Solution to Homework Assignment 6

Solution to Problem 1: (a) For FM wave, since

$$\int_{-\infty}^{t} m(\tau)d\tau = \frac{A}{2T_0}t^2 \text{ for } t \in [0, T_0),$$

and repeats for every T_0 seconds for $t \geq T_0$. For $t \in [0, T_0)$,

$$s_{FM}(t) = \cos\left(2\pi f_c t + 2\pi k_p \frac{A}{2T_0}t^2\right) = \cos(8\pi t + 10\pi t^2).$$

The FM wave is show in the Figure 1. Specifically, the FM signal is a cos-wave with increasing instantaneous frequency (the instantaneous frequency value starts from f_c) for $t \in (0, T_0)$. Then the instantaneous frequency becomes f_c again at $t = T_0$ and keeps increases for $t \in (T_0, 2T_0)$.

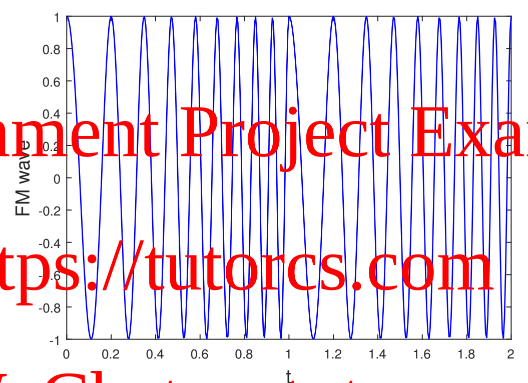

Figure 1: FM wave in Problem 1.

(b) Since the message signal has a period of $T_0 = 1$, it frequency is 1Hz. Thus its fifth harmonic frequency is 5Hz. From the problem, the bandwidth of $m(t)$ is defined by the fifth harmonic frequency of $m(t)$. That is, $W = 5$Hz. For FM wave,

$$\beta = \frac{k_f \max|m(t)|}{W} = 2$$

$$B_T = 2(\beta + 1)W = 2(2 + 1) * 5 = 30\text{Hz}$$

Solution to Problem 2: (a) From Carson's rule, $B_T = 2(\beta + 1)f_m$ for single-tone angle modulation. For FM modulation,

$$\beta = \frac{k_f \max|m(t)|}{f_m} = \frac{100000}{1000} = 100$$

$$B_T = 2(\beta + 1)f_m = 2(100 + 1) * 1000 = 202{,}000\text{Hz}$$

For PM modulation,

$$\beta = k_p \max|m(t)| = 10$$

$$B_T = 2(\beta + 1)f_m = 2(10 + 1) * 1000 = 22,000\text{Hz}$$

(b) If the message amplitude doubles, i.e. $m(t) = 2\sin(2000\pi t)$, both β in FM modulation and PM modulation double. That is, for FM modulation,

$$\beta = \frac{k_f \max|m(t)|}{f_m} = \frac{100000 * 2}{1000} = 200$$

$$B_T = 2(\beta + 1)f_m = 2(200 + 1) * 1000 = 402,000\text{Hz}$$

For PM modulation,

$$\beta = k_p \max|m(t)| = 10 * 2 = 20$$

$$B_T = 2(\beta + 1)f_m = 2(20 + 1) * 1000 = 42,000\text{Hz}$$

(c) If the message frequency doubles, i.e. $m(t) = \sin(4000\pi t)$, for FM modulation,

$$\beta = \frac{k_f \max|m(t)|}{f_m} = \frac{100000}{2000} = 50$$

$$B_T = 2(\beta + 1)f_m = 2(50 + 1) * 2000 = 204,000\text{Hz}$$

For PM modulation,

$$\beta = k_p \max|m(t)| = 10$$

$$B_T = 2(\beta + 1)f_m = 2(10 + 1) * 2000 = 44,000\text{Hz}$$

(d) Doubling the amplitude of $m(t)$ roughly doubles the bandwidth of both FM and PM waves. Doubling the frequency of $m(t)$ (expanding the spectrum of $M(f)$ by a factor 2) has hardly any effect on the FM wave bandwidth. However, it roughly doubles the bandwidth of PM wave, indicating that PM spectrum is sensitive to the shape of the baseband spectrum. FM spectrum is relatively insensitive to the spectrum $M(f)$.

Solution to Problem 3: The phase-modulated wave is

$$\begin{aligned} s(t) &= A_c \cos[2\pi f_c t + k_p A_m \cos(2\pi f_m t)] \\ &= A_c \cos(2\pi f_c t)\cos(\beta\cos(2\pi f_m t)) - A_c \sin(2\pi f_c t)\sin(\beta\cos(2\pi f_m t)). \end{aligned}$$

Since $\beta \ll 1$, we have

$$\begin{aligned} s(t) &\approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_c t)\cos(2\pi f_m t) \\ &= A_c \cos(2\pi f_c t) - \frac{\beta A_c}{2}\sin[2\pi(f_c - f_m)t] - \frac{\beta A_c}{2}\sin[2\pi(f_c + f_m)t]. \\ S(f) &\approx \frac{A_c}{2}[\delta(f - f_c) + \delta(f + f_c)] - \\ &\quad \frac{\beta A_c}{4j}[\delta(f - f_c + f_m) - \delta(f + f_c - f_m)] - \frac{\beta A_c}{4j}[\delta(f - f_c - f_m) - \delta(f + f_c + f_m)]. \end{aligned}$$

Solution to Problem 4: The output signal $v_0(t)$ can be calculated as

$$\begin{aligned}
v_0(t) &= a_1 A_c \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right) \\
&\quad + \frac{a_2 A_c^2}{2} + \frac{a_2 A_c^2}{2} \cos\left(4\pi f_c t + 4\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right) \\
&\quad + \frac{3a_3 A_c^3}{4} \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right) + \frac{a_3 A_c^3}{4} \cos\left(6\pi f_c t + 6\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right) \\
&= \frac{a_2 A_c^2}{2} + \left(a_1 A_c + \frac{3a_3 A_c^3}{4}\right) \cos\left(2\pi f_c t + 2\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right) \\
&\quad + \frac{a_2 A_c^2}{2} \cos\left(4\pi f_c t + 4\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right) + \frac{a_3 A_c^3}{4} \cos\left(6\pi f_c t + 6\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right).
\end{aligned}$$

(a) From Carson's rule, the bandwidth of $s(t)$ is approximately $2(\Delta f_{\max} + W)$. Thus the frequency band of $s(t)$ is approximately $[f_c - (\Delta f_{\max} + W), f_c + (\Delta f_{\max} + W)]$. Similarly by noticing that the maximum frequency deviation of the term $\cos\left(2n\pi f_c t + 2n\pi k_f \int_{-\infty}^{t} m(\tau)d\tau\right)$ is $n\Delta f_{\max}$, the frequency bands of the terms of $v_0(t)$ are respectively 0, $[f_c - (\Delta f_{\max} + W), f_c + (\Delta f_{\max} + W)]$, $[2f_c - (2\Delta f_{\max} + W), 2f_c + (2\Delta f_{\max} + W)]$, $[3f_c - (3\Delta f_{\max} + W), 2f_c + (3\Delta f_{\max} + W)]$.

To recover $s(t)$ with a band-pass filter, the frequency band of the second term needs to be separated from the rest terms, i.e., we need

$$f_c + (\Delta f_{\max} + W) < 2f_c - (2\Delta f_{\max} + W) \text{ and } f_c + (\Delta f_{\max} + W) < 3f_c - (3\Delta f_{\max} + W),$$

which is

$$f_c > 3\Delta f_{\max} + 2W.$$

(b) The pass band of the band-pass filter can be $[f_c - (\Delta f_{\max} + W) - B, f_c + (\Delta f_{\max} + W) + B]$, where $B < [f_c - (3\Delta f_{\max} + 2W)]/2$.