Chapter 4. Angle Modulation

Modulation: the process by which *some characteristics* of a carrier wave is varied in accordance with an information-bearing signal m(t).

- **Carrier:** used to facilitate the transmission of messages, e.g., sinusoid waves.

$$c(t) = A_c \cos(2\pi f_c t + \theta)$$

- Amplitude modulation: the amplitude A_C is varied in accordance with $m(t)$.

- Angle modulation: the angle $2\pi f_c t + \theta$ is varied in accordance with $m(t)$.

4.1 Fundamental Theories of Angle Modulation **(Haykin & Moher 4.1, 4.3)**

4.2 Properties of Angle Modulation **(Haykin & Moher 4.2,)**

4.3 Spectral Analysis of FM **(Haykin & Moher 4.4, partial 4.5 & 4.6)**

4.4 Generation and Demodulation of FM **(Haykin & Moher partial 4.7 & 4.8)**

4.1 Fundamental Theories of Angle Modulation

- **A sinusoidal carrier wave:**

$$c(t) = A_c \cos(2\pi f_c t),$$

where f_c is the carrier frequency, A_c is the carrier amplitude.

- **Message signal/information-bearing signal:** $m(t)$

Angle-modulated wave:

$$s(t) = A_c \cos[\theta_i(t)],$$

$\theta_i(t)$: angle of the modulated wave. Varies in accordance with the message signal $m(t)$.

If no message signal (or $m(t) = 0$), $\theta_i(t) = 2\pi f_c t + \phi_c$.

Without loss of generality, ϕ_c is assumed to be 0.

Angle modulated signal:

$$s(t) = A_c \cos[\theta_i(t)],$$

The angle $\theta_i(t)$ is a function of time and changes by 2π radians.

Instantaneous frequency of the modulated signal:

$$f_i(t) = \lim_{\Delta t \to 0} \frac{\theta_i(t+\Delta t) - \theta_i(t)}{2\pi \Delta t} = \frac{1}{2\pi} \frac{d\theta_i(t)}{dt}.$$

Hz: cycles/s

rads/s

That is: $\theta_i(t) = 2\pi \int_0^t f_i(\tau) d\tau$

Phase modulation (PM): the **angle** is varied linearly with the message signal $m(t)$ (k_p : the *phase-sensitivity factor*)

PM modulated wave:

$$s(t) = A_c \cos[2\pi f_c t + k_p m(t)].$$

- Instantaneous phase: $\theta_i(t) = 2\pi f_c t + k_p m(t).$

  - Maximum phase deviation: $\Delta\theta_{\max} = k_p \max_t |m(t)|.$

- Instantaneous frequency: $f_i(t) = f_c + \frac{k_p}{2\pi}\frac{dm(t)}{dt}.$

  - Maximum frequency deviation: $\Delta f_{\max} = \frac{k_p}{2\pi} \max_t \left|\frac{dm(t)}{dt}\right|.$

Frequency modulation (FM): the **instantaneous frequency** is varied linearly with the message signal $m(t)$ (k_f: the *frequency-sensitivity factor*)

FM modulated wave:

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

- Instantaneous phase: $\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau.$

  - Maximum phase deviation: $\Delta\theta_{\max} = 2\pi k_f \max_t \left|\int_0^t m(\tau)\tau\right|.$

- Instantaneous frequency: $f_i(t) = f_c + k_f m(t).$

  - Maximum frequency deviation: $\Delta f_{\max} = k_f \max_t |m(t)|.$

Example: modulation of a single-tone message

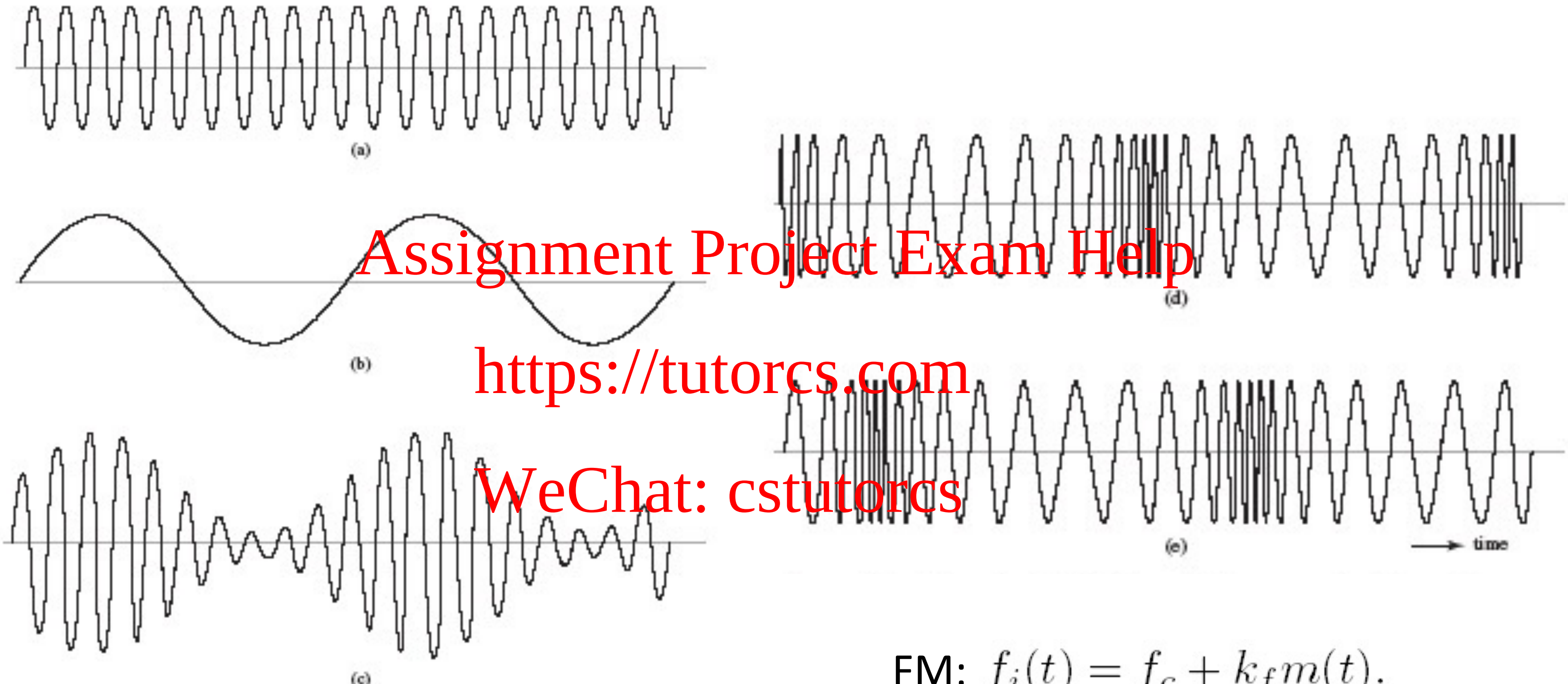
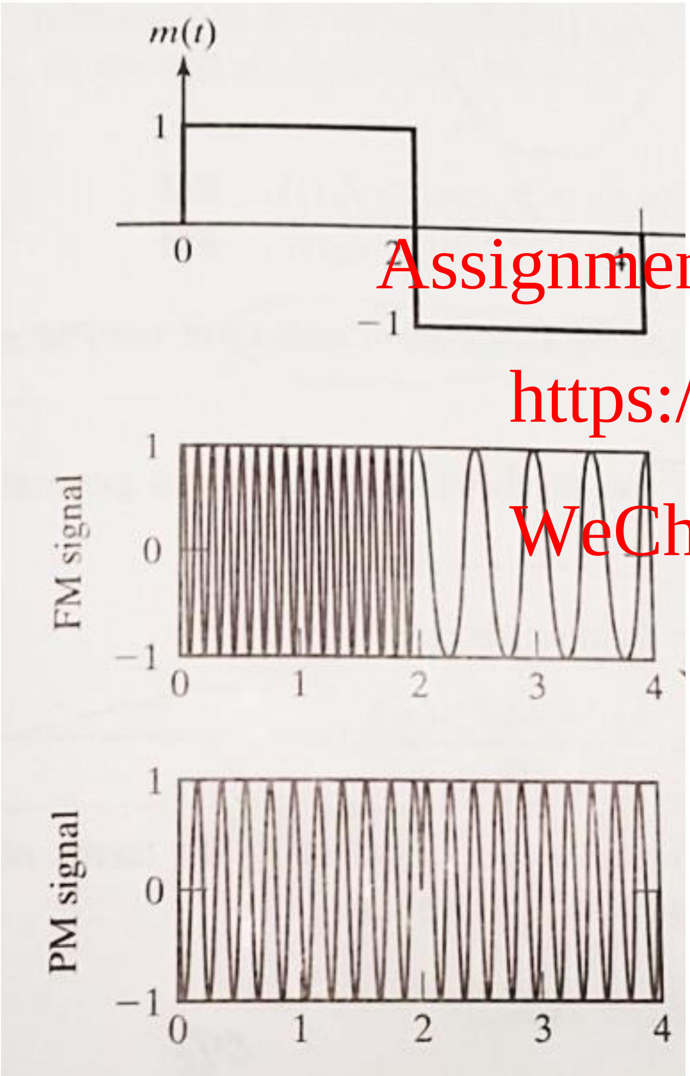


FIGURE 4.1 Illustration of AM, PM, and FM waves produced by a single tone. (*a*) Carrier wave. (*b*) Sinusoidal modulating signal. (*c*) Amplitude-modulated signal. (*d*) Phase-modulated signal. (*e*) Frequency modulated signal.

FM: $f_i(t) = f_c + k_f m(t).$

PM: $f_i(t) = f_c + \dfrac{k_p}{2\pi}\dfrac{dm(t)}{dt}.$

Example: Given frequency and phase modulations of rectangular messages in the figure, find f_c, k_f, and k_p.



For FM signal:

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

For PM signal:

$$s(t) = A_c \cos[2\pi f_c t + k_p m(t)].$$

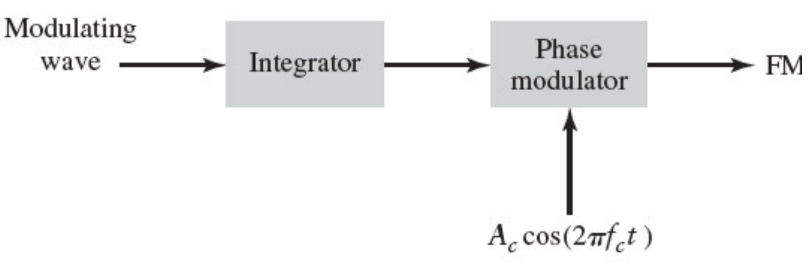
Relationship between PM and FM

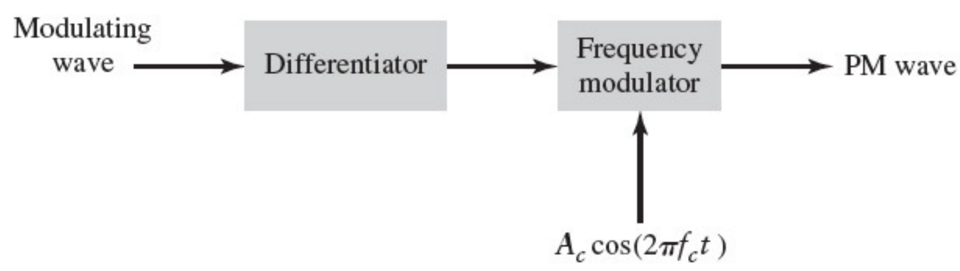
An FM wave $s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$

- can be seen as a PM wave produced by the modulating message: $\int_0^t m(\tau)d\tau$, where $k_p = 2\pi k_f$.

A PM wave $s(t) = A_c \cos[2\pi f_c t + k_p m(t)]$

- can be seen as an FM wave produced by the modulating message: $\frac{dm(t)}{dt}$, where $k_f = k_p/(2\pi)$.





4.2 Properties of Angle Modulation

1. **Constant transmit power.**

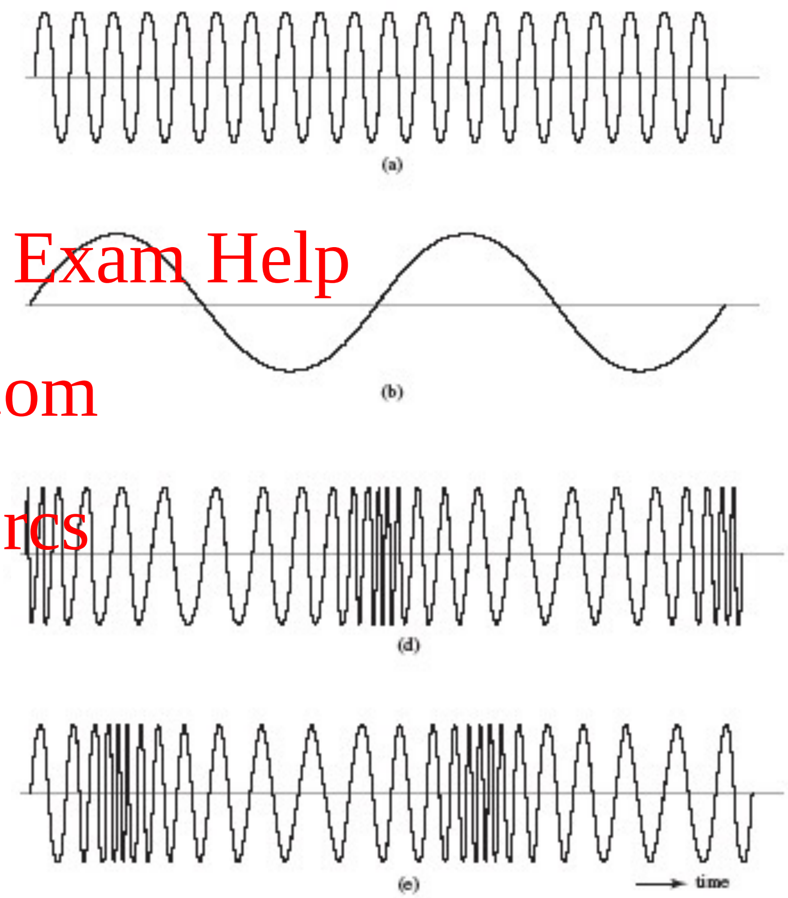
   Amplitude of modulated wave is fixed: A_c

   Average transmit power of modulated wave:

$$P_{\text{Ave}} = \frac{1}{2}A_c^2.$$

2. **Irregularity of zero-crossings.**

   For angle modulation, the message resides in the zero-crossings of the modulated wave.



3. **Non-linear modulation process.**

   Violates the principle of superposition.

   Difficult to analyze.

4. **Visualization difficulty of message waveform.**

5. **Tradeoff of transmission bandwidth and performance**

   Less sensitive to noise compared to AM,

   but with increased bandwidth.

   Offers a tradeoff.

Example: Given an angle-modulated signal $s(t) = 100\cos[2\pi f_c t + 4\sin(2000\pi t)]$ where $f_c = 10MHz$. (a) Determine the average transmit power; (b) Determine the max phase and max frequency derivations; (c) Is this an FM or PM signals? If yes, please find corresponding $m(t)$.

4.3 Spectral Analysis of FM

FM is non-linear. How to conduct the spectral analysis?

Step 1: Consider single-tone modulation

Step 2: Gain insights for the general case, and approximations are necessary.

Spectral analysis of single-tone modulation

Message signal:

$$m(t) = A_m \cos(2\pi f_m t)$$

FM wave:

$$\begin{aligned} s(t) &= A_c \cos\left[2\pi f_c t + 2\pi k_f A_m \int_0^t \cos(2\pi f_m \tau) d\tau\right] \\ &= A_c \cos\left[2\pi f_c t + \frac{k_f A_m}{f_m} \sin(2\pi f_m t)\right]. \\ &= A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)]. \end{aligned}$$

where $\beta = \frac{\Delta f_{max}}{f_m} = \frac{k_f A_m}{f_m}$ is the FM modulation index.

Spectral analysis of <u>single-tone modulation</u> (continued)

Claim: in time domain, the single-tone FM modulated wave can be written as:

$$s(t) = A_c \sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c + nf_m)t]$$

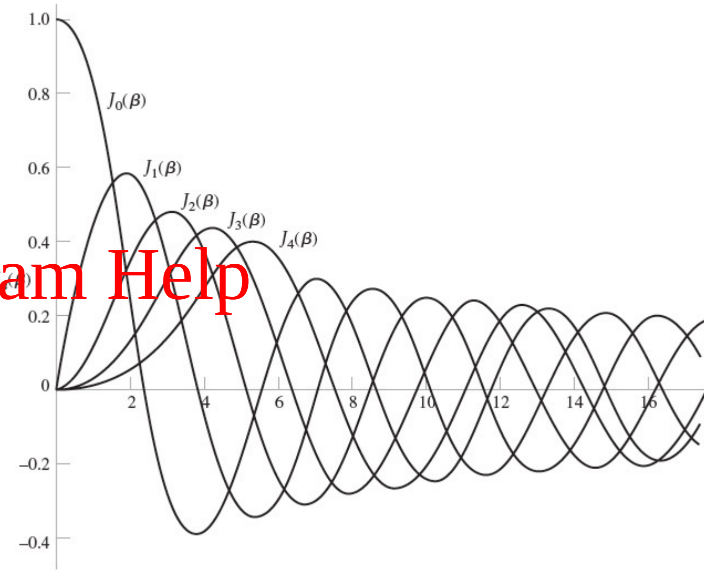


FIGURE 4.6 Plots of the Bessel function of the first kind, $J_n(\beta)$, for varying order

Where the nth order Bessel function of the first kind:

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[j(\beta \sin x - nx)]dx,$$

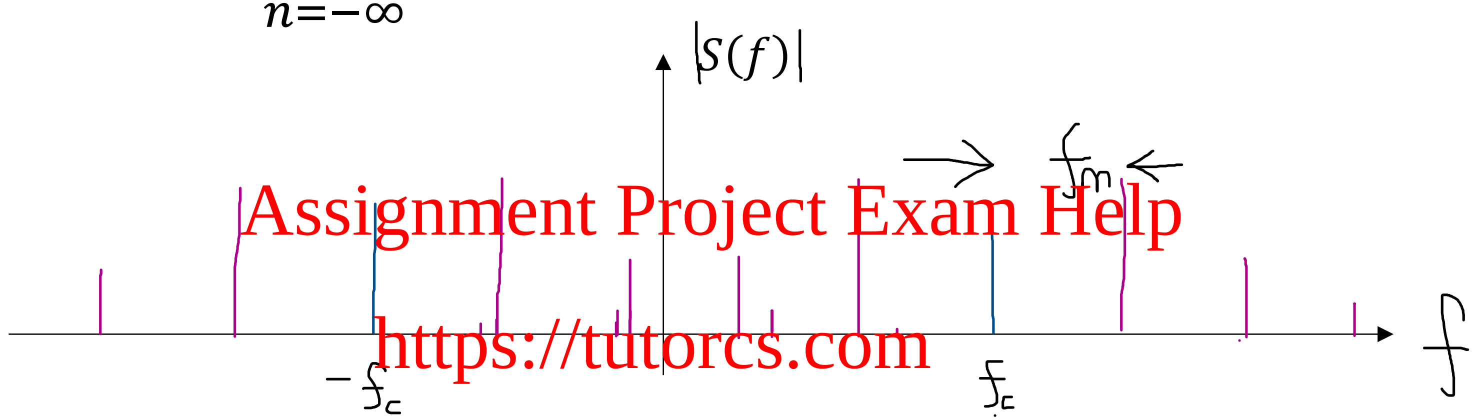
and $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$ for arbitrary β.

Frequency representation:

$$S(f) = \frac{1}{2}A_c \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$

Properties of the spectrum of single-tone FM wave:

$$S(f) = \frac{1}{2}A_c \sum_{n=-\infty}^{\infty} J_n(\beta)[\delta(f - f_c - nf_m) + \delta(f + f_c + nf_m)]$$



1. **Infinite bandwidth**: the spectrum contains a carrier component and an infinite set of side frequencies located symmetrically on both side of the carrier $\pm f_c$ at frequency separations of f_m.
2. The amplitude of the carrier component $\frac{1}{2}A_c J_0(\beta)$ varies with β.

3. Narrow-band FM ($\beta \ll 1$):

$$
\begin{aligned}
s(t) &= A_c \cos[2\pi f_c t + \beta \sin(2\pi f_m t)] \\
&= A_c \cos(2\pi f_c t) \cos[\beta \sin(2\pi f_m t)] - A_c \sin(2\pi f_c t) \sin[\beta \sin(2\pi f_m t)] \\
&\approx A_c \cos(2\pi f_c t) - \beta A_c \sin(2\pi f_m t) \sin(2\pi f_c t).
\end{aligned}
$$

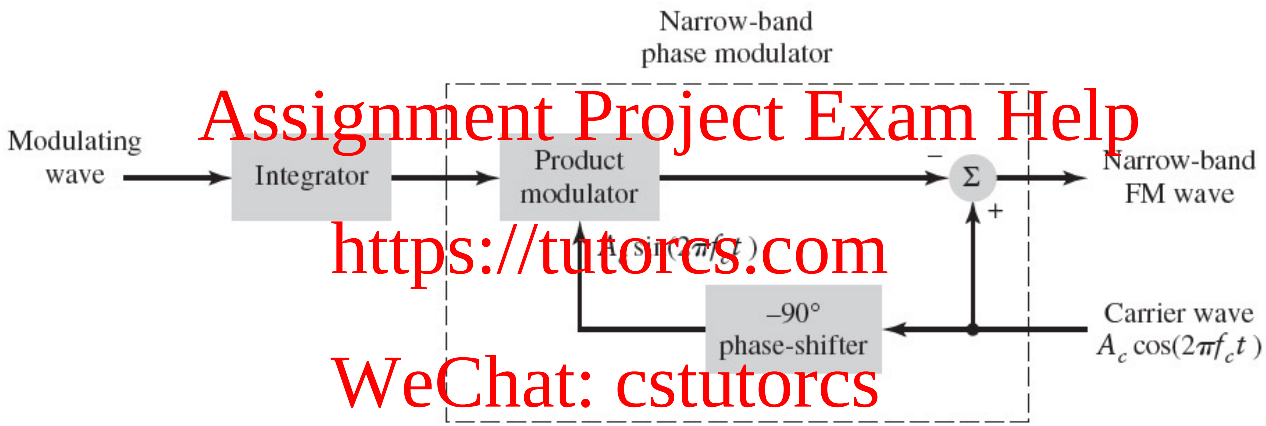


FIGURE 4.4 Block diagram of an indirect method for generating a narrow-band FM wave.

$$
S(f) = \frac{A_c}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] - \frac{\beta A_c}{4}\left[\delta(f - f_c + f_m) + \delta(f + f_c - f_m)\right] + \frac{\beta A_c}{4}\left[\delta(f - f_c - f_m) + \delta(f + f_c + f_m)\right]
$$

Narrow-band FM Bandwidth: approximately $2 f_m$

Summary:

Strictly speaking, FM wave has infinite bandwidth.

In practice, FM wave is effectively limited to a finite number of significant side-frequencies compatible with a specific amount of distortion.

For single-tone FM wave,

Carson's rule (for both narrow-band FM and wideband FM) for the bandwidth of the modulated wave:

$$B_T \approx 2(\beta + 1)f_m = 2(k_f A_m + f_m) = 2\Delta f_{\max}\left(1 + \frac{1}{\beta}\right)$$

For PM, define the modulation index as

$$\beta = \Delta\theta_{\max} = k_p A_m.$$

The same result on bandwidth can be obtained.

For <u>the general case</u>, define *<u>modulation index</u>*

$$\beta = \Delta\theta_{\max} = k_p \max_t |m(t)| \quad \text{for PM}$$

$$\beta = \frac{\Delta f_{\max}}{W} = \frac{k_f \max_t |m(t)|}{W} \quad \text{for FM}$$

where W is the message bandwidth.

Generalized Carson's rule for the bandwidth of angle modulated wave:

$$B_T \approx 2(\beta + 1)W$$

- Increase in the message amplitude increases the bandwidth.
- Increase in the message frequency increases the bandwidth.

Example: Given $m(t) = 10sinc(10^4 t)$. Determine the bandwidth of an FM-modulated signal with $k_f = 4000$.

4.4 Generation and Demodulation of FM

Generation: Direct method.

- Voltage-controlled oscillator: an sinusoidal oscillator which is directly controlled by the message signal.

Advantage: Direct and straightforward.

Can have large frequency deviations.

Disadvantage: Prone to frequency drift (Requiring frequency stabilization circuit).

Generation: Indirect method.

First produce a narrow-band FM, which is followed by frequency multiplier to increase the frequency deviation to the desired level.

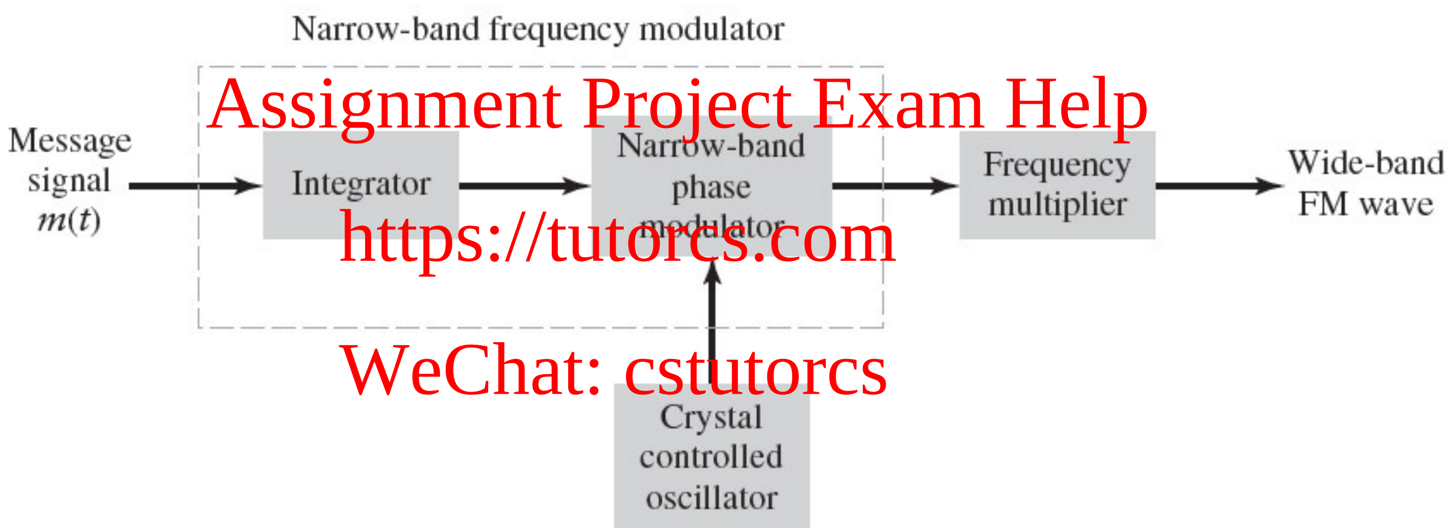


FIGURE 4.10 Block diagram of the indirect method of generating a wide-band FM wave.

Less prone to frequency-drift.

Frequency-multiplier

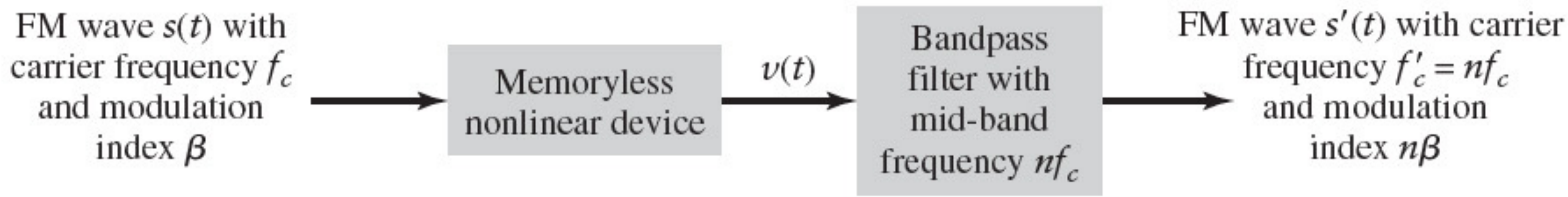


FIGURE 4.11 Block diagram of frequency multiplier.

$$v(t) = a_1 s(t) + a_2 s^2(t) + \cdots + a_n s^n(t).$$

$$s(t) = A_c \cos(2\pi f_c t + \theta(t)), \text{ where } \theta(t) = 2\pi k_p \int_0^t m(\tau)d\tau.$$

$$v(t) = a_1 A_c \cos(2\pi f_c t + \theta(t)) + \cdots + a_n \left[A_c \cos(2\pi f_c t + \theta(t))\right]^n.$$

With large enough f_c, the term in *v(t)* whose frequency is around nf_c takes the following form:

$$c a_n A_c^n \cos(2\pi n f_c t + n\theta(t))$$

This is also the output of the bandpass filter, i.e., the desired FM wave. $s'(t) = A_c \cos\left[2\pi f'_c t + 2\pi k'_f \int_0^t m(\tau)d\tau\right]$. $f'_c = nf_c,\ k'_f = nk_f.$

Demodulation: To recover the message from the FM wave.

Frequency Discriminator.

FM wave:

$$s(t) = A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right]$$

$$\frac{ds(t)}{dt} = -2\pi A_c[f_c + k_f m(t)] \sin\left[2\pi f_c t + 2\pi k_f \int_0^t m(\tau)d\tau\right].$$

Envelop of $\frac{ds(t)}{dt}$: $2\pi A_c[f_c + k_f m(t)]$

After DC blocker and scaling: $m(t)$ is obtained.