Linear Algebra

Matrix Algebra

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Matrix Addition and Scalar Multiplication

Topics and Objectives

Topics

We will explore the following concepts in this video.

- identity and zero matrices
- matrix algebra: sums and scalar multiples

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- apply matrix algebra and the zero and identity matrices to solve and analyze matrix equations

Definitions: Zero and Identity Matrices

- A **zero matrix** is any matrix whose every entry is zero.

$$0_{2\times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad 0_{2\times 1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- The $n \times n$ **identity matrix** has ones on the main diagonal, otherwise all zeros.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note: any matrix with dimensions $n \times n$ is **square**. Zero matrices need not be square, identity matrices must be square.

Matrix Addition and Scalar Multiples

Suppose A and B are $m \times n$ matrices. $a_{i,j}$ is the entry of A in row i and column j, and $b_{i,j}$ is the entry of B in row i and column j.

- The entries of $A + B$ are $a_{i,j} + b_{i,j}$.
- If $c \in \mathbb{R}$, then the entries of cA are $ca_{i,j}$.

For example, if

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + c \begin{pmatrix} 7 & 4 & 7 \\ 0 & 0 & k \end{pmatrix} = \begin{pmatrix} 15 & 10 & 17 \\ 4 & 5 & 16 \end{pmatrix}$$

What are the values of c and k?

Properties of Sums and Scalar Multiples

Scalar multiples and matrix addition have the expected properties.

If $r, s \in \mathbb{R}$ are scalars, and A, B, C are $m \times n$ matrices, then

1. $A + 0_{m\times n} = A$
2. $(A + B) + C = A + (B + C)$
3. $r(A + B) = rA + rB$
4. $(r + s)A = rA + sA$
5. $r(sA) = (rs)A$

Summary

We explored the following concepts in this video.

- the identity and zero matrices
- matrix algebra: sums and scalar multiplies

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Matrix Multiplication

Topics and Objectives

Topics

We will explore the following concepts in this video.

- matrix multiplication

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- apply matrix algebra and the zero and identity matrices to solve and analyze matrix equations

Matrix Multiplication

Definition

Let A be a $m \times n$ matrix, and B be a $n \times p$ matrix. The product is AB a $m \times p$ matrix, equal to

$$AB = A \begin{pmatrix} \vec{b}_1 & \cdots & \vec{b}_p \end{pmatrix} = \begin{pmatrix} A\vec{b}_1 & \cdots & A\vec{b}_p \end{pmatrix}$$

Example

Compute the following product.

$$C = AB = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \end{pmatrix}$$

Row Column Rule for Matrix Multiplication

The Row Column Rule is a convenient way to calculate the product AB that many students have encountered in pre-requisite courses.

Row Column Method

If $A \in \mathbb{R}^{m \times n}$ has rows $\vec{a}_i$, and $B \in \mathbb{R}^{n \times p}$ has columns $\vec{b}_j$, each element of the product $C = AB$ is the dot product $c_{ij} = \vec{a}_i \cdot \vec{b}_j$.

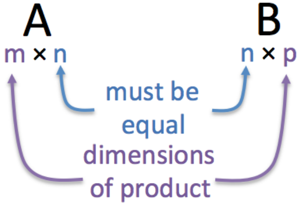
Example

Compute the following using the row-column method.

$$C = AB = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 3 & 4 & 0 \end{pmatrix}$$

Matrix Dimensions and Matrix Multiplication

Note: the dimensions of A and B determine whether AB is defined, and what its dimensions will be.



Properties of Matrix Multiplication

Let A, B, C be matrices of the sizes needed for the matrix multiplication to be defined, and A is a $m \times n$ matrix.

1. (Associative) $(AB)C = A(BC)$
2. (Left Distributive) $A(B + C) = AB + AC$
3. (Right Distributive) $(A + B)C = AC + AC$
4. (Identity for matrix multiplication) $I_m A = AI_n$

Warnings:

1. (non-commutative) In general, $AB \neq BA$.
2. (non-cancellation) $AB = AC$ does not mean $B = C$.
3. (Zero divisors) $AB = 0$ does not mean that either $A = 0$ or $B = 0$.

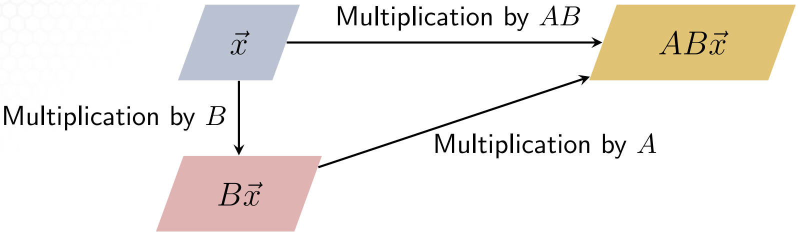
Examples

Suppose $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

1. Give an example of a 2×2 matrix that does not commute with A.
2. Construct any non-zero matrices B and C so that $B \neq C$ but $AB = AC$.

The Associative Property

If $C = \vec{x}$, then the associative property is: $(AB)\vec{x} = A(B\vec{x})$. Schematically:



The matrix product $AB\vec{x}$ can be obtained by either: multiplying by matrix AB, or by multiplying by B then by A. This means that matrix multiplication corresponds to **composition of the linear transformations**.

Summary

We explored the following concepts in this video.

- the identity and zero matrices
- matrix algebra: sums and products, scalar multiplies

There are several ways of multiplying matrices that we explore in this class. This video introduced two methods.

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Matrix Transpose and Powers

Topics and Objectives

Topics

We will explore the following concepts in this video.

- the transpose of a matrix
- matrix powers

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- apply the matrix transpose and matrix powers to solve and analyze matrix equations

The Transpose of a Matrix

A^T is the matrix whose columns are the rows of A.

Example

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 2 \end{pmatrix}^T =$$

Properties of the Matrix Transpose

1. $(A^T)^T =$
2. $(A + B)^T =$
3. $(rA)^T =$
4. $(AB)^T =$

Matrix Powers

For $n \times n$ matrix and positive integer k, A^k is the product of k copies of A.

$$A^k = AA \dots A$$

Example: Compute C^2.

$$C = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

Example

Define

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Which of these operations are defined, and what are the dimensions of the result?

1. $A + 3C^2$

2. $A(AB)^T$

3. $A + ABCB^T$

Summary

We explored the following concepts in this video.

- use of the matrix transpose, matrix powers, to solve and analyze matrix equations

For example, we can determine whether a particular expression involving matrices is defined and what the dimensions of the product will be.

Linear Algebra

Matrix Algebra

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The Inverse of a 2×2 Matrix

Topics and Objectives

We will explore the following concepts in this video.

- the inverse of a 2×2 matrix

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- compute the inverse of a 2×2 matrix and use it to solve a linear system

An Algorithm with Limitations

"Your scientists were so preoccupied with whether or not they could, they didn't stop to think if they should."
- Spielberg and Crichton, Jurassic Park, 1993 film

The algorithm we introduce in this section **could** be used to compute an inverse of an $n \times n$ matrix. At the end of this section of the course, we will discuss some of the problems with our algorithm and why it can be difficult to compute a matrix inverse.

The Matrix Inverse

Definition

$A \in \mathbb{R}^{n\times n}$ is **invertible** (or **non-singular**) if there is a $C \in \mathbb{R}^{n\times n}$ so that

$$AC = CA = I_n.$$

If there is, we write $C = A^{-1}$.

A matrix that is not invertible is **singular**.

The Inverse of a 2×2 Matrix

There is a formula for computing the inverse of a 2×2 matrix.

Theorem

The 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is non-singular if and only if $ad - bc \neq 0$, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example

State the inverse of the matrix $\begin{pmatrix} 2 & 5 \\ -3 & -7 \end{pmatrix}$.

Solving a Linear System

Use a matrix inverse to solve the linear system.

$$\begin{aligned} 3x_1 + 4x_2 &= 7 \\ 5x_1 + 6x_2 &= 7 \end{aligned}$$

Summary

We explored the following concepts in this video.

- use of the inverse of a 2×2 matrix to solve a linear system

In the next set of videos we will explore a method to compute the inverse of an $n \times n$ matrix and some of its limitations.

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The Inverse of an $n \times n$ Matrix

Topics and Objectives

Topics

We will explore the following concepts in this video.

- an algorithm for computing the inverse of a square matrix

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- compute the inverse of an $n \times n$ matrix, and use it to solve linear systems

The Matrix Inverse

Recall the following theorem.

Theorem

$A \in \mathbb{R}^{n \times n}$ has an inverse if and only if for all $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique solution. And, in this case, $\vec{x} = A^{-1}\vec{b}$.

This theorem gives us a method for solving a linear systems with n equations and n variables. But how do we construct the inverse of an $n \times n$ matrix?

An Algorithm for Computing A^{-1}

Suppose $A \in \mathbb{R}^{n\times n}$. We can use the following algorithm to compute A^{-1}.

1. Row reduce the augmented matrix $(A \mid I_n)$ to RREF.
2. If reduction has form $(I_n \mid B)$ then A is invertible and $B = A^{-1}$. Otherwise, A is not invertible.

Example

Compute the inverse of $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}$.

Why Does Our Algorithm Produce A^{-1}?

Suppose A is a 3×3 matrix and $A^{-1} = (\vec{x}_1\ \vec{x}_2\ \vec{x}_3)$. The first column of A^{-1} is

$$\vec{x}_1 = A^{-1}\vec{e}_1$$

This implies:

$$A\vec{x}_1 = \vec{e}_1, \quad \text{or} \quad (A \,|\, \vec{e}_1)$$

Thus:

- If we row reduce to RREF, we obtain the first column of the inverse, $\vec{x}_1$.
- Each column of A^{-1} is found by reducing $A\vec{x}_i = \vec{e}_i$.

.

Why Does This Work?

We can think of our algorithm as simultaneously solving n linear systems:

$$
\begin{gathered}
A\vec{x}_1 = \vec{e}_1 \\
A\vec{x}_2 = \vec{e}_2 \\
\vdots \\
A\vec{x}_n = \vec{e}_n
\end{gathered}
$$

Each column of A^{-1} is $A^{-1}\vec{e}_i = \vec{x}_i$.

Another perspective on constructing A^{-1} uses elementary matrices.

Summary

We explored the following concepts in this video.

- a method for constructing the inverse of an $n \times n$ matrix, A^{-1}, that could be used to solve a linear system

Our algorithm will have limitations but the concept of a matrix inverse is something that is widely used in applications of linear algebra, even if it is not used in practice.

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Elementary Matrices

Topics and Objectives

Topics

We will explore the following concepts in this video.

- the inverse of a matrix, its algebraic properties, and its relation to solving systems of linear equations
- elementary matrices and their role in calculating the matrix inverse

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- apply the formal definition of an inverse, and its algebraic properties, to solve and analyze linear systems
- construct elementary matrices and characterize row operations with them

Properties of the Matrix Inverse

A and B are invertible $n \times n$ matrices.

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$ (Non-commutative!)
- $(A^T)^{-1} = (A^{-1})^T$

Example

True or false: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Elementary Matrices

An elementary matrix, E, is one that differs by I_n by one row operation. Recall our elementary row operations:

- swap rows
- multiply a row by a non-zero scalar
- add a multiple of one row to another

We can represent each operation by a matrix multiplication with an **elementary matrix**.

Example

Suppose

$$E\begin{pmatrix}1 & 1 & 1\\ -2 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1 & 1\\ 0 & 3 & 2\\ 0 & 0 & 1\end{pmatrix}$$

By inspection, what is E? How does it compare to I_3?

Theorem

Returning to understanding why our algorithm works, we apply a sequence of row operations to A to obtain I_n:

$$(E_k \cdots E_3 E_2 E_1)A = I_n$$

Thus, $E_k \cdots E_3 E_2 E_1$ is the inverse matrix we seek.

Our algorithm for calculating the inverse of a matrix is the result of the following theorem.

Theorem

Matrix A is invertible if and only if it is row equivalent to the identity. In this case, the any sequence of elementary row operations that transforms A into I, applied to I, generates A^{-1}.

Using The Inverse to Solve a Linear System

- We could use A^{-1} to solve a linear system,

$$A\vec{x} = \vec{b}$$

  We would calculate A^{-1} and then: $\vec{x} = A^{-1}\vec{b}$.
- As our textbook points out, A^{-1} is seldom used: computing it can take a very long time, and is prone to numerical error.
- So why did we learn how to compute A^{-1}? Later on in this course, we use elementary matrices and properties of A^{-1} to derive results.
- A recurring theme of this course: just because we **can** do something a certain way, does not meant that we **should**.

Summary

We explored the following concepts in this video.

- elementary matrices and their relationship to row operations
- properties of the matrix inverse

Elementary matrices are used a few times throughout this course to describe the processes behind algorithms.

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The Invertible Matrix Theorem

Topics and Objectives

Topics

We will explore the following concepts in this video.

- the invertible matrix theorem, which is a review/synthesis of many of the concepts we have introduced

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize the invertibility of a matrix using the Invertible Matrix Theorem
- construct and give examples of matrices that are/are not invertible

Equivalent Expressions

"A synonym is a word you use when you can't spell the other one."
- Baltasar Gracián

The theorem we introduce in this section of the course gives us many ways of saying the same thing. Depending on the context, some will be more convenient than others.

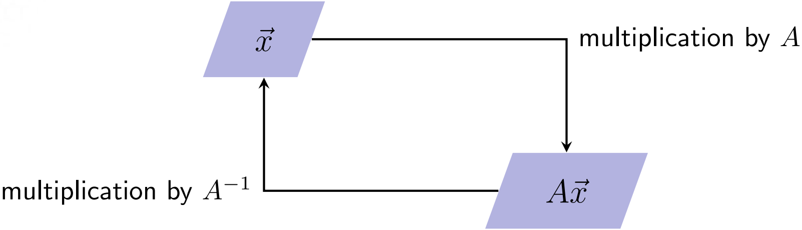
The Invertible Matrix Theorem

Let A be an $n \times n$ matrix. These statements are all equivalent.

a) A is invertible.
b) A is row equivalent to I_n.
c) A has n pivotal columns (all columns are pivotal).
d) $A\vec{x} = \vec{0}$ has only the trivial solution.
e) The columns of A are linearly independent.
f) The equation $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in \mathbb{R}^n$.
g) The columns of A span $\mathbb{R}^n$.
h) There is a $n \times n$ matrix C so that $CA = I_n$ (A has a left inverse.)
i) There is a $n \times n$ matrix D so that $AD = I_n$ (A has a right inverse.)
j) A^T is invertible.

Invertibility and Composition

The diagram below gives us another perspective on the role of A^{-1}.



The matrix inverse A^{-1} transforms Ax back to $\vec{x}$. This is because:

$$A^{-1}(A\vec{x}) = (A^{-1}A)\vec{x} =$$

The Invertible Matrix Theorem: Final Notes

- Items (h) and (i) of the invertible matrix theorem (IMT) lead us directly to the following theorem.

> **Theorem**
>
> If A and B are $n \times n$ matrices and $AB = I$, then A and B are invertible, and $B = A^{-1}$ and $A = B^{-1}$.

- The IMT is a set of equivalent statements. They divide the set of all square matrices into two separate classes: invertible, and non-invertible.
- As we progress through this course, we will be able to add additional equivalent statements to the IMT (that deal with determinants, eigenvalues, etc).

Example 1: Identifying Whether a Matrix is Invertible

Is this matrix invertible?

$$\begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ 0 & -1 & -1 \end{pmatrix}$$

Example 2: Constructing an Expression for the Inverse

Suppose A is an invertible square matrix and

$$A^2 + 4A = I$$

Give an expression for A^{-1}.

Example 3: Matrix Completion

If possible, fill in the missing elements of the matrices below with numbers so that each of the matrices are singular. If it is not possible to do so, state why.

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & & 1 \end{pmatrix}$$

Matrix Completion Problems

- The previous example is an example of a matrix completion problem (MCP).
- MCPs are great questions for recitations, midterms, exams.
- the **Netflix Problem** is another example of an MCP.

> Given a **ratings matrix** in which each entry (i, j) represents the rating of movie j by customer i if customer i has watched movie j, and is otherwise missing, predict the remaining matrix entries in order to make recommendations to customers on what to watch next.

Summary

We explored the following concepts in this video.

- characterizing the invertibility of a matrix using the Invertible Matrix Theorem
- construct and give examples of matrices that are/are not invertible

As we go through the course we will add more equivalent statements to this theorem.

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Partitioned Matrices and Matrix Multiplication

Topics and Objectives

Topics

We will explore the following concepts in this video.

- partitioned matrices (or block matrices)
- matrix multiplication with partitioned matrices

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- apply partitioned matrices to solve problems regarding matrix multiplication

Multiple Perspectives

"Mathematics is not about numbers, equations, computations, or algorithms. Mathematics is about understanding."
- William Paul Thurston

Multiple perspectives of the same concept is a theme of this course; each perspective deepens our understanding. In this section we explore another way of representing matrices and their algebra that gives us another way of thinking about them.

What is a Partitioned Matrix?

This matrix:

$$A = \begin{pmatrix} 3 & 1 & 4 & 1 & 0 \\ 1 & 6 & 1 & 0 & 1 \\ 0 & 0 & 0 & 4 & 2 \end{pmatrix}$$

can also be written as:

$$A = \begin{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ 1 & 6 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 4 & 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$$

We partitioned our matrix into four **blocks**, each of which has different dimensions.

Why are Partitioned Matrices Useful?

Partitioned matrices can give a succinct representation of a matrix.

$$\begin{pmatrix} 1 & 0 & 0 & * & \cdots & * \\ 0 & 1 & 0 & * & \cdots & * \\ 0 & 0 & 1 & * & \cdots & * \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} I_3 & F \\ 0 & 0 \end{pmatrix}$$

This can be useful when studying something called the **null space** of A, which we will define later in this course.

Recall: the Row Column Method

Recall the **row column** method for matrix multiplication.

Theorem

Let A be $m \times n$ and B be $n \times p$ matrix. Then, the (i, j) entry of AB is

$$\text{row}_i\, A \cdot \text{col}_j\, B.$$

This is the **Row Column Method** for matrix multiplication.

Example

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} =$$

The Row Column Method for Block Matrices

Partitioned matrices can be multiplied using this method, as if each block were a scalar (*provided each block has appropriate dimensions so that products are defined*).

Example

Compute the matrix product using the given partitioning.

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 0 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} I_2 & X \end{pmatrix} \begin{pmatrix} U \\ Y \end{pmatrix}$$

Where $X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, U = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \end{pmatrix}$.

Summary

We explored the following concepts in this video.

- using partitioned matrices to solve problems regarding matrix multiplication

Although not part of our course, matrix partitioning is often used to help derive new algorithms because they give a more concise representation of a matrix and of operations on matrices.

Linear Algebra

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Partitioned Matrices and the Matrix Inverse

Topics and Objectives

Topics

We will explore the following concepts in this video.

- partitioned matrices (or block matrices)
- expressions for the inverse of a partitioned matrix

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- apply partitioned matrices to solve problems regarding matrix invertibility

The Row Column Method for Block Matrices

Partitioned matrices can be multiplied using this method, as if each block were a scalar (*provided each block has appropriate dimensions so that products are defined*).

How might we use this approach to determine an expression for the inverse of a matrix?

Using the Row Column Method to Construct an Inverse

Recall, using our formula for a 2×2 matrix, $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^{-1} = \frac{1}{ac} \begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$.

Example: Suppose $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$, and $C \in \mathbb{R}^{n \times n}$ are invertible matrices. Construct the inverse of $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$.

Summary

We explored the following concepts in this video.

- using partitioned matrices to solve problems regarding matrix invertibility and matrix multiplication

Although not part of our course, matrix partitioning is often used to help derive new algorithms because they give a more concise representation of a matrix and of operations on matrices.

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Solving Linear Systems with the LU Factorization

Topics and Objectives

Topics

We will explore the following concepts in this video.

- triangular matrices
- the LU factorization of a matrix
- using the LU factorization to solve a system

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- identify and construct triangular matrices
- apply the LU factorization to solve systems of equations

Mathematical Ingenuity

"Mathematical reasoning may be regarded rather schematically as the exercise of a combination of two facilities, which we may call intuition and ingenuity."
- Alan Turing

The use of the LU Decomposition to solve linear systems was one of the areas of mathematics that Turing helped develop. The decomposition is widely used to solve linear systems of equations.

Motivation

- Recall that we **could** solve $A\vec{x} = \vec{b}$ by using

$$\vec{x} = A^{-1}\vec{b}$$

- This requires computation of the inverse of an $n \times n$ matrix, which is especially difficult for large n.
- Instead we could solve $A\vec{x} = \vec{b}$ with Gaussian Elimination, but this is not efficient for large n
- There are more efficient and accurate methods for solving linear systems that rely on matrix factorizations.

Matrix Factorizations

- A **matrix factorization**, or **matrix decomposition** is a factorization of a matrix into a product of matrices.
- Factorizations can be useful for solving $A\vec{x} = \vec{b}$, or understanding the properties of a matrix.
- We explore a few matrix factorizations throughout this course.
- In this section, we factor a matrix into **lower** and into **upper** triangular matrices.

Triangular Matrices

Rectangular matrix A is **upper triangular** if $a_{i,j} = 0$ for $i > j$. Examples:

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Rectangular matrix A is **lower triangular** if $a_{i,j} = 0$ for $i < j$. Examples:

$$\begin{pmatrix} 1 & 0 & 0 \\ 3 & 2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 4 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}$$

Can you name a matrix that is both upper and lower triangular?

The LU Factorization

Theorem

If A is an $m \times n$ matrix that can be row reduced to echelon form without row exchanges, then $A = LU$. L is a lower triangular $m \times m$ matrix with 1's on the diagonal, U is an **echelon** form of A.

Example: If $A \in \mathbb{R}^{3\times 2}$, the LU factorization has the form:

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{pmatrix} \begin{pmatrix} * & * \\ 0 & * \\ 0 & 0 \end{pmatrix}$$

Using the LU Decomposition to Solve a Linear System

Goal: given rectangular matrix A and vector $\vec{b}$, we wish to solve $A\vec{x} = \vec{b}$ for $\vec{x}$.

Algorithm

To solve $A\vec{x} = \vec{b}$ for $\vec{x}$:

1. Construct the LU decomposition of A to obtain L and U.
2. Set $U\vec{x} = \vec{y}$. Forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$.
3. Backwards solve for $\vec{x}$ in $U\vec{x} = \vec{y}$.

Example of Solving a Linear System Given $A = LU$

Example: Solve the linear system $A\vec{x} = \vec{b}$, given the LU decomposition of A.

$$A = LU = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 3 \\ 2 \\ 0 \end{pmatrix}$$

Summary

We explored the following concepts in this video.

- triangular matrices
- using the LU factorization to solve linear systems

To solve $A\vec{x} = LU\vec{x} = \vec{b}$, we follow this process:

1. construct the LU factorization
2. forward solve for $\vec{y}$ in $L\vec{y} = \vec{b}$
3. backwards solve for $\vec{x}$ in $U\vec{x} = \vec{y}$

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Computing the LU Factorization

Topics and Objectives

Topics

We will explore the following concepts in this video.

- the LU factorization of a matrix
- why the LU factorization works

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- compute an LU factorization of a matrix

Why We Can Compute the LU Factorization

Suppose A can be row reduced to echelon form U without interchanging rows.

$$E_p \cdots E_1 A = U$$

where the E_j are matrices that perform elementary row operations. Because we did not swap rows, each E_j happens to be lower triangular and invertible. Example:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

Therefore,

$$A = \underbrace{E_1^{-1} \cdots E_p^{-1}}_{=L} U = LU.$$

An Algorithm for Computing LU

To compute the LU decomposition:

1. Reduce A to an echelon form U by a sequence of row replacement operations, if possible.
2. Place entries in L such that the same sequence of row operations reduces L to I.

Example

Compute the LU factorization of A.

$$A = \begin{pmatrix} 4 & -3 & -1 & 5 \\ -16 & 12 & 2 & -17 \\ 8 & -6 & -12 & 22 \end{pmatrix}$$

Final Notes on Computing LU

- There are other definitions of the LU factorization that you may encounter in future courses or applications.
- There are several other ways of computing this decomposition.
- The only row operation we use to construct L and U: *replace a row with a multiple of a row above it*.
- As for the other two row operations:
  - Multiplying a row by a non-zero scalar is not needed.
  - We cannot swap rows: more advanced linear algebra and numerical analysis courses address this limitation.

Summary

We explored the following concepts in this video.

- why we can construct $A = LU$ when A can be reduced to echelon form without row swaps
- constructing the LU decomposition using the following process
  1. reduce A to an echelon form U by a sequence of row replacement operations, if possible
  2. place entries in L such that the same sequence of row operations reduces L to I

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The Leontif Input-Output Model

Topics and Objectives

Topics

We will explore the following concepts in this video.

- the Leontief Input-Output model as a simple example of a model of an economy

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

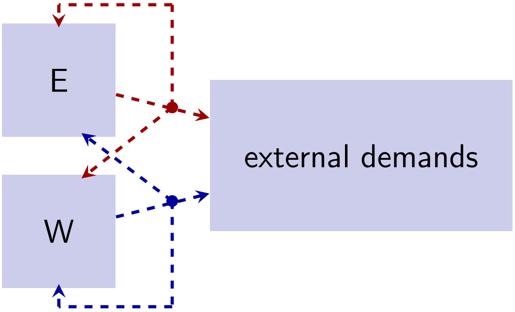
- apply matrix algebra and inverses to construct and solve and analyze Leontif Input-Output problems

Understanding the Underlying Concepts

"Computers and robots replace humans in the exercise of mental functions in the same way as mechanical power replaced them in the performance of physical tasks."
- Wassily Leontif, 1983

Students in linear algebra are, of course, required to demonstrate an understanding of underlying concepts behind procedures and algorithms. This is in part because computers are continuing to take on a much larger role in performing calculations.

Example: An Economy with Two Sectors



- this economy contains two sectors: electricity (E), and water (W)
- the *external demands* do not produce E and W
- how might we represent this economy with a set of linear equations?

The Leontif Model: Output Vector

Suppose economy has N sectors, with outputs measured by $\vec{x} \in \mathbb{R}^N$.

$$\begin{aligned} \vec{x} &= \text{output vector} \\ x_i &= \text{entry } i \text{ of vector } \vec{x} \\ &= \text{number of units produced by sector } i \end{aligned}$$

The Leontif Model: Internal Consumption

The **consumption matrix**, C, describes how units are consumed by sectors to produce output.

Two equivalent ways of defining entries of C.

- sector i **sends** a proportion of its units to sector j, call it $c_{i,j}x_i$
- sector j **requires** a proportion of the units created by sector i, call it $c_{i,j}x_i$

Entries of C are $c_{i,j}$, with $c_{i,j} \in [0,1]$, and

$$\begin{aligned} C\vec{x} &= \text{units consumed} \\ \vec{x} - C\vec{x} &= \text{units left after internal consumption} \end{aligned}$$

Example With Three Sectors

An economy contains three sectors, E, W, M.

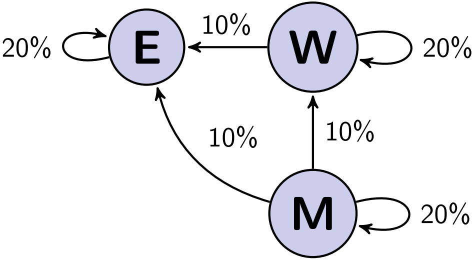
For every 100 units of output,

- E requires 20 units from E, 10 units from W, and 10 units from M
- W requires 0 units from E, 20 units from W, and 10 units from M
- M requires 0 units from E, 0 units from W, and 20 units from M

If the output vector is $\vec{x} = \begin{pmatrix} x_E \\ x_W \\ x_M \end{pmatrix}$, construct the consumption matrix for this economy.

Sketching a Graph for the Economy

Although not strictly necessary, it can help to sketch the graph for the economy.



Solution: Creating C

Our consumption matrix is

$$C = \frac{1}{10}\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

Note:

- total output for each sector is the sum along the outgoing edges for each sector, which generates rows of C
- elements of C represent percentages with no units, they have values between 0 and 1
- our output vector has units

The Leontif Model: Demand

There is also an external demand given by $\vec{d} \in \mathbb{R}^N$. We ask if there is an $\vec{x}$ such that

$$\vec{x} - C\vec{x} = \vec{d}$$

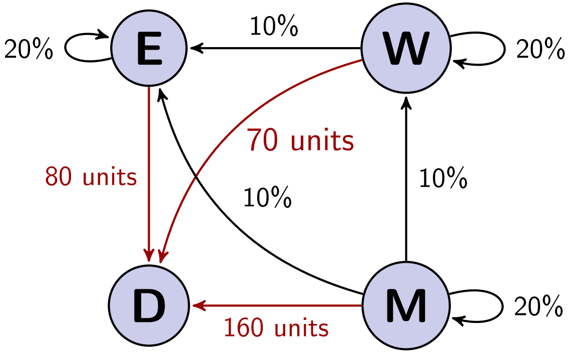
We can re-write this as

$$(I - C)\vec{x} = \vec{d}$$

This matrix equation is the **Leontief Input-Output Model**. Solving for $\vec{x}$ gives the output that meets external demand exactly.

Example Revisited

Now suppose there is an external demand: what production level is required to satisfy a final demand of 80 units of E, 70 units of W, and 160 units of M?



Solution

The production level would be found by solving:

$$(I - C)\vec{x} = \vec{d}$$

$$\frac{1}{10}\begin{pmatrix} 8 & 0 & 0 \\ -1 & 8 & 0 \\ -1 & -1 & 8 \end{pmatrix}\vec{x} = \begin{pmatrix} 80 \\ 70 \\ 160 \end{pmatrix}$$

$$\begin{aligned} 8x_1 &= 800 \quad &\Rightarrow \quad & x_1 = 100 \\ -x_1 + 8x_2 &= 700 \quad &\Rightarrow \quad & x_2 = 100 \\ -x_1 - x_2 + 8x_3 &= 1600 \quad &\Rightarrow \quad & x_3 = 1800/8 = 225 \end{aligned}$$

The output that balances demand with internal consumption is $\vec{x} = \begin{pmatrix} 100 \\ 100 \\ 225 \end{pmatrix}$.

Summary

We explored the following concepts in this video.

- setting up and solving Leontif input-output models

This video explored an application of systems of equations.

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Homogeneous Coordinates

Topics and Objectives

Topics

We will explore the following concepts in this video.

- homogeneous coordinates in 2D
- translations and composite transforms in 2D

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- construct a data matrix to represent points in $\mathbb{R}^2$ using homogeneous coordinates
- construct and apply transformation matrices to represent composite transforms in 2D using homogeneous coordinates

Motivating Questions

How can we represent translations, and rotations about arbitrary points, using linear transforms?

- transformations of the form $T(\vec{x}) = A\vec{x}$ were explored earlier in this course
- we introduced rotations about the origin, but not about arbitrary points
- we also did not explore the transform $(x, y) \to (x + h, y + k)$

Homogeneous Coordinates

Translations of points in $\mathbb{R}^n$ does not correspond directly to a linear transform. **Homogeneous coordinates** are used to model translations using matrix multiplication.

Homogeneous Coordinates in $\mathbb{R}^2$

Each point (x, y) in $\mathbb{R}^2$ can be identified with the point (x, y, H), $H \neq 0$, on the plane in $\mathbb{R}^3$ that lies H units above the xy-plane.

Note: we often we set $H = 1$.

Homogeneous Coordinates Example

A translation of the form $(x, y) \to (x + h, y + k)$ can be represented as a matrix multiplication with homogeneous coordinates:

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x + h \\ y + k \\ 1 \end{pmatrix}$$

A Composite Transform with Homogeneous Coordinates

Triangle S is determined by three data points, $(1,1), (2,4), (3,1)$.

Transform T rotates points by $\pi/2$ radians counterclockwise about the point $(0,1)$.

a) Represent the data with a matrix, D. Use homogeneous coordinates.
b) Use matrix multiplication to determine the image of S under T.
c) Sketch S and its image under T.

Summary

We explored the following concepts in this video.

- homogeneous coordinates
- constructing composite transforms that apply translations and rotations about arbitrary points in $\mathbb{R}^2$

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3D Transformations

Topics and Objectives

Georgia Tech

Topics

We will explore the following concepts in this video.

- homogeneous coordinates in 3D
- translations and composite transforms in 3D

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- construct a data matrix to represent points in $\mathbb{R}^3$ with homogeneous coordinates
- construct and apply transformation matrices to represent composite transforms in 3D using homogeneous coordinates

3D Homogeneous Coordinates

Homogeneous coordinates in 3D are analogous to our 2D coordinates.

Homogeneous Coordinates in $\mathbb{R}^3$

$(X, Y, Z, 1)$ are homogeneous coordinates for (x, y, z) in $\mathbb{R}^3$

Homogeneous Coordinates Example

A translation of the form $(x, y, z) \to (x+h, y+k, z+l)$ can be represented as a matrix multiplication with homogeneous coordinates:

$$\begin{pmatrix} 1 & 0 & 0 & h \\ 0 & 1 & 0 & k \\ 0 & 0 & 1 & l \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ z+l \\ 1 \end{pmatrix}$$

3D Transformation Matrices

Construct matrices for the following transformations.

a) A translation in $\mathbb{R}^3$ specified by the vector $\vec{p} = \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$.

b) A rotation in $\mathbb{R}^3$ about the x_2-axis by π radians.

c) A projection onto the plane $x_3 = 4$.

For any of the above, it is ok to leave your answer as a product of matrices.

Summary

We explored the following concepts in this video.

- constructing composite transforms that apply translations and rotations about arbitrary points in $\mathbb{R}^3$

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Subsets and Subspaces

Topics and Objectives

Topics

We will explore the following concepts in this video.

- subsets and subspaces
- set builder notation

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- determine whether a subset of $\mathbb{R}^n$ is a subspace

Subsets of $\mathbb{R}^n$

Definition

A **subset of** $\mathbb{R}^n$ is any collection of vectors that are in $\mathbb{R}^n$.

Examples

- the span of the columns of a 3×4 matrix is a subset of $\mathbb{R}^3$
- the set of all vectors of the form $\begin{pmatrix} 1 \\ k \end{pmatrix}$ is a subset of $\mathbb{R}^2$

Subspaces in $\mathbb{R}^n$

Definition

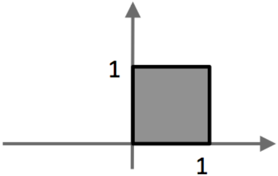
A subset H of $\mathbb{R}^n$ is a **subspace** if it is closed under scalar multiplies and vector addition. That is: for any $c \in \mathbb{R}$ and for $\vec{u}, \vec{v} \in H$,

1. $c\,\vec{u} \in H$
2. $\vec{u} + \vec{v} \in H$

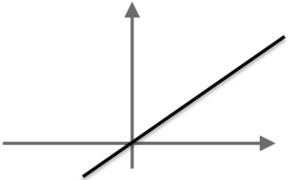
Note that condition 1 implies that the zero vector must be in H.

Example: Subspaces

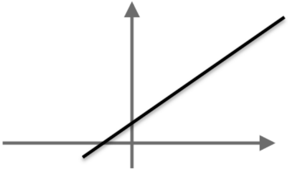
Which of the following subsets could be a subspace of $\mathbb{R}^2$?



a) the unit square



b) a line passing through the origin



c) a line that doesn't pass through the origin

Example: Set Builder Notation

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab = 0 \right\}$.

1. Give an example of at least two vectors that are in V.
2. Give an example of at least two vectors that are not in V.
3. Is the zero vector in V?
4. Is V a subspace?

Summary

We explored the following concepts in this video.

- identifying whether a subset is a subspace

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Column Space and Null Space

Topics and Objectives

Topics

We will explore the following concepts in this video.

- the column space and null space of a matrix

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- determine whether a vector is a column or null space of a matrix, or identify a vector in that subspace
- construct a matrix whose column and/or null space is given

The Column Space and the Null Space of a Matrix

Recall: for $\vec{v}_1, \ldots, \vec{v}_p \in \mathbb{R}^n$, that $\mathrm{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is: the set of all possible linear combinations of the vectors $\vec{v}_j$.

This is a **subspace**, spanned by $\vec{v}_1, \ldots, \vec{v}_p$.

Definition

Given an $m \times n$ matrix $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$

- The **column space of** A, $\mathrm{Col}\, A$, is the subspace of $\mathbb{R}^m$ spanned by $\vec{a}_1, \ldots, \vec{a}_n$.
- The **null space of** A, $\mathrm{Null}\, A$, is the subspace of $\mathbb{R}^n$ spanned by the set of all vectors $\vec{x}$ that solve $A\vec{x} = \vec{0}$.

Example: Column Space

Is $\vec{b}$ in the column space of A?

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 3 \\ 3 \\ -4 \end{pmatrix}$$

Example: Null Space

Using the matrix on the previous slide: is $\vec{v}$ in the null space of A?

$$A = \begin{pmatrix} 1 & -3 & -4 \\ -4 & 6 & -2 \\ -3 & 7 & 6 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -5\lambda \\ -3\lambda \\ \lambda \end{pmatrix}, \quad \lambda \in \mathbb{R}$$

Example: Column and Null Space

Give an example of a vector in the column space of A, and a vector in the null space of A.

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Example Construction

Give an example of a matrix whose column space is spanned by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and whose null space is spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Summary

We explored the following concepts in this video.

- determining whether a vector is a column or null space of a matrix
- identifying vectors in the column and null space of a matrix
- constructing a matrix whose column and/or null space is given

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The Basis of a Subspace

Topics and Objectives

Topics

We will explore the following concepts in this video.

- a basis for a subspace

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- construct a basis for a subspace (for example, a basis for $\mathrm{Col}(A)$)

Basis

Definition

A **basis** for a subspace H is a set of linearly independent vectors in H that span H.

Example

The set $H = \{\vec{x} \in \mathbb{R}^4 \mid x_1 - 3x_2 - 5x_3 + 7x_4 = 0\}$ is a subspace.

a) H is a null space for what matrix A?

b) Construct a basis for H.

Example

Construct a basis for Null$A$ and a basis for Col$A$.

$$A = \begin{pmatrix} -3 & 6 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 2 & -4 & 5 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Summary

We explored the following concepts in this video.

- constructing a basis for a subspace
- constructing a basis for the column and/or null space of a matrix

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Coordinate Systems

Topics and Objectives

Topics

We will explore the following concepts in this video.

- change of basis
- coordinates relative to a basis

Objectives

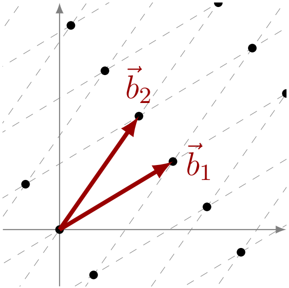
Students should be able to do the following after watching this video and completing the assigned homework.

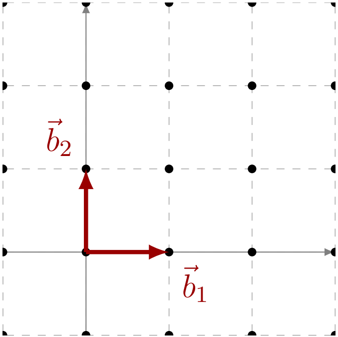
- calculate the coordinates of a vector in a given basis

Choice of Basis

Key idea: There are many possible choices of basis for a subspace. Our choice can give us dramatically different properties.

Example: sketch $\vec{b}_1 + \vec{b}_2$ for the two different coordinate systems below.





Definition: Coordinate Vector

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ be a basis for a subspace H. If $\vec{x}$ is in H, then **coordinates of $\vec{x}$ relative $\mathcal{B}$** are the weights (scalars) $c_1, \ldots, c_p$ so that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_p\vec{b}_p$$

And

$$[x]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix}$$

is the **coordinate vector of $\vec{x}$ relative to $\mathcal{B}$**, or the **$\mathcal{B}$-coordinate vector of $\vec{x}$**

Example: Coordinate Vector

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ 5 \end{bmatrix}$. Verify that $\vec{x}$ is in the span of $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, and calculate $[\vec{x}]_{\mathcal{B}}$.

Summary

We explored the following concepts in this video.

- calculating the coordinates of a vector with a given basis for a subspace

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The Dimension of a Susbpsace

Topics and Objectives

Topics

We will explore the following concepts in this video.

- dimension of a subspace

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- characterize a subspace using the concept of dimension (or cardinality)

Dimension of a Subspace

Definition

The **dimension** (or cardinality) of a non-zero subspace H, $\dim H$, is the number of vectors in a basis of H. We define $\dim\{\vec{0}\} = 0$.

Note that the zero vector cannot be a basis vector. The dimension of the set that only contains the zero vector is zero.

Example

The dimensions of the column space for each matrix below is 2.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Examples: Dimension of a Subspace

Fill in the blanks.

1. $\dim \mathbb{R}^n = \boxed{}$.
2. $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\dim(\mathrm{Col}\, A) = \boxed{}$.
3. $\dim(\mathrm{Null}\, A)$ is the number of $\boxed{}$.
4. $\dim(\mathrm{Col}\, A)$ is the number of $\boxed{}$.
5. $H = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$ has dimension $\boxed{}$.

Dimension of a Subspace

Theorem

Suppose H is a p-dimensional subspace of $\mathbb{R}^n$. Any set of p independent vectors that are in H are automatically a basis for H.

Example

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Two bases for the column space of A are:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Summary

We explored the following concepts in this video.

- the dimension of a subspace
- characterizing a subspace using the concept of dimension

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Rank and Invertibility

Topics and Objectives

Topics

We will explore the following concepts in this video.

- the rank of a matrix
- invertibility and rank

Objectives

Students should be able to do the following after watching this video and completing the assigned homework.

- apply the rank, basis, and matrix invertibility theorems to describe matrices and subspaces

Rank

Definition

The **rank** of a matrix is the dimension of its column space.

Example: Compute rank(A) and dim(Nul(A)).

$$A = \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & -7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank, Basis, and Invertibility Theorems

Rank Theorem

If a matrix A has n columns, then $\operatorname{Rank} A + \dim(\operatorname{Nul} A) = n$.

Example Construction

If possible give an example of a 2×3 matrix A, that is in RREF and has the given properties.

a) $\text{rank}(A) = 3$

b) $\text{rank}(A) = 2$

c) $\dim(\text{Null}A) = 2$

d) $\text{Null}A = \{\vec{0}\}$

Invertibility

Let A be a $n \times n$ matrix. These conditions are equivalent.

1. A is invertible.
2. The columns of A are a basis for $\mathbb{R}^n$.
3. $\operatorname{Col} A = \mathbb{R}^n$.
4. $\operatorname{rank} A = \dim(\operatorname{Col} A) = n$.
5. $\operatorname{Null} A = \{\vec{0}\}$.

Example Construction

If possible give an example of a matrix A, that is in RREF and has the given properties.

a) A is 2×2, invertible, and rank$A = 1$.

b) A is 4×4, invertible, and rank$A =$ dim(Null$A$)).

Summary

We explored the following concepts in this video.

- the rank of a matrix
- applying rank, basis, and matrix invertibility theorems to describe matrices and subspaces