Solution to Homework Assignment 3

Solution to Problem 1: $T_0 = 3$ and $f_0 = 1/3$.

(a) Consider the periodic signal $x_{T_0}(t) = g_{T_0}(t) + 2$. Take one period of $x_{T_0}(t)$ for $0 \le t < 3$ and call it $x(t)$. We have $x(t) = 3\text{rect}\left(\frac{t-1}{2}\right)$. Thus

$$x(t) \rightleftharpoons X(f) = 6\,\text{sinc}(2f)e^{-j2\pi f}.$$

$$X_{T_0}(f) = 2\sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{2n}{3}\right) e^{-j\frac{2n\pi}{3}}\delta\left(f - \frac{n}{3}\right).$$

Since $g_{T_0}(t) = x_{T_0}(t) - 2$,

$$G_{T_0}(f) = X_{T_0}(f) - 2\delta(f) = 2\sum_{n=-\infty, n\neq 0}^{\infty} \text{sinc}\left(\frac{2n}{3}\right) e^{-j\frac{2n\pi}{3}}\delta\left(f - \frac{n}{3}\right).$$

Thus, the power spectral density of $g_{T_0}(t)$ can be calculated as

$$S_{g_{T_0}}(f) = |G_{T_0}(f)|^2 = 4\sum_{n=-\infty, n\neq 0}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right)\delta\left(f - \frac{n}{3}\right).$$

(b) From inverse Fourier transform, the autocorrelation function is

$$\begin{aligned}
R_{g_{T_0}}(\tau) &= 4\sum_{n=-\infty, n\neq 0}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) e^{j\frac{2\pi n}{3}\tau} \\
&= 4\left[\sum_{n=-\infty}^{-1} \text{sinc}^2\left(\frac{2n}{3}\right) e^{j\frac{2\pi n}{3}\tau} + \sum_{n=1}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) e^{j\frac{2\pi n}{3}\tau}\right] \\
&= 4\left[\sum_{n=1}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) e^{-j\frac{2\pi n}{3}\tau} + \sum_{n=1}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right) e^{j\frac{2\pi n}{3}\tau}\right] \\
&= 8\sum_{n=1}^{\infty} \text{sinc}^2\left(\frac{2n}{3}\right)\cos\left(\frac{2\pi n}{3}\tau\right)
\end{aligned}$$

Notice that the imaginary part of $R_{g_{T_0}}(\tau)$ is zero. That is, $R_{g_{T_0}}(\tau)$ is a real valued function. The autocorrelation function can be drawn by Matlab in Figure 1 (by taking 60 terms of the summation, 30 for positive n and 30 negative n). We can see that the autocorrelation function is a periodic triangular-plus-square wave with period 3.

Solution to Problem 2: Since

$$400\pi e^{-200\pi t}u(t) \rightleftharpoons \frac{400\pi}{200\pi + j2\pi f} = \frac{2}{1 + j\frac{f}{100}},$$

$$\cos(20{,}000\pi t) = \frac{1}{2}\left[\delta(f - 10{,}000) + \delta(f + 10{,}000)\right],$$

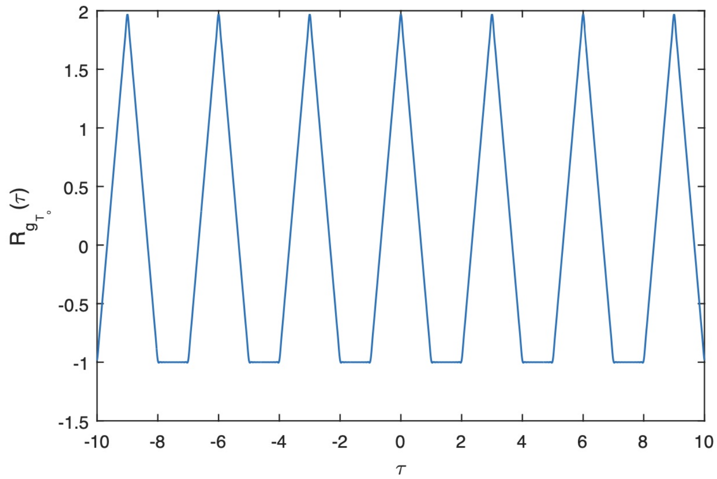

Figure 1: Solution to Problem 1.

we have

$$H_1(f) = \frac{2}{1 + j\frac{f}{100}}$$

$$H_2(f) = \mathbf{F}[h_2(t)] = \frac{1}{1 + j\frac{f-10{,}000}{100}} + \frac{1}{1 + j\frac{f+10{,}000}{100}}.$$

(b) Since $|H_1(f)|$ has 1 peaks at 0Hz (maximum magnitude which is 2 at $f = 0$kHz and decreasing magnitude on both sides of the peak). It is a low-pass filter. For the 3-dB cut-off frequencies:

$$|H_1(f_{3dB})| = \frac{2}{\sqrt{2}} \Rightarrow \frac{2}{\sqrt{1 + f_{3dB}^2/10000}} = \frac{2}{\sqrt{2}} \Rightarrow f = \pm 100 \text{ HZ}.$$

The 3-dB bandwidth of the low-pass filter is $[0, 100]$Hz.

(c) $H_2(f)$ is the shifted version of $H_1(f)$ in the frequency domain. It has 2 peaks at 10kHz and -10kHz (maximum magnitude which is 1 at $f = \pm 10$kHz and decreasing magnitude on both sides of the two peaks). Hence, $H_2(t)$ is a band-pass filter centered at $f = \pm 10$kHz. The 3-dB frequency pass band of the filter is $[9, 900, 10, 100]$Hz. That is, the 3-dB cutoff frequencies are 9,900 Hz and 10,100 Hz.

Solution to Problem 3:

(a) Since the RC circuits are in cascade, the impulse response of the overall system can be obtained by convolving the impulse responses of all sub-systems. Thus, the transfer function of the overall system is

$$H(f) = \prod_{i=1}^{N} H_i(f) = \frac{1}{(1 + j2\pi f RC)^N}.$$

Hence, the magnitude response is

$$|H(f)| = [H(f)H^*(f)]^{1/2} = \frac{1}{[1 + 4\pi^2(RC)^2 f^2]^{N/2}}.$$

(b)

$$|H(f)| = \frac{1}{[1+T^2f^2/N]^{N/2}} = \frac{1}{[1+T^2f^2/N]^{\frac{N}{T^2f^2}\frac{T^2f^2}{2}}} = e^{-f^2T^2/2}.$$

Solution to Problem 4: (a)

$$G(f) = \frac{2}{1+(2\pi f)^2},$$

Thus

$$\Psi_g(f) = |G(f)|^2 = \frac{4}{[1+(2\pi f)^2]^2}.$$

The energy spectral density of the output is

$$\Psi_y(f) = |H(f)|^2\Psi_{g_1}(f) = \frac{4}{1+(2\pi f)^2}.$$

Thus

$$E_y = \int_{-\infty}^{\infty} \frac{4}{1+(2\pi f)^2}df = \frac{2}{\pi}\arctan(2\pi f)|_{-\infty}^{\infty} = 2.$$

(b) The energy spectral density of the output is

$$\Psi_y(f) = |H(f)|^2\Psi_g(f) = \begin{cases} \frac{4}{1+(2\pi f)^2} & 0 < |f| \leq f_0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus

$$E_y = 2\int_{-f_0}^{f_0} \frac{4}{1+(2\pi f)^2}df = \frac{4}{\pi}\arctan(2\pi f_0).$$

This solution demonstrates that different phase responses of LTI systems does not change the energy spectral density of the output. Rather, the magnitude responses effect output's energy spectral density.

Solution to Problem 5: For the input signal $x(t)$, from the figure, $H(f) = rect(f/4) + rect(f/2)$. So

$$\begin{aligned} \Psi_x(f) &= |X(f)|^2 = [rect(f/4) + recr(f/2)]^2 \\ &= rect^2(f/4) + rect^2(f/2) + 2rect(f/4)rect(f/2) \\ &= rect(f/4) + 3rect(f/2), \\ R_x(\tau) &= \mathcal{F}^{-1}[\Psi_x(f)] = 4sinc(4\tau) + 6sinc(2\tau), \\ E_x &= R_x(0) = 10. \end{aligned}$$

For the output signal $y(t) = x(t) * h(y)$, where $H(f) = rect(f/2)$.So,

$$\begin{aligned} Y(f) &= X(f)H(f) = 2rect(f/2), \\ \Psi_y(f) &= |Y(f)|^2 = 4recr(f/2), \\ R_Y(\tau) &= \mathcal{F}^{-1}[\Psi_y(f)] = 8sinc(2\tau) \\ E_y &= R_x(0) = 8 \end{aligned}$$