Chapter 2. Fourier Representation of Signals and Systems – *Review*

Summary: *Time- and frequency-domain representations of signals, properties of signals, time- and frequency-domain analysis of systems (signal transmission), Fourier transform, Fourier series.*

Textbook coverage:

2.1 Fourier Transform **(Haykin & Moher 2.1)**

2.2 Properties of Fourier Transform **(Haykin & Moher 2.2)**

2.3 Fourier Series and Fourier Transform of Periodic Signals **(Haykin & Moher 2.4 and 2.5)**

2.4 Transmission of Signals through Linear Time-Invariant Systems **(Haykin & Moher 2.6)**

2.5 Filters **(Haykin & Moher 2.7 and more)**

2.6 Energy Spectral Density and Autocorrelation Function for Energy Signals **(Haykin & Moher 2.8)**

2.7 Power Spectral Density and Autocorrelation Function for Power Signal **(Haykin & Moher 2.9 and more)**

Signal: A set of data that is functions of one or more independent variables.

- Examples: speech signal, air pressure as a function of time, etc.
- Focus on single-variable continuous signal: A function of time $g(t)$.

Fundamental signals:

- Dirac delta function: $\delta(t) = 0 \; for \; t \neq 0 \;\; and \;\; \int_{-\infty}^{\infty} \delta(t)dt = 1$

$$\int_{-\infty}^{\infty} g(t)\delta(t - t_0)dt = g(t_0);$$

- Signum function: $\operatorname{sgn}(t) = \begin{cases} 1 & , \; t > 0 \\ 0 & , \; t = 0 \\ -1 & , \; t < 0 \end{cases}$

- Unit step function: $\mathrm{u}(t) = \begin{cases} 1 & , \; t > 0 \\ 1/2 & , \; t = 0 \\ 0 & , \; t < 0 \end{cases}$

__Fundamental signals (cont):__

- Unit rectangular function: $$\mathrm{rect}(t) = \begin{cases} 1, & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0, & t < -\frac{1}{2} \text{ or } t > \frac{1}{2} \end{cases}$$

- Unit triangle function: $$\Delta(t) = \begin{cases} 0 & |t| \ge \frac{1}{2} \\ 1 - 2|t| & |t| < \frac{1}{2} \end{cases}$$

- Sinc function: $$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

- Complex exponential function: $\mathrm{x(t)} = e^{jw_0 t} = \cos(w_0 t) + j\sin(w_0 t)$

Periodic signal: $g(t) = g(t + nT)$

- T: fundamental period (the smallest period) of signal $g(t)$

Symmetric signal

- Even signal: $g(t) = g(-t)$ - Odd signal: $g(t) = -g(-t)$

Q: is $\mathrm{x(t)} = e^{jw_0 t}$ a periodic signals?

Signal: A set of data that is functions of one or more independent variables.

- Examples: speech signal, air pressure as a function of time, etc.
- Focus on single-variable continuous signal: A function of time g(t).

Signal energy & power

- Energy: $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$
- Power: $P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 dt = \lim_{T\to\infty} (E_g/T)$
- Energy signal: $0 < E_g < \infty$
- Power signal: $0 < P_g < \infty$

Q1: Are periodic signals are energy signals?

Q2: Is there any signal that is both energy signal and power signal?

Q3: Is there any signal that is neither energy signal nor power signal?

System: An entity that processes its inputs to produce outputs.

$$\text{input } g(t) \longrightarrow \boxed{\text{System}} \longrightarrow \text{output } y(t)$$

- Processing the input signal to modify it or to extract information from it.
- Focus on single-input-single-output, *linear time-invariant (LTI)* systems.

A system is **linear** if the principle of superposition holds, i.e., its response to the weighted sum of a number of inputs is equal to the weighted sum of its responses when each input is applied individually.

If $g_1(t) \to y_1(t), g_2(t) \to y_2(t)$,

then for any a_1, a_2, $a_1 g_1(t) + a_2 g_2(t) \to a_1 y_1(t) + a_2 y_2(t)$.

Q: Are following systems linear?

(a) $g(t) \to e^{g(t)}$

(b) $g(t) \to g(t) + 1$

A system is **time invariant** if its response to a time-shifted input by any amount of time is equal to the time-shift of its response to the input with the same amount of time.

If $g(t) \rightarrow y(t),$ then for any T, $g(t-T) \rightarrow y(t-T)$.

Q: Are following systems time invariant? $x(t)$ and $y(t)$ represent input and output of the system.

(a) $g(t) \rightarrow g(t) + 1$

(b) $g(t) \rightarrow \sin(t) g(t)$

A system is linear time-invariant (LTI) system if both conditions hold.

example: $g(t) \rightarrow g(t) + 1$ is not a LTI system as it is non-linear.

Consider linear time-invariant (LTI) systems.

The **unit impulse response** of a system, $h(t)$, is its output when the input is the unit impulse function $\delta(t)$.

$$\delta(t) \rightarrow h(t)$$

For an LTI system, its output to any input $g(t)$ is the **convolution** of the input and its unit impulse response:

$$y(t) = g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau) h(t - \tau) d\tau$$

A system is **Causal** if it does not respond before the excitation is applied.

- For a causal LTI system, $h(t) = 0$ for $t < 0$.

A system is **stable** if the output is bounded for all bounded input.

- For a stable LTI system, $\int_{-\infty}^{\infty} |h(t)| dt < \infty$

input $g(t)$ $\longrightarrow$ $\boxed{\text{LTI system } h(t)}$ $\longrightarrow$ output $y(t) = g(t) * h(t)$

Why frequency-domain representation/analysis?

- Understand a signal via its frequency components (Fourier transform, Fourier series)

- Understand a system's response for different frequency components (Frequency response).

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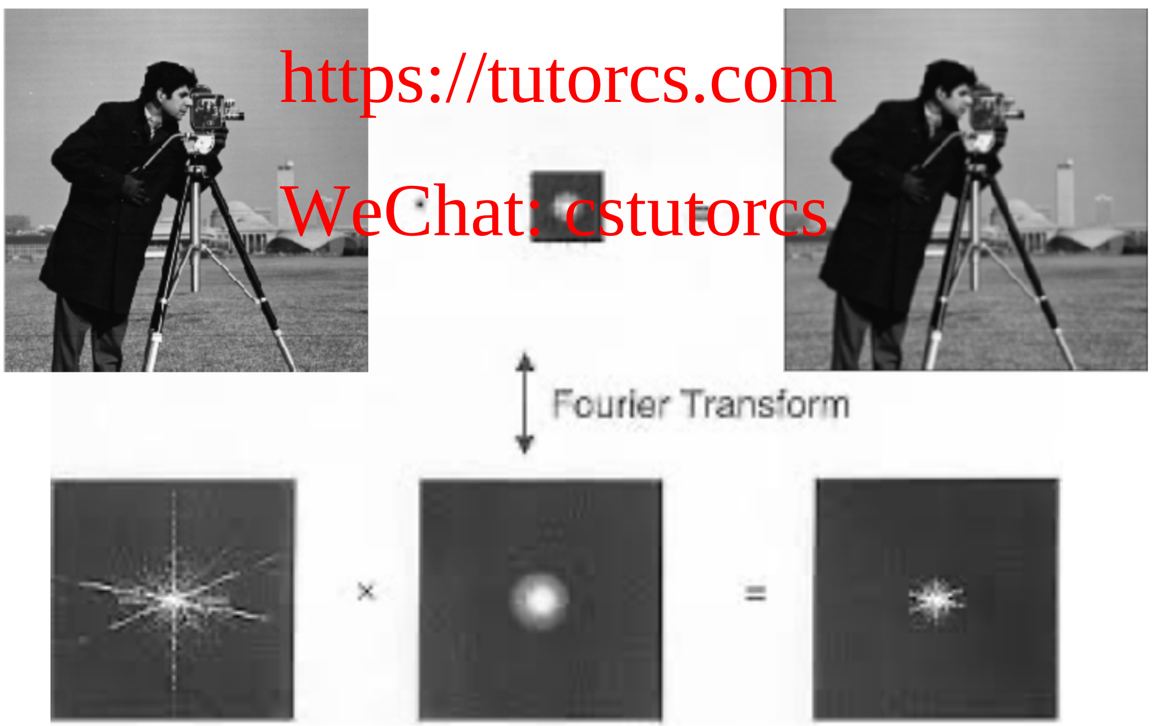


Example from R. Nowak "Intro to digital image processing".

2.1 Fourier Transform

Represent a non-periodic signal as a continuous sum of infinitesimal 'simple' exponential functions

Def. The Fourier transform of a nonperiodic signal $g(t)$ (if Fourier transformable) is:

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi f t)dt.$$

Given the Fourier transform $G(f)$, the original time-domain signal can be recovered by the following inverse Fourier transform:

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi f t)df.$$

$g(t)$ and $G(f)$ constitute a **Fourier transform pair.**

f : frequency in Hertz (Hz).

$\omega = 2\pi f$: frequency in radian/second (rad/sec).

$$G(f) = \int_{-\infty}^{\infty} g(t) \exp(-j2\pi f t) dt. \rightleftharpoons g(t) = \int_{-\infty}^{\infty} G(f) \exp(j2\pi f t) df.$$

Notation:

$$G(f) = \mathbf{F}[g(t)] \quad g(t) = \mathbf{F}^{-1}[G(f)] \quad g(t) \rightleftharpoons G(f)$$

Lower case letter: time function

Upper case letter: frequency function

Q1: Please compute F[$\delta(t)$].

Q2: Given $G(f) = \delta(f), g(t) = ?$

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi f t)dt. \quad \rightleftharpoons \quad g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi f t)df.$$

Table 1: Basic Fourier transform pairs

Time-domain function $g(t)$	Fourier transform $G(f)$
1	$\delta(f)$
$\delta(t)$	1
$u(t)$	$\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$
$\text{rect}\left(\frac{t}{T}\right)$	$T\,\text{sinc}(fT)$
$2W\text{sinc}(2Wt)$	$\text{rect}\left(\frac{f}{2W}\right)$
$\Delta\left(\frac{t}{T}\right)$	$\frac{T}{2}\text{sinc}^2\left(\frac{fT}{2}\right)$
$e^{-at}u(t) \quad (a > 0)$	$\frac{1}{a+j2\pi f}$
$e^{-a\|t\|} \quad (a > 0)$	$\frac{2a}{a^2+(2\pi f)^2}$
$\sin(2\pi f_c t)$	$\frac{1}{2j}[\delta(f - f_c) - \delta(f + f_c)]$
$\cos(2\pi f_c t)$	$\frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$

$G(f)$: Generally, a complex function of frequency f.

$$G(f) = |G(f)| \exp[j\theta(f)] = |G(f)| \angle\theta(f)$$

Amplitude spectrum (magnitude spectrum): $|G(f)|$

Phase spectrum: $\angle\theta(f)$

Prove: for a real-value function $g(t)$,

- $G(-f) = G^*(f)$ → conjugate symmetry
- $|G(-f)| = |G(f)|$ → even function of f
- $\theta(-f) = -\theta(f)$ → odd-symmetric w.r.t. f

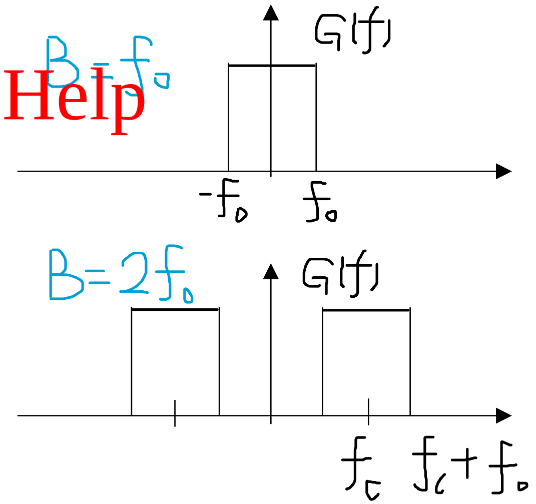
Bandwidth of a signal: The difference between the highest and lowest frequencies of the spectral components of a signal.

- It is convention to state the bandwidth as the range of positive frequencies.
- A signal cannot be strictly limited in both time and frequency.

For <u>band-limited signals:</u>

- <u>Baseband signal:</u> energy centered around zero-frequency.

- <u>Bandpass signal:</u> energy centered around a frequency far away from zero.



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For pulse-like signals:

- main lobe bandwidth for symmetric signals
- 3-dB bandwidth
- 95% essential bandwidth of a signal: Range of the frequency that contains 95% of the energy.

2.2 Properties of Fourier Transform

1. **Linearity**: $c_1 g_1(t) + c_2 g_2(t) \rightleftharpoons c_1 G_1(f) + c_2 G_2(f)$

2. **Time scaling**: $g(t) \rightleftharpoons G(f)$ then $g(at) \rightleftharpoons \frac{1}{|a|} G(\frac{f}{a})$ (Dilation) $\boxed{a \neq 0}$

3. **Conjugation**: $g^*(t) \rightleftharpoons G^*(-f)$

4. **Duality**: $g(t) \rightleftharpoons G(f)$ then $G(t) \rightleftharpoons g(-f)$

5. **Time shifting**: $g(t - t_0) \rightleftharpoons e^{-j2\pi f t_0} G(f)$

6. **Frequency shifting:** $g(t) \exp(j2\pi f_c t) \rightleftharpoons G(f - f_c)$ for any constant f_c.

7. **Areas under $g(t)$ and $G(f)$:**

$$\int_{-\infty}^{+\infty} g(t)dt = G(0) \text{ and } \int_{-\infty}^{+\infty} G(f)df = g(0)$$

8. **Differentiation and integration:**

$$\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f) \quad \text{and} \quad \int_{-\infty}^{t} g(\tau)d\tau \rightleftharpoons \frac{1}{j2\pi f}G(f) + \frac{1}{2}G(0)\delta(f)$$

9. **Convolution and Modulation:**

$$g_1(t) * g_2(t) \rightleftharpoons G_1(f) \cdot G_2(f) \text{ and } g_1(t) \cdot g_2(t) \rightleftharpoons G_1(f) * G_2(f)$$

10. **Parseval's theorem:**

$$E_g = \int_{-\infty}^{+\infty} |g(t)|^2 dt = \int_{-\infty}^{+\infty} |G(f)|^2 df$$

Energy Spectral Density & Autocorrelation Function for Energy Signals

For an energy signal $g(t)$, i.e. $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt < \infty$, its **energy spectral density** is defined as:

$$\Psi_g(f) = |G(f)|^2, \text{ where } g(t) \rightleftharpoons G(f).$$

Wiener-Khitchine Relation: energy signal's autocorrelation function and energy spectral density function form a Fourier transform pair:

$$R_g(\tau) \rightleftharpoons \Psi_g(f),$$

where the **autocorrelation function** $R_g(\tau) = \int_{-\infty}^{\infty} g(t) g^*(t-\tau) dt$.

Energy spectral density tells how signal energy spreads over frequencies, so we have the Parseval's Theorem:

$$E_g = \int_{-\infty}^{\infty} |G(f)|^2 df = \int_{-\infty}^{\infty} \Psi_g(f) df = R_g(0) = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

Parseval's Power theorem & Power Spectral Density of Power Signals

For a power signal g(t), i.e. $P_g = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|g(t)|^2 dt < \infty$, define

$$g_T(t) = \begin{cases} g(t) & -\frac{T}{2} \le t \le T/2 \\ 0. & otherwise \end{cases}$$

Then $g_T(t)$ is an energy signal.

From $g_T(t) \rightleftharpoons G_T(f)$, **Parseval's power theorem** is

$$P_g = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|g(t)|^2 dt = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}|G_T(f)|^2 df$$

$$= \int_{-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{T}|G_T(f)|^2\, df = \int_{-\infty}^{\infty}S_g(f)df$$

Define **power spectral density (PSD)** of $g(t)$ as:

$$S_g(f) = \lim_{T\to\infty} \frac{1}{T} |G_{T_0}(f)|^2$$

PSD tells how signal power spreads over frequencies.

The **auto-correlation function** of power signal $g(t)$ is

$$R_g(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g(t) g^*(t-\tau) dt$$

Similarly, we have $R_g(\tau) \rightleftarrows S_g(f)$ and $P_g = \int_{-\infty}^{\infty} S_g(f) df = R_g(0)$

2.3 Fourier Series and Fourier Transform of Periodic Signals

$g_{T_0}(t)$: A periodic signal with period T_0.

Fundamental frequency: $f_0 = \frac{1}{T_0}$

A periodic signal $g_{T_0}(t)$ can be represented as a sum of complex exponential (complex exponential Fourier series):

$$g_{T_0}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_0 t)$$

where $c_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} g_{T_0}(t) e^{-j2\pi n f_0 t} dt.$

Fourier transform of $g_{T_0}(t)$ is:

$$G_{T_0}(f) = \sum_{n=-\infty}^{+\infty} c_n \delta(f - n f_0)$$

For periodic signal $g_{T_0}(t)$ with period T_0

Fundamental freq. $f_0 = \frac{1}{T_0}$

FS: $g_{T_0}(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$, $C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_{T_0}(t) e^{-j2\pi n f_0 t} dt$

$\Rightarrow$ FT: $G_{T_0}(f) = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_0)$

To find C_n for $\cos(2\pi f_0 t)$ by definition:

$$C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \cos(2\pi f_0 t)\, e^{-j2\pi n f_0 t} dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \frac{1}{2}\left[e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right] e^{-j2\pi n f_0 t} dt$$

$$= \frac{1}{2T_0}\int_{-T_0/2}^{T_0/2} e^{j2\pi f_0 (1-n)t} + e^{-j2\pi f_0 (n+1)t} dt$$

$$= \frac{1}{2T_0}\left[\int_{-T_0/2}^{T_0/2} e^{j2\pi f_0 (1-n)t} dt + \int_{-T_0/2}^{T_0/2} e^{-j2\pi f_0 (n+1)t} dt\right]$$

$$= \frac{1}{2T_0}\left[\frac{e^{j2\pi f_0(1-n)t}}{j2\pi f_0 (1-n)}\Bigg|_{-T_0/2}^{T_0/2} + \frac{e^{-j2\pi f_0 (n+1)t}}{-j2\pi f_0 (n+1)}\Bigg|_{-T_0/2}^{T_0/2}\right]$$

$$= \frac{1}{2T_0}\left[\frac{e^{j2\pi f_0 (1-n)\frac{T_0}{2}} - e^{-j2\pi f_0 (1-n)\frac{T_0}{2}}}{j2\pi f_0 (1-n)} - \frac{e^{-j2\pi f_0 (n+1)T_0/2} - e^{j2\pi f_0 (n+1) T_0/2}}{j2\pi f_0 (n+1)}\right]$$

$f_0 = \frac{1}{T_0} \Rightarrow f_0 T_0 = 1$

$$= \frac{1}{2T_0}\left[\frac{e^{j\pi(1-n)} - e^{-j\pi(1-n)}}{j2\pi f_0 (1-n)} - \frac{e^{-j\pi(n+1)} - e^{j\pi(n+1)}}{j2\pi f_0 (n+1)}\right]$$

$$= \frac{1}{2T_0}\left[\frac{2j\sin[\pi(1-n)]}{j2\pi f_0 (1-n)} + \frac{2j\sin[\pi(n+1)]}{j2\pi f_0 (n+1)}\right]$$

$$= \frac{\sin[\pi(1-n)]}{2\pi(1-n)} + \frac{\sin[\pi(n+1)]}{2\pi(n+1)}$$

$$= \begin{cases} 0 & n \neq \pm 1 \\ \frac{\sin[\pi\cancel{(1-n)}^{0}]}{2\pi\cancel{(1-n)}_{0}}\Big|_{n=1} + 0 & n = 1 \\ 0 + \frac{\sin[\pi\cancel{(1+n)}^{0}]}{2\pi\cancel{(1+n)}^{0}} & n = -1 \end{cases}$$

$$= \begin{cases} 0 & n \neq \pm 1 \\ \frac{-\cancel{\pi}\cos[\pi(1-n)]}{-2\cancel{\pi}}\Big|_{n=1} & n = 1 \\ \frac{\cancel{\pi}\cos[\pi(1+n)]}{2\cancel{\pi}}\Big|_{n=-1} & n = -1 \end{cases}$$

$$= \begin{cases} 0 & n \neq \pm 1 \\ \frac{1}{2} & n = 1 \\ \frac{1}{2} & n = -1 \end{cases}$$

$$\Rightarrow \cos(2\pi f_0 t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi f_0 n t} = \frac{1}{2}e^{j2\pi f_0 t} + \frac{1}{2}e^{-j2\pi f_0 t}$$

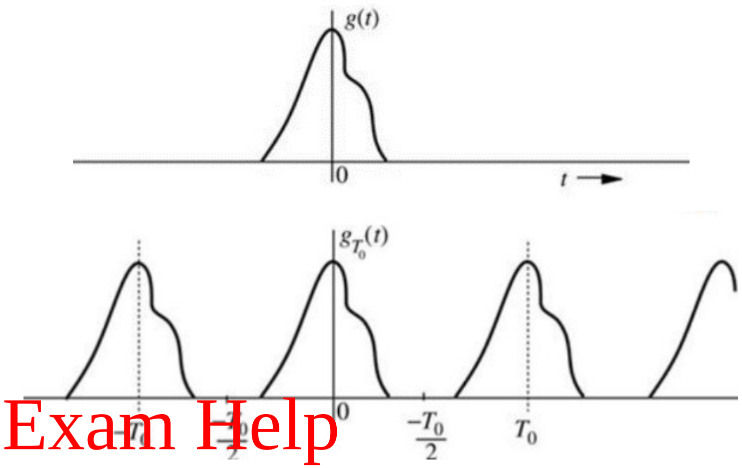
$$\Rightarrow \mathcal{F}[\cos(2\pi f_0 t)] = \sum_{n=-\infty}^{\infty} C_n \delta(f - nf_0) = \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0) \quad //.$$

Relation between FT and FS

Define two signals $g(t)$ and $g_{T_0}(t)$:

$$g_{T_0}(t) = \sum_{m=-\infty}^{\infty} g(t - mT_0)$$

$$g(t) = \begin{cases} g_{T_0}(t) & -\frac{T_0}{2} \le t \le \frac{T_0}{2} \\ 0 & \text{otherwise} \end{cases}.$$



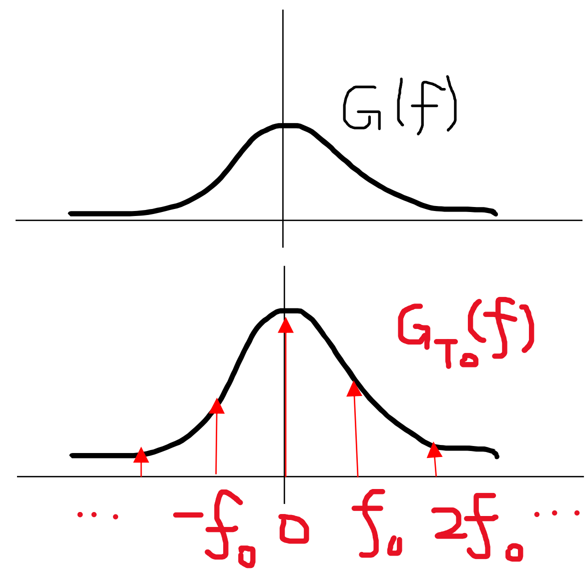
Let $g(t) \rightleftharpoons G(f)$ $\quad g_{T_0}(t) \rightleftharpoons G_{T_0}(f)$,

we have $\quad c_n = \frac{1}{T_0} G(nf_0) = f_0 G(nf_0).$

That is,

$$G_{T_0}(f) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0).$$

Repetition in the time domain results in sampling in the frequency domain.



Power spectral density for periodic signal $g_{T_0}(t)$ with period T_0.

$$G_{T_0}(f) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0). \qquad f_0 = \frac{1}{T_0}$$

Then its power spectral density is

$$S_{g_{T_0}}(f) = \sum_{n=-\infty}^{\infty} |c_n|^2 \delta(f - nf_0) = \sum_{n=-\infty}^{\infty} f_0^2 |G(nf_0)|^2 \delta(f - nf_0)$$

Parseval's power theorem for periodic signal:

$$\frac{1}{T}\int_{-T/2}^{T/2} |g(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

2.4 Transmission of signals through LTI systems

System: An entity that processes its inputs to produce outputs.

$$\text{input } g(t) \longrightarrow \boxed{\text{System}} \longrightarrow \text{output } y(t)$$

- Processing the input signal to modify it or to extract information from it.
- Focus on single-input-single-output, *linear time-invariant (LTI)* system.

The **unit impulse response** of a system, $h(t)$, is its output when the input is the unit impulse function $\delta(t)$.

$$\delta(t) \rightarrow h(t)$$

For an LTI system, its output to any input $g(t)$ is the **convolution** of the input and its unit impulse response:

$$y(t) = g(t) * h(t) = \int_{-\infty}^{\infty} g(\tau)h(t-\tau)d\tau$$

For an LTI system, its **frequency response** is the Fourier transform of its unit impulse response:

$$H(f) = \int_{-\infty}^{\infty} h(t) \exp(-j2\pi f t) dt. \quad h(t) \rightleftharpoons H(f)$$

- Magnitude response $|H(f)|$
- Phase response $\theta(f)$, or $\angle H(f)$

The output of an LTI system with any input $g(t)$ in freq. domain:

$$Y(f) = H(f)G(f) \rightleftharpoons y(t) = g(t) * h(t)$$

Given any LTI system's input-output pair $g(t) \rightarrow y(t)$, the impulse response of the system can be specified in two steps;

- Step 1: $H(f) = \frac{Y(f)}{G(f)}$
- Step 2: $h(t) = \mathcal{F}^{-1}(H(f)$

Filters

- Filters are systems that are designed to remove some unwanted components or features from a signal.
  - Low-pass filter
  - High-pass filter
  - Band-pass filter
  - Band-reject or notch filters

- Cutoff frequencies: frequencies beyond which the filter will not pass the signal components.

- 3-dB frequencies: the frequency at which the power-transfer ratio of the filter drops to half of the maximum, i.e., the energy spectral density (or PSD) at this frequency is half (3dB drop) of its maximum.

Effect of filtering (LTI system) on energy spectral density

$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$Y(f) = H(f)G(f) \Rightarrow \Psi_y(f) = |H(f)|^2 \Psi_g(f).$$

The energy spectral density of the output equals the energy spectral density of the input multiplied by the squared amplitude of frequency response of the LTI system.

Effect of filtering (LTIC system) on power spectral density

$$g(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$S_y(f) = |H(f)|^2 S_g(f)$$