice* edges: edges within a time slice
- *Persistence* edges: inter time-slice edges of the form $X \rightarrow X'$ (influence persists). A variable for which such an edge exists is called a *persistent* variable.
- Given an initial state, we can *unroll* the network over a desired interval

Hidden Markov Models (HMMs)

- A special case of a DBN
  - A single state variable and a single observation variable



State-Observation Models

- An alternative to the DBN view of a temporal model
- State evolves on its own
  - Follows a Markovian *transition* model: next state is conditionally independent of previous states given the current state
  - Transition model P(X' | X)
- Given the current state, observations are conditionally independent of the rest of the state sequence (past and future)
  - Observation model P(O | X)
- 2-TBN where O' are all leaf nodes, parents of O' only in X'
- Any 2-TBN can be transformed into such a representation (construction given in the book) but may hide some structure.
- 2 special cases
  - Linear Dynamical Systems
  - Hidden Markov Models (HMMs)

Linear Dynamical Systems

- All variables are continuous; dependencies are linear Gaussian

$P(\mathbf{X}^{(t)} \mid \mathbf{X}^{(t-1)}) = N\,(A\mathbf{X}^{(t-1)}; Q)$

$P(O^{(t)} \mid \mathbf{X}^{(t)}) = N\,(H\mathbf{X}^{(t)}; R)$

(note the deterministic analog)

If X is *n*-vector, O is an *m*-vector, A is *n* x *n* matrix, H is *n* x *m*

N is normal distribution, Q and R are noise matrices

Such systems commonly studied in estimation theory and control theory

A common objective is to estimate **X** given a sequence of observations **O**

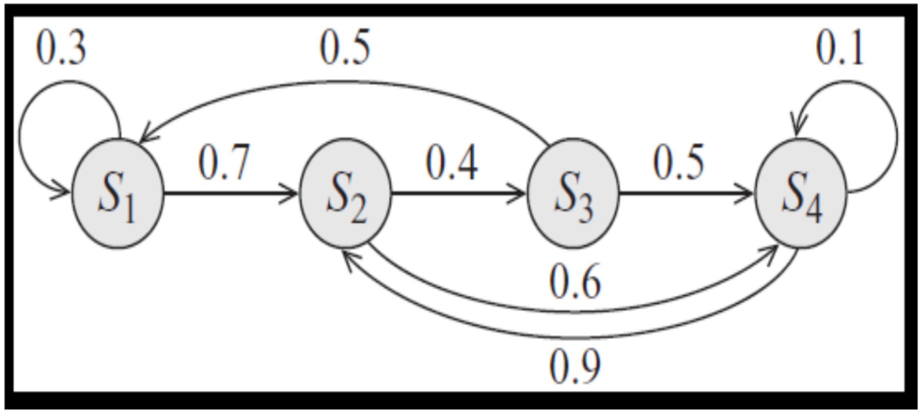
=> Kalman-Bucy Filter provides the optimal estimate

Extensions to non-linear dynamics

Hidden Markov Models (HMMs)

- One of the most important tools for many practical problems, e.g. speech recognition
- An HMM is a DBN but the transition model P(S '|S) is typically sparse
- State transitions can be represented as a graph
  - Note: this graph is NOT a Bayesian network, nodes do not represent random variables but *states* (or possible values of a state variable)
  - Arrows represent possible transitions and their probabilities
  - *Probabilistic finite state automation*
- Observation model is not encoded in the graph; it is associated with states separately

HMM Example



	s_1	s_2	s_3	s_4
s_1	0.3	0.7	0	0
s_2	0	0	0.4	0.6
s_3	0.5	0	0	0.5
s_4	0	0.9	0	0.1

Speech Recognition

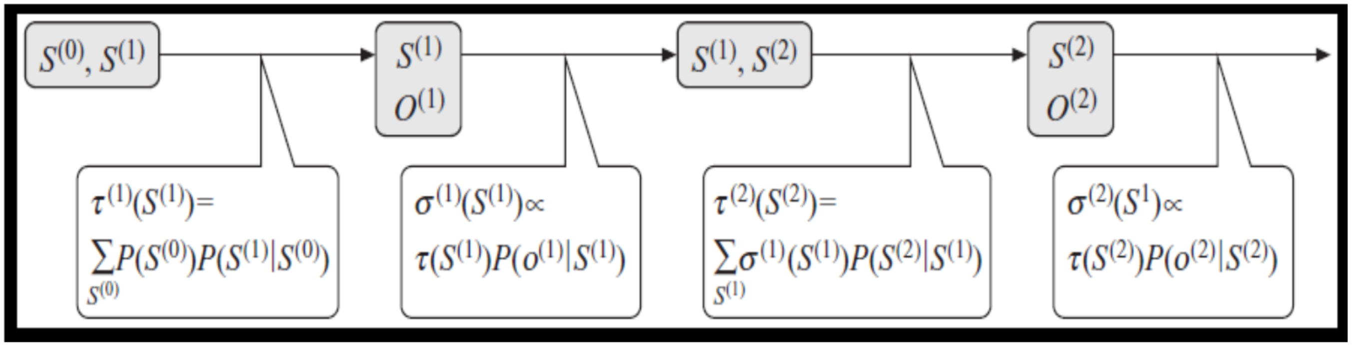
- Speech Recognition: identify sequence of words from acoustic signal.
- *Speech understanding* requires inference of *meaning*
  - Speech recognition is a necessary first step
- Prime example of utility of HMMs
- Given *signal*, find *words* that maximize P(*words*|*signal*)
  - **P**(*words*|*signal*) = **P**(*signal*|*words*)* **P**(*words*)
- **P**(*words*) is the *language* model
  - Probability of words and word sequences
- **P**(*signal*|*words*) is the acoustic model
- Slides from Russel-Norvig, AIMA book (section 23.5)
  - Details of speech will not be tested on the Exam.
  - See separate slides in Speech HMM.pdf

Inference Tasks in Temporal Models

- *Filtering*: distribution of current state, given all the observations so far: $\mathbf{P}\,(\,\boldsymbol{X}^{(t)} \mid O^{(1:t)}\,)$
- *Prediction*: probability distribution over X at time t' > t, given $O^{(1:t)}$.
- *Smoothing*: Compute probabilities of a state given all the evidence: $\mathbf{P}\,(\,\boldsymbol{X}^{(t)} \mid O^{(1:T)}\,)$
- *Most likely trajectory* (sequence of states) given all the observations: arg max $_{\xi(0:T)}$ $\mathbf{P}\,(\,\boldsymbol{\xi}^{(0:T)} \mid O^{(1:T)}\,)$
  - Viterbi algorithm solves the last query

Exact Inference: State-Observation Models

- Can view unrolled model as any other graph: construct a clique tree for it and perform inference on it



- Filtering requires just the forward pass: normalize at each step
- Prediction is inference without evidence node (prediction can be from the start or after some time t)
- Smoothing requires a backward pass also: multiply messages and normalize
- Trajectory: solve the MAP problem
- Book derives recursive formula for forward pass but omits details of others. Instead, we look at more details from the Russell-Norvig book (slides should be self-contained)

Filtering Equations

- Notation: belief state $\sigma^{(t)}(\boldsymbol{X}^{(t)}) = \mathbf{P}\ (\boldsymbol{X}^{(t)} \mid O^{(1:t)})$
- Goal: compute $\sigma^{(t+1)}$ from $\sigma^{(t)}$ (recursive algorithm)
- Notation: $\sigma^{(.t+1)}(\boldsymbol{X}^{(t+1)}) = \mathbf{P}\ (\boldsymbol{X}^{(t+1)} \mid O^{(1:t)})$; .t says evidence up to time *t* only (prior belief state)
- $\sigma^{(.t+1)}(\boldsymbol{X}^{(t+1)}) = \mathbf{P}\ (\boldsymbol{X}^{(t+1)} \mid O^{(1:t)}) =$

$$= \Sigma_{xt}\ \mathbf{P}\ (\boldsymbol{X}^{(t+1)} \mid \boldsymbol{X}^{(t)}, O^{(1:t)})\ \mathbf{P}\ (\boldsymbol{X}^{(t)} \mid O^{(1:t)})$$

$$= \Sigma_{xt}\ \mathbf{P}\ (\boldsymbol{X}^{(t+1)} \mid \boldsymbol{X}^{(t)}))\ \sigma^{(t)}(\boldsymbol{X}^{(t)})$$

- Add the effect of $O^{(t+1)}$:

$$\begin{aligned}
\sigma^{(t+1)}(\boldsymbol{X}^{(t+1)}) &= P(\boldsymbol{X}^{(t+1)} \mid o^{(1:t)}, o^{(t+1)}) \\
&= \frac{P(o^{(t+1)} \mid \boldsymbol{X}^{(t+1)}, o^{(1:t)}) P(\boldsymbol{X}^{(t+1)} \mid o^{(1:t)})}{P(o^{(t+1)} \mid o^{(1:t)})} \\
&= \frac{P(o^{(t+1)} \mid \boldsymbol{X}^{(t+1)}) \sigma^{(\cdot t+1)}(\boldsymbol{X}^{(t+1)})}{P(o^{(t+1)} \mid o^{(1:t)})}.
\end{aligned}$$

- Basically, multiply by observation probability and normalize

Next Class

- Read sections 15.2 and 15.3 of the KF book