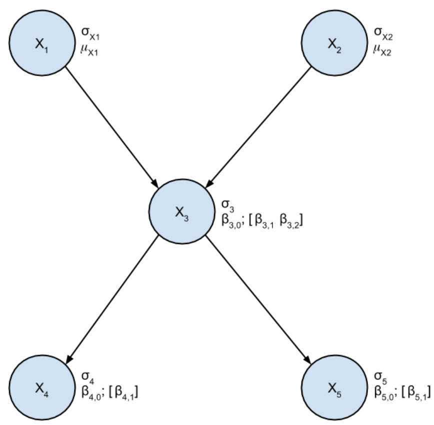
1 Question 1

For this network, find answer to the query of $p(X_1|X_4 = x_4, X_5 = x_5)$. Use the sum-product variable elimination algorithm, as one would for the discrete case, but applying to Gaussian distributions this time. It is recommended, but not essential, to use the canonical form to represent the distributions. As distributions are not given in numerical form but in symbolic form, your answer will also be in symbolic form.



Solution:
Factors in the network C_1, C_2, C_3, C_4, C_5, all canonical forms.

1. Factor 1: $C_1(X_1; K_1, h_1, g_1) = C_1(X_1; \frac{1}{\sigma_{x_1}^2}, \frac{\mu_{x_1}}{\sigma_{x_1}^2}, -\frac{\mu_{x_1}^2}{2\sigma_{x_1}^2} - \frac{1}{2}\ln(2\pi\sigma_{x_1}^2))$

2. Factor 2: $C_2(X_2; K_2, h_2, g_2) = C_2(X_2; \frac{1}{\sigma_{x_2}^2}, \frac{\mu_{x_2}}{\sigma_{x_2}^2}, -\frac{\mu_{x_2}^2}{2\sigma_{x_2}^2} - \frac{1}{2}\ln(2\pi\sigma_{x_2}^2))$

3. Factor 3: $C_3(X_3, X_1, X_2; K_3, h_3, g_3)$

   (a) $g_3 = -\frac{\beta_{3,0}^2}{2\sigma_3^2} - \frac{1}{2}\ln(2\pi\sigma_3^2)$

   (b) $h_3 = [\frac{\beta_{3,0}}{\sigma_3^2} \quad -\beta_{3,1}\frac{\beta_{3,0}}{\sigma_3^2} \quad -\beta_{3,2}\frac{\beta_{3,0}}{\sigma_3^2}]^T$

   (c) $K_3 = \begin{bmatrix} \frac{1}{\sigma_3^2} & -\frac{1}{\sigma_3^2}\beta_{3,1} & -\frac{1}{\sigma_3^2}\beta_{3,2} \\ -\frac{1}{\sigma_3^2}\beta_{3,1} & \frac{1}{\sigma_3^2}\beta_{3,1}^2 & \frac{1}{\sigma_3^2}\beta_{3,2}\beta_{3,1} \\ -\frac{1}{\sigma_3^2}\beta_{3,2} & \frac{1}{\sigma_3^2}\beta_{3,1}\beta_{3,2} & \frac{1}{\sigma_3^2}\beta_{3,2}^2 \end{bmatrix}$

4. Factor 4: $C_4(X_4, X_3; K_4, h_4, g_4)$

   (a) $g_4 = -\frac{\beta_{4,0}^2}{2\sigma_4^2} - \frac{1}{2}\ln(2\pi\sigma_4^2)$

   (b) $h_4 = [\frac{\beta_{4,0}}{\sigma_4^2} \quad -\beta_{4,1}\frac{\beta_{4,0}}{\sigma_4^2}]^T$

   (c) $K_4 = \begin{bmatrix} \frac{1}{\sigma_4^2} & -\beta_{4,1}\frac{1}{\sigma_4^2} \\ -\beta_{4,1}\frac{1}{\sigma_4^2} & \beta_{4,1}^2\frac{1}{\sigma_4^2} \end{bmatrix}$

5. Factor 5: $C_5(X_5, X_3; K_5, h_5, g_5)$

   (a) $g_5 = -\frac{\beta_{5,0}^2}{2\sigma_5^2} - \frac{1}{2}\ln(2\pi\sigma_5^2)$

   (b) $h_5 = [\frac{\beta_{5,0}}{\sigma_5^2} \quad -\beta_{5,1}\frac{\beta_{5,0}}{\sigma_5^2}]^T$

   (c) $K_5 = \begin{bmatrix} \frac{1}{\sigma_5^2} & -\beta_{5,1}\frac{1}{\sigma_5^2} \\ -\beta_{5,1}\frac{1}{\sigma_5^2} & \beta_{5,1}^2\frac{1}{\sigma_5^2} \end{bmatrix}$

Next, we must reduce factors C_4 and C_5 to evidence, creating C_4' and C_5'.

1. $C_4'(X_3; K_4', h_4', g_4')$ is C_4 reduced to evidence $X_4 = x_4$. Using Equation 14.6, we have:

   (a) $g_4' = g_4 + \frac{\beta_{4,0}}{\sigma_4^2}x_4 - \frac{1}{2}x_4^2\frac{1}{\sigma_4^2}$

   (b) $h_4' = -\beta_{4,1}\frac{\beta_{4,0}}{\sigma_4^2} + \beta_{4,1}\frac{1}{\sigma_4^2}x_4$

   (c) $K_4' = \beta_{4,1}^2\frac{1}{\sigma_4^2}$

2. $C_5'(X_3; K_5', h_5', g_5')$ is C_5 reduced to evidence $X_5 = x_5$. Using Equation 14.6, we have:

   (a) $g_5' = g_5 + \frac{\beta_{5,0}}{\sigma_5^2}x_5 - \frac{1}{2}x_5^2\frac{1}{\sigma_5^2}$

   (b) $h_5' = -\beta_{5,1}\frac{\beta_{5,0}}{\sigma_5^2} + \beta_{5,1}\frac{1}{\sigma_5^2}x_5$

   (c) $K_5' = \beta_{5,1}^2\frac{1}{\sigma_5^2}$

Now, we use factors $C_1, C_2, C_3, C_4', C_5'$ to find $P(X_1, X_4 = x_4, X_5 = x_5)$. Renormalizing gives us $P(X_1|X_4 = x_4, X_5 = x_5)$.

$$f_6(X_3, X_1, X_2) = p(X_2)\ p(X_3 \mid X_1, X_2) = C(K_6,\ h_6,\ g_6) = \tag{14}$$

$$= C(\begin{pmatrix} K_{6\_X3X3} & K_{6\_X3X1} & K_{6\_X3X2} \\ K_{6\_X1X3} & K_{6\_X1X1} & K_{6\_X1X2} \\ K_{6\_X2X3} & K_{6\_X2X1} & K_{6\_X2X2} \end{pmatrix}, \begin{pmatrix} h_{6\_X3} \\ h_{6\_X1} \\ h_{6\_X2} \end{pmatrix}, g_6) \tag{15}$$

$$C(\begin{pmatrix} \frac{1}{\mathrm{s3}^2} & -\frac{\mathrm{b31}}{\mathrm{s3}^2} & -\frac{\mathrm{b32}}{\mathrm{s3}^2} \\ -\frac{\mathrm{b31}}{\mathrm{s3}^2} & \frac{\mathrm{b31}^2}{\mathrm{s3}^2} & \frac{\mathrm{b31\,b32}}{\mathrm{s3}^2} \\ -\frac{\mathrm{b32}}{\mathrm{s3}^2} & \frac{\mathrm{b31\,b32}}{\mathrm{s3}^2} & \frac{1}{\mathrm{sx2}^2} + \frac{\mathrm{b32}^2}{\mathrm{s3}^2} \end{pmatrix}; \begin{pmatrix} \frac{\mathrm{b30}}{\mathrm{s3}^2} \\ -\frac{\mathrm{b30\,b31}}{\mathrm{s3}^2} \\ \frac{\mathrm{mx2}}{\mathrm{sx2}^2} - \frac{\mathrm{b30\,b32}}{\mathrm{s3}^2} \end{pmatrix}; -\log\left(\mathrm{s3}\sqrt{2\,\pi}\right) - \log\left(\mathrm{sx2}\sqrt{2\,\pi}\right) - \frac{\mathrm{b30}^2}{2\,\mathrm{s3}^2} - \frac{\mathrm{mx2}^2}{2\,\mathrm{sx2}^2}) \tag{16}$$

Now, we should marginalize X_2.
Using equation (14.5) from the book:

$$K' = K_{xx} - K_{xy} K_{yy}^{-1} K_{yx} \tag{17}$$

$$h' = h_x - K_{xy} K_{yy}^{-1} h_y \tag{18}$$

$$g' = g + \frac{1}{2}(log(|2\pi K_{yy}^{-1}|) + h_y^T K_{yy}^{-1} h_y) \tag{19}$$

$$f_7(X_3, X_1) = \int_{X_2} f_6(X_3, X_1, X_2) dX_2 = C(K_7,\ h_7,\ g_7) \tag{20}$$

$$K_7 = \begin{pmatrix} K_{6\_X3X3} & K_{6\_X3X1} \\ K_{6\_X1X3} & K_{6\_X1X1} \end{pmatrix} - \begin{pmatrix} K_{6\_X3X2} \\ K_{6\_X1X2} \end{pmatrix} K_{6\_X2X2}^{-1} \begin{pmatrix} K_{6\_X2X3} & K_{6\_X2X1} \end{pmatrix} \tag{21}$$

$$h_7 = \begin{pmatrix} h_{6\_X3} \\ h_{6\_X1} \end{pmatrix} - \begin{pmatrix} K_{6\_X3X2} \\ K_{6\_X1X2} \end{pmatrix} K_{6\_X2X2}^{-1} h_{6\_X2} \tag{22}$$

$$g_7 = g_6 + \frac{1}{2}(log(|2\pi K_{6\_X2X2}^{-1}|) + h_{6\_X2} K_{6\_X2X2}^{-1} h_{6\_X2}) \tag{23}$$

Now we can eliminate X_3:
Padded factors:

$$f_4'(X_3, X_1) = C(K_4',\ h_4',\ g_4')$$
$$= C(\begin{pmatrix} \frac{\mathrm{b41}^2}{\mathrm{s4}^2} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{\mathrm{b41\,e4}}{\mathrm{s4}^2} - \frac{\mathrm{b40\,b41}}{\mathrm{s4}^2} \\ 0 \end{pmatrix}, \frac{\mathrm{b40\,e4}}{\mathrm{s4}^2} - \frac{\mathrm{b40}^2}{2\,\mathrm{s4}^2} - \frac{\mathrm{e4}^2}{2\,\mathrm{s4}^2} - \log\left(\mathrm{s4}\sqrt{2\,\pi}\right)) \tag{24}$$

$$f_5'(X_3, X_1) = C(K_5',\ h_5',\ g_5')$$
$$= C(\begin{pmatrix} \frac{\mathrm{b51}^2}{\mathrm{s5}^2} & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \frac{\mathrm{b51\,e5}}{\mathrm{s5}^2} - \frac{\mathrm{b50\,b51}}{\mathrm{s5}^2} \\ 0 \end{pmatrix}, \frac{\mathrm{b50\,e5}}{\mathrm{s5}^2} - \frac{\mathrm{b50}^2}{2\,\mathrm{s5}^2} - \frac{\mathrm{e5}^2}{2\,\mathrm{s5}^2} - \log\left(\mathrm{s5}\sqrt{2\,\pi}\right)) \tag{25}$$

Now, the new multiplied factor:

$$f_8(X_3, X_1) = f_4'(X_3, X_1)\ f_5'(X_3, X_1)\ f_7(X_3, X_1) = C(K_8,\ h_8,\ g_8) \tag{26}$$

, where

$$f_8(X_3, X_1) = C(K_8,\ h_8,\ g_8) = C(K_4' + K_5' + K_7,\ h_4' + h_5' + h_7,\ g_4' + g_5' + g_7) \tag{27}$$

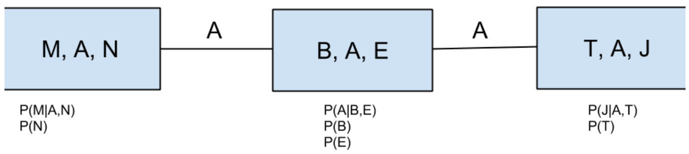
$$= C(\begin{pmatrix} K_{8\_X3X3} & K_{8\_X3X1} \\ K_{8\_X1X3} & K_{8\_X1X1} \end{pmatrix}, \begin{pmatrix} h_{8\_X3} \\ h_{8\_X1} \end{pmatrix}, g_8) \tag{28}$$

Now, we can marginalize:

$$f_9(X_1) = \int_{X_2} f_8(X_3, X_1) dX_1 = C(K_9,\ h_9,\ g_9) \tag{29}$$

Finally:
Non normalized distribution (since evidences were introduced):

$$\hat{p}(X_1|\ X_4 = e4,\ X_5 = e5) = p(X_1)f_9(X_1) = C(K_1 + K_9, h_1 + h_9, g_1 + g_9) = C(K_1', h_1', g_1') \tag{33}$$



2 Question 2

2.1 Part a

2.2 Part b

2.3 Part c

Let $\mathbf{X} = \{A, B, E, J, M, N, T\}$

$argmax_{\mathbf{x}} P(\mathbf{X}) =$
$\{A = a_0, B = b_0, E = e_0, J = j_0, M = m_0, N = n_0, T = t_1\}$

$P(a_0, b_0, e_0, j_0, m_0, n_0, t_1) = 0.5733$

2.4 Part d

Let $\mathbf{E} = \{J, M\}$

Let $\mathbf{W} = \{A, N, T\}$

Let $\mathbf{Y} = \{B, E\}$

$argmax_{\mathbf{y}} \Sigma_{\mathbf{W}} P(\mathbf{y}, \mathbf{W}|J = j_1, M = m_0) = \{B = b_0, E = e_0\}$

$P(b_0, e_0, j_1, m_0) = 0.028$
$P(b_0, e_0|j_1, m_0) = 0.9831$