Solution to Homework Assignment 2

Solution to Problem 1:

(a) proof $\frac{d}{dt} g(t) \rightleftharpoons j2\pi f G(f)$

Given $g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$,

$$\frac{d}{dt} g(t) = \frac{d}{dt} \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} dt = \int_{-\infty}^{\infty} G(f) \frac{d}{dt} e^{j2\pi f t} df$$

$$= \int_{-\infty}^{\infty} G(f)\, j2\pi f\, e^{j2\pi f t} df = j2\pi f \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df$$

$$= j2\pi f\, G(f) \quad //.$$

(b) Proof $g_1(t) \rightleftharpoons G_1(f)$, $g_2(t) \rightleftharpoons G_2(f)$, then $g_1(t) * g_2(t) \rightleftharpoons G_1(f) G_2(f)$

$$\mathcal{F}[g_1(t) * g_2(t)] = \int_{-\infty}^{\infty} [g_1(t) * g_2(t)] e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) d\tau \right] e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(\tau) g_2(t-\tau) e^{-j2\pi f t} d\tau\, dt$$

Let $s = t-\tau$, then $t = s+\tau$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_1(\tau) g_2(s) e^{-j2\pi f (s+\tau)} d\tau\, d(s+\tau)$$

$$= \left[\int_{-\infty}^{\infty} g_1(\tau) e^{-j2\pi f \tau} d\tau \right] \left[\int_{-\infty}^{\infty} g_2(s) e^{-j2\pi f s} ds \right]$$

$$= G_1(f) G_2(f) \quad //.$$

(c) Proof $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} |G(f)|^2 df$.

From the def. of Energy, $E_g = \int_{-\infty}^{\infty} |g(t)|^2 dt$.

$$\int_{-\infty}^{\infty} |g(t)|^2 dt = \int_{-\infty}^{\infty} g(t) g^*(t)\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df \right] \left[\int_{-\infty}^{\infty} G(s) e^{j2\pi s t} ds \right]^* dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df \right] \left[\int_{-\infty}^{\infty} G^*(s) e^{-j2\pi s t} ds \right] dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(f) G^*(s) e^{j2\pi (f-s) t} df\, ds \right] dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(f) G^*(s) \left[\int_{-\infty}^{\infty} e^{j2\pi (f-s) t} dt \right] df\, ds$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(f) G^*(s)\, \delta(s-f)\, df\, ds$$

$$= \int_{-\infty}^{\infty} G(f) G^*(f)\, df$$

$$= \int_{-\infty}^{\infty} |G(f)|^2 df \quad //.$$

$\int_{-\infty}^{\infty} e^{j2\pi (f-s) t} dt$

$= \int_{-\infty}^{\infty} e^{j2\pi f t} e^{-j2\pi s t} dt$

$= \mathcal{F}[e^{j2\pi f t}]$

Given $1 \rightleftharpoons \delta(s)$,

$1 e^{j2\pi f t} \rightleftharpoons \delta(s-f)$

Freq. shift property!

Solution to Problem 2:

(a) From the FT pair table, we get

$$G(f) = \frac{2}{1 + (2\pi f)^2},$$

Thus

$$\Psi_g(f) = |G(f)|^2 = \frac{4}{[1 + (2\pi f)^2]^2}.$$

(b) $g_1(t) = g(t-2)$. Thus $G_1(f) = G(f)e^{-j4\pi f}$ (time-shifting property). Since $\exp(-j4\pi f)$ has unit amplitude for all f, we have $\Psi_{g_1}(f) = \Psi_g(f)$, meaning that the signal $g_1(t)$ has the same energy spectral density as the signal $g(t)$.

Solution to Problem 3: Define $g(t) = \begin{cases} \cos t & 0 \le t < \pi \\ 0 & \text{otherwise} \end{cases}$.

$$G(f) = \int_0^{\pi} \cos t e^{-j2\pi f t} dt = \frac{e^{-j2\pi f t}}{1 + (-j2\pi f)^2}[-j2\pi f \cos t + \sin t]\Big|_0^{\pi} = \frac{j2\pi f(1 + e^{-j2\pi^2 f})}{1 - 4\pi^2 f^2}.$$

Notice that $T_0 = \pi$ and $f_0 = 1/\pi$. Thus

$$G_{T_0}(f) = f_0 \sum_{n=-\infty}^{\infty} G(nf_0)\delta(f - nf_0) = \sum_{n=-\infty}^{\infty} \frac{j4n}{\pi(1 - 4n^2)} \delta\left(f - \frac{n}{\pi}\right).$$

The frequency spectra are drawn as below: