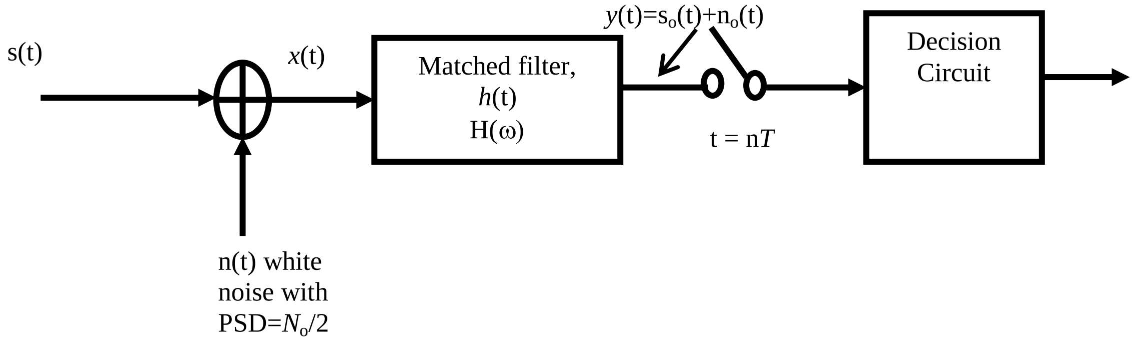
The matched filter(-1940)

Since the signal is going to be contaminated by noise generated outside and inside electrical systems, we wish to find the receiver that will best identify the wanted signal, i.e.

we want the receiver which will give the maximum signal to noise ratio at the time T when we want to make a decision whether the signal is present or not.



Let us define the signal to noise ratio $(SNR)_o$ as: (SNR is usually defined in terms of the average power of the signal to the average power of noise)

$$SNR_o = \left.\frac{|s_o(t)|_{max}^2}{\text{average noise power}}\right|_{t=T} \tag{5}$$

To find the optimum impulse response for the matched filter, $h(t)$ consider the output of the filter $y(t)$ due to the signal component given in the frequency domain by

$$s_o(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega).H(\omega).e^{j\omega t}\,d\omega \tag{6}$$

The noise output power is found by integrating the power spectral density of noise at the output of the filter given by multiplying the input power spectral density of noise $N_o/2$ with the magnitude squared of the filter response

$$N = \frac{1}{2\pi}\frac{N_o}{2}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega \tag{7}$$

As shown in the figure the optimum receiver only needs to maximise the output peak of the signal with respect to the average noise power at the instant of sampling, which occurs at multiples of T i.e.

$$\frac{|s_o(t)|_{max}^2}{N} = \left.\frac{\left|\int_{-\infty}^{\infty} S(\omega).H(\omega).e^{j\omega t}\,d\omega\right|^2}{2\pi\frac{N_o}{2}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega}\right|_{t=T} = \frac{\left|\int_{-\infty}^{\infty} S(\omega).H(\omega).e^{j\omega T}\,d\omega\right|^2}{2\pi\frac{N_o}{2}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega} \tag{8}$$

To maximise equation 8, use can be made of Schwartz inequality for integrals of complex functions, which states that the area under the product of two complex functions is smaller than or equal to the area under the magnitude squared of each function. Assuming that $H(\omega)$ is one function and $S(\omega)\exp(j\omega T)$ is the second function, then

$$\frac{\left|\int_{-\infty}^{\infty} S(\omega).H(\omega).\exp(j\omega T)d\omega\right|^2}{2\pi \frac{N_o}{2}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega} \le \frac{\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega \int_{-\infty}^{\infty}|S(\omega)|^2 d\omega}{2\pi \frac{N_o}{2}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega} \tag{9}$$

For $(SNR)_o$ to be a maximum, the equality should hold. This occurs when one function is the complex conjugate of the other i.e.

$$H(\omega) = kS^*(\omega)\exp(-j\omega T) \tag{10}$$

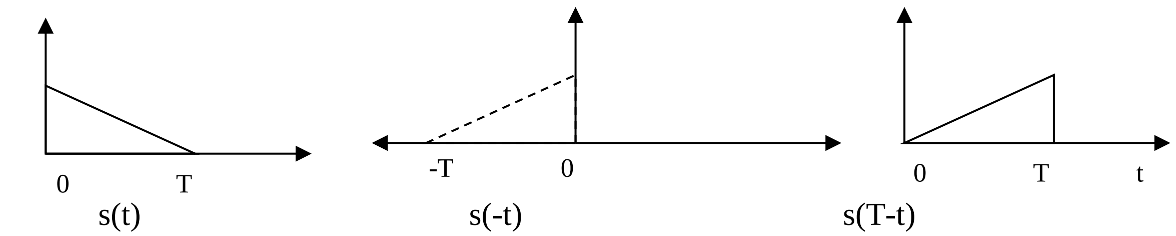
Using Fourier transform properties this corresponds to a filter with an impulse response given by

$$h(t) = ks(T-t) \tag{11}$$

Thus the SNR at the output of the matched filter is given by

$$(SNR_o)\big|_{t=T} = \frac{\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega \int_{-\infty}^{\infty}|S(\omega)|^2 d\omega}{2\pi \frac{N_o}{2}\int_{-\infty}^{\infty}|H(\omega)|^2 d\omega} = \frac{\int_{-\infty}^{\infty}|S(\omega)|^2 d\omega}{2\pi \frac{N_o}{2}} \tag{12}$$

Therefore, in the matched filter detector, the received signal is passed through a filter specially designed to match the transmitted waveform *s*(t), by having an impulse response, which is the mirror image of the transmitted signal and delayed by *T* to ensure that the filter's response is causal. Example:



The matched filter output in the time domain can be expressed as follows:

$$y(t) = s(t)*h(t) = \int_{-\infty}^{\infty} s(\lambda).h(t-\lambda)d\lambda = \int_{-\infty}^{\infty} s(\lambda)s(T-(t-\lambda))d\lambda$$

$$y(t) = \int_{-\infty}^{\infty} s(\lambda)s(T-t+\lambda)d\lambda = \Re_{ss}(T-t) \text{ which is the auto-correlation function}$$

$$\text{at } t=T,\ y(T) = \Re_{ss}(0) \tag{13}$$

Note:

1) The autocorrelation of a function s(t) is:

$$\Re_{ss}(\tau) = \int_{-\infty}^{\infty} \mathrm{s}(t).\mathrm{s}(t+\tau).\mathrm{dt}$$

i.e. we take the function s(t), shift it by τ, multiply the original signal with the time shifted signal and then find the area and repeat this for all time shifts. The area under the product is a maximum when $\tau = 0$ which is equal to the Energy of the signal, E, i.e. when they are identical $\Rightarrow$ R_{ss} gives the similarity of the function with itself as a function of time shift.

$$\Re_{ss}(0) = \int_{-\infty}^{\infty} \mathrm{s}(t).\mathrm{s}(t).\mathrm{dt} = \int_{-\infty}^{\infty} s^2(t)dt = E \qquad (14)$$

This can be deduced by assuming that s(t) represents a voltage or current in a 1 ohm resistor. Then the square value of s(t) represents the instantaneous power and the integral of the instantaneous power with time is equal to the energy.

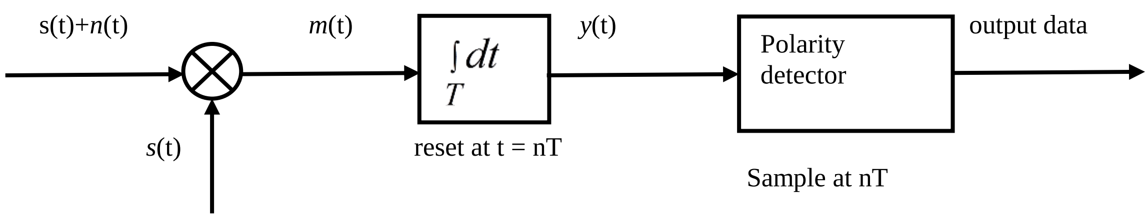
2) The matched filter impulse response is the mirror image of the signal. Since convolution is found by taking the mirror image of one signal, by generating an impulse response which is mirror image brings back the impulse response in phase again with the signal thus the autocorrelation function at time of coincidence gives the maximum output.

3) The matched filter is not useful for analogue modulation because it gives us the best output at t=T but does not care about fidelity since for analogue modulation s(t) varies randomly. In digital modulation we only care about identifying the 1's and 0's and do not care about reproducing the information waveform that existed at the transmitter. The 1's and 0's can be used if needed to regenerate the analogue signal.

4) An equivalent to the matched filter is the correlation detector which can be found as follows:

$$y(t) = s(t) * h(t) = \int_{-\infty}^{\infty} s(\lambda).h(t-\lambda)d\lambda = \int_{-\infty}^{\infty} s(\lambda)s(T-(t-\lambda))d\lambda$$

$$y(T) = \int_{0}^{T} s(\lambda)s(\lambda)d\lambda = \Re_{ss}(0) = E$$



This is of course a synchronous or coherent form of detection because the incoming signal is multiplied by an identical signal or replica.

5) The output SNR

$$(SNR_o)\Big|_{t=T} = \frac{\int_{-\infty}^{\infty} |S(\omega)|^2 d\omega}{2\pi \frac{N_o}{2}} \tag{15}$$

Using Parseval's theorem which states:

$$\frac{\int_{-\infty}^{\infty} |S(\omega)|^2 d\omega}{2\pi} = \int_{-\infty}^{\infty} |s(t)|^2 dt = E$$

The SNR at the output becomes equal to

$$\boxed{(SNR_o)\Big|_{t=T} = \frac{E}{\frac{N_o}{2}}} \tag{16}$$

The above result is very important. It means that in evaluating the ability of a matched filter receiver to combat white Gaussian noise we find that all signals which have the same energy are equally effective.

6) The output of the matched filter in the frequency domain is written as

$$Y(\omega) = H(\omega).S(\omega) = S^*(\omega)e^{-j\omega T}.S(\omega) = |S(\omega)|e^{j\phi(\omega)}.|S(\omega)|e^{-j\phi(\omega)}e^{-j\omega T}$$

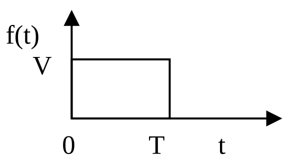
and the signal in the time domain can then be found by taking the inverse Fourier transform

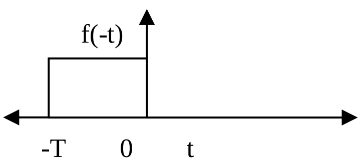
$$\boxed{\begin{aligned} y(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |S(\omega)|e^{j\phi(\omega)}.|S(\omega)|e^{-j\phi(\omega)}e^{-j\omega T}e^{j\omega t} d\omega = \\ &\int_{-\infty}^{\infty} |S(\omega)|.|S(\omega)|e^{-j\omega T}e^{j\omega t} d\omega, \text{ and at } t = T, \text{ it becomes } \int_{-\infty}^{\infty} |S(\omega)|.|S(\omega)|e^{-j\omega T}e^{j\omega T} d\omega = \int_{-\infty}^{\infty} |S(\omega)|^2 d\omega \end{aligned}}$$

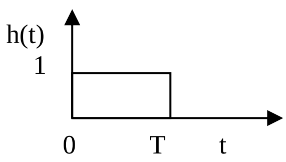
i.e. all the frequency components are brought in phase since the output is only dependent on the magnitude of the spectrum and not its phase. That is all the frequency components add constructively to give a maximum at time *T*.

Example:

Design a matched filter for f(t):

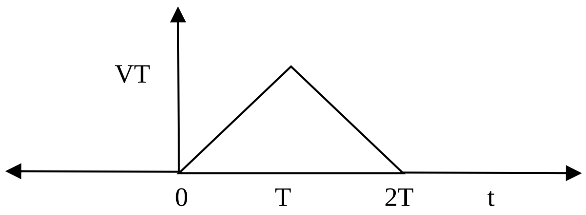




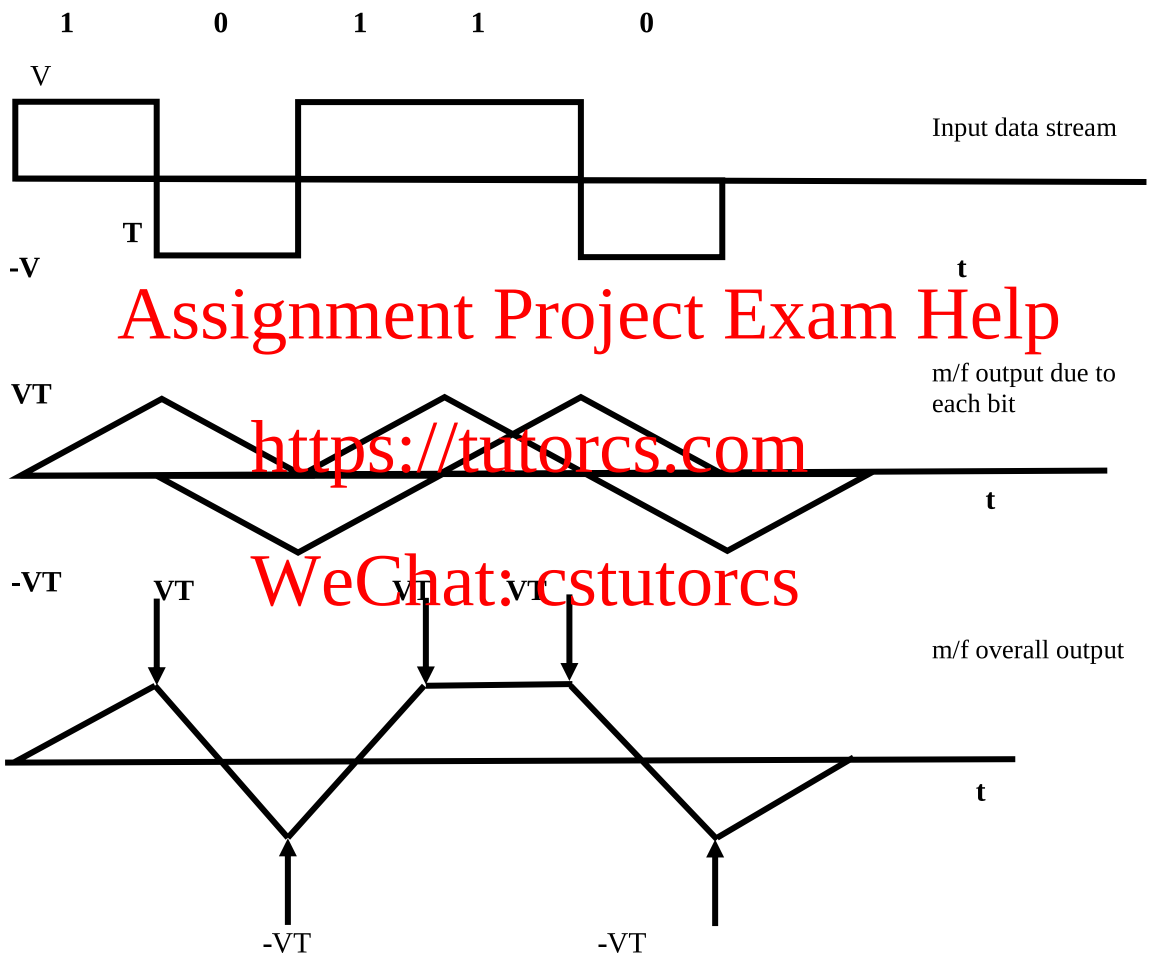


Let *k*=1/V, h(t)=(1/V) f(T-t)

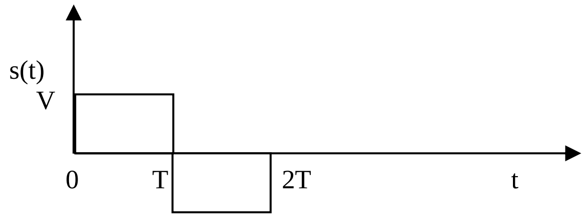
the output of the filter:

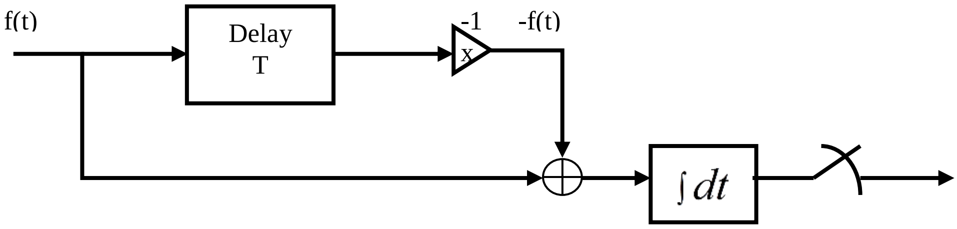


Now suppose we represent binary 1 by +*v* and binary 0 by -*v*, then the output of the matched filter can be found by superposition.



The above can be also implemented by generating the signal shown below, s(t) and then integrating the area as in the corresponding block diagram.





Note that the value at t=T, in the above implementation, the output is only equal to VT rather than the expected output which should be equal to the energy of the signal. This can be compensated for by a scaling factor. In the ideal case the SNR at the output should be equal to:

$$(SNR_o)\big|_{t=T} = \frac{E}{\frac{N_o}{2}} = \frac{V^2 T}{\frac{N_o}{2}}$$