Solution to Homework Assignment 1

Solution to Problem 1:

(a)

$$G(f) = \begin{cases} e^{-j2\pi f t_0} & f \in [-f_0, f_0] \\ 0 & \text{elsewhere} \end{cases}.$$

By using the definition of inverse FT:

$$f(t) = \int_{-f_0}^{f_0} e^{-j2\pi f t_0} e^{j2\pi f t} df = \int_{-f_0}^{f_0} e^{j2\pi f(t-t_0)} df = \frac{1}{j2\pi(t-t_0)} e^{j2\pi f(t-t_0)}|_{-f_0}^{f_0}$$
$$= \frac{\sin[2\pi f_0(t-t_0)]}{\pi(t-t_0)} = 2f_0 \text{sinc}[2f_0(t-t_0)].$$

(b) $G(f) = \text{rect}\left(\frac{f}{2f_0}\right)$. Thus $g(t) = 2f_0\text{sinc}(2f_0 t)$.
Or from the definition of inverse FT:

$$f(t) = \int_{-f_0}^{f_0} e^{j2\pi f t} df = \frac{1}{j2\pi t} e^{j2\pi f t}|_{-f_0}^{f_0} = \frac{\sin(2\pi f_0 t)}{\pi t} = 2f_0 \text{sinc}(2f_0 t).$$

Solution to Problem 2: From the plot,

$$g_1(t) = \sin t[u(t) - u(t-\pi)].$$

By definition:

$$\begin{aligned}
G_1(f) &= \int_0^{\pi} \sin(t) e^{-j2\pi f t} dt \\
&= -\int_0^{\pi} e^{-j2\pi f t} d\cos(t) \\
&= -\cos(t) e^{-j2\pi f t}|_0^{\pi} + \int_0^{\pi} \cos(t) de^{-j2\pi f t} \\
&= 1 + e^{-j2\pi^2 f} + (-j2\pi f) \int_0^{\pi} cos(t) e^{-j2\pi f t} dt \\
&= 1 + e^{-j2\pi^2 f} + (-j2\pi f) \int_0^{\pi} e^{-j2\pi f t} d\sin(t) \\
&= 1 + e^{-j2\pi^2 f} + (-j2\pi f)\left[\sin(t) e^{-j2\pi f t}|_0^{\pi} - \int_0^{\pi} \sin(t) de^{-j2\pi f t}\right] \\
&= 1 + e^{-j2\pi^2 f} - (-j2\pi f)^2 \int_0^{\pi} \sin(t) e^{-j2\pi f t} dt \\
&= 1 + e^{-j2\pi^2 f} + 4\pi^2 f^2 G_1(f)
\end{aligned}$$

After move the $4\pi^2 f^2 G_1(f)$ to the left of the equation, we get

$$G_1(f) - 4\pi^2 f^2 G_1(f) = 1 + e^{-j2\pi^2 f}.$$

Hence,

$$G_1(f) = \frac{1+e^{-j2\pi^2 f}}{1-4\pi^2 f^2}$$

By properties: Notice that

$$g_1(t) = \sin t\, u(t) + \sin(t-\pi)u(t-\pi).$$
$$\sin t\, u(t) \Longleftrightarrow \frac{\pi}{2j}\left[\delta\left(f-\frac{1}{2\pi}\right) - \delta\left(f+\frac{1}{2\pi}\right)\right] + \frac{1}{1-(2\pi f)^2}.$$

By using the time-shifting property,

$$\sin(t-\pi)\, u(t-\pi) \Longleftrightarrow \left\{\frac{\pi}{2j}\left[\delta\left(f-\frac{1}{2\pi}\right) - \delta\left(f+\frac{1}{2\pi}\right)\right] + \frac{1}{1-(2\pi f)^2}\right\} e^{-j2\pi^2 f}.$$

By linearity of FT, we have

$$G(f) = \left\{\frac{\pi}{2j}\left[\delta\left(f-\frac{1}{2\pi}\right) - \delta\left(f+\frac{1}{2\pi}\right)\right] + \frac{1}{1-(2\pi f)^2}\right\}(1+e^{-j2\pi^2 f}).$$

As $\delta\left(f \pm \frac{1}{2\pi}\right)(1+e^{-j2\pi^2 f}) = \delta\left(f \pm \frac{1}{2\pi}\right)(1+e^{\pm j\pi}) = 0$, we have

$$F(\omega) = \frac{1+e^{-j2\pi^2 f}}{1-4\pi^2 f^2}.$$

For the second signal,

$$g_2(t) = e^{-at}[u(t) - u(t-T)].$$

By definition,

$$G_2(f) = \int_0^T e^{-at} e^{-j2\pi f t}\, dt = -\frac{1}{a+j2\pi f} e^{-(a+j2\pi f)t}\Big|_0^T = \frac{1-e^{-(a+j2\pi f)T}}{a+j2\pi f}$$

Notice that

$$g_2(t) = e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t-T).$$
$$e^{-at}u(t) \Longleftrightarrow \frac{1}{a+j2\pi f}.$$

By using the time-shifting property,

$$e^{-a(t-T)}u(t-T) \Longleftrightarrow \frac{1}{a+j2\pi f}e^{-j2\pi f T}.$$

By the linearity of FT, we have

$$F(\omega) = \frac{1-e^{-(a+j2\pi f)T}}{a+j2\pi f}.$$

Solution to Problem 3:

(a) Since

$$g(t)\sin(2\pi f_c t) = \frac{1}{2j}g(t)\left[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right],$$

from frequency-shifting property and linearity,

$$g(t)\sin(2\pi f_c t) \rightleftharpoons \frac{1}{2j}[G(f-f_c) - G(f+f_c)].$$

(b) Since $2+\cos(2\pi f_0 t) \rightleftharpoons 2\delta(f) + \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)$, by using the property in (a), we have

$$\begin{aligned} S(f) = \frac{1}{2j}\Bigg[& 2\delta(f-100) + \frac{1}{2}\delta(f-100-f_0) + \frac{1}{2}\delta(f-100+f_0) \\ & -2\delta(f+100) - \frac{1}{2}\delta(f+100-f_0) - \frac{1}{2}\delta(f+100+f_0)\Bigg]. \end{aligned}$$

The spectrum is as in Figure 1.

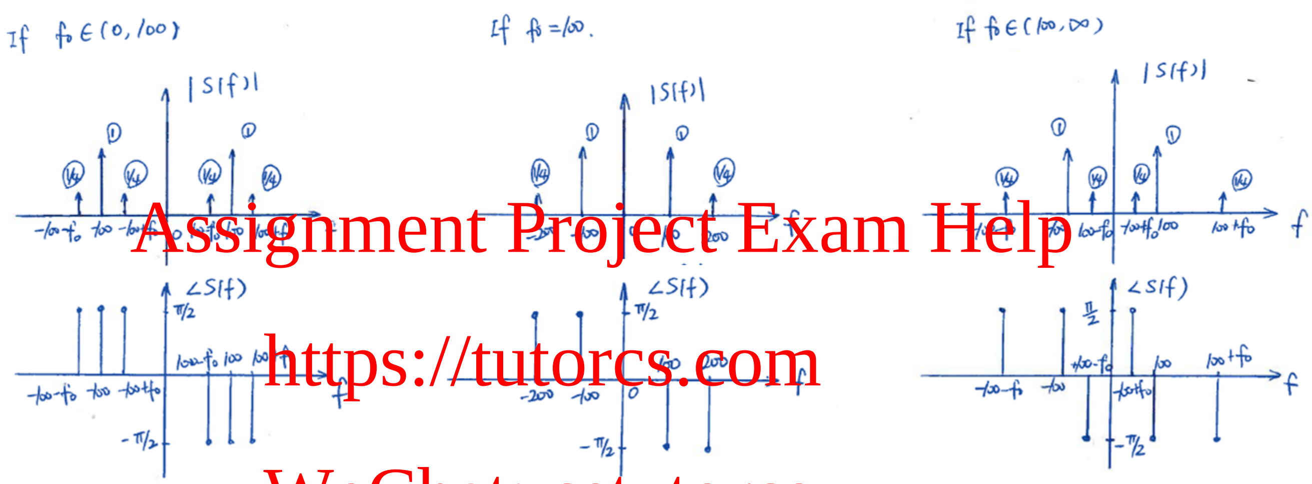

Figure 1: Spectra for Problem 3.

Solution to Problem 4:

(a) The signals $u(t), u_e(t), u_o(t)$ are represented by the following figure.

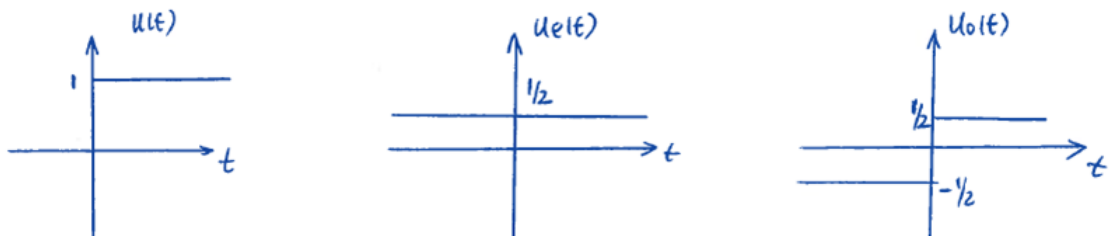

Figure 2: Signals for Problem 4.

(b)We have

$$u_e(t) = \frac{1}{2} \rightleftharpoons \frac{1}{2}\delta(f).$$

$$u_o(t) = \frac{1}{2}\text{sgn}\,(t) \rightleftharpoons \frac{1}{j2\pi f}.$$

$$u(t) \rightleftharpoons \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}.$$