Bit error rate

It is useful to be able to compare the performance of different systems under similar conditions. For binary transmission, this is best achieved by measuring the bit error rate for a range of values of signal energy and white noise power.

In particular, we define:

E: energy per bit of data.
N_0: single sided noise power density.

Error rate = (number of bits received in error)/(total number of bits received). An error rate curve is usually generated by plotting the error rate against E/N_0.

For a theoretical analysis of systems considered so far, we can evaluate the probability of error in detecting a '0' or a '1' in the presence of additive white Gaussian noise with power spectral density $N_0/2$ (DSS) which can be used for comparison.

Noise properties: So far, we said that most sources of noise can be considered white. However noise is not deterministic i.e. we cannot determine future values of noise by observing the present or past values. Such processes are random i.e. unpredictable. For random processes, it is usual to obtain probability density functions which enable us to say that the probability of noise being between v_1 and v_2 is equal to a particular value.

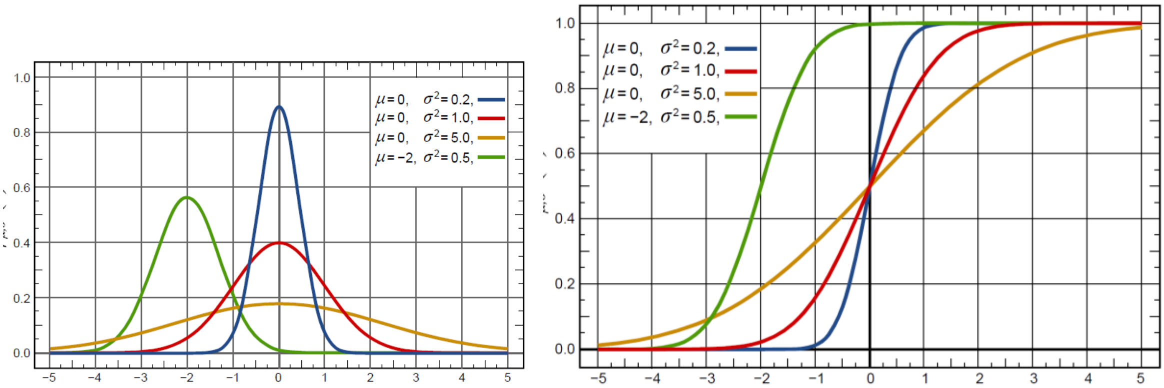
For a Gaussian process the p.d.f. is given by

$$p(v) = \frac{1}{\sqrt{2\pi \overline{v_n^2}}} \exp - \left(\frac{(v-\bar{v})^2}{2\overline{v_n^2}} \right) = \frac{1}{\sqrt{2\pi}\sigma_v} \exp - \left(\frac{(v-\bar{v})^2}{2\sigma_v} \right) \qquad -\infty < v < \infty$$

where $\bar{v}$ is the average or mean of the process, and $\sigma^2{}_v$ is the variance which gives us an idea about the spread of the values of v about the mean and $\sigma^2{}_v = \overline{v^2{}_n}$ and

$$p(v_1 < v < v_2) = \int_{v_1}^{v_2} p(v)dv$$

The figure below shows the PDF of noise and its cumulative distribution, CDF, obtained by taking the integral of the PDF for different values of the mean μ and the variance σ. The figure shows that the peak of the PDF coincides with the mean and that as the variance increases the peak of the distribution becomes smaller. This preserves the total area under the curve which should always be equal to 1.



(a) (b)

(a) Gaussian PDF for different mean and variance values, (b) Corresponding CDF (Cumulative Distribution Function) which is equal to the Area under the PDF

The Gaussian distribution satisfies the 68%-95%-99.7% rule, which is also known as the three-sigma rule, or empirical rule, which states that for a normal distribution, nearly all values lie within 3 standard deviations of the mean.

For white noise, the mean is equal to zero. This gives:

$$p(v) = \frac{1}{\sqrt{2\pi\,\overline{v_n^2}}}\exp\left[-\frac{v^2}{2\overline{v_n^2}}\right]$$

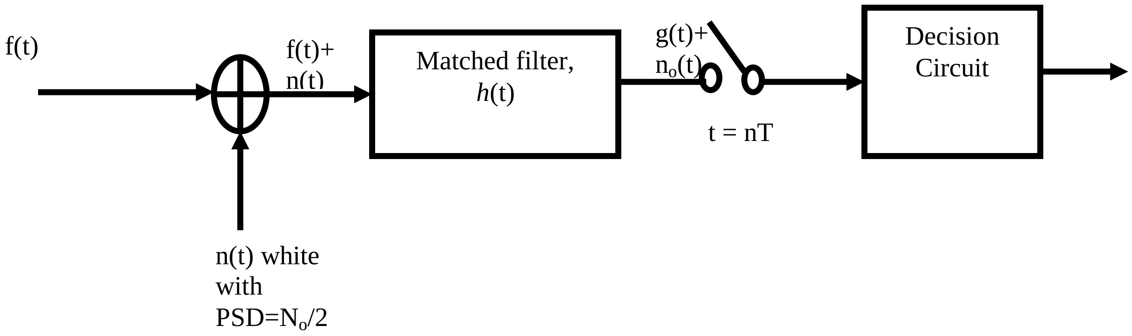
σ^2_v : is the mean square value of noise

σ_v : is the *rms* value of noise voltage

Thus the noise power, N= σ^2_v assuming a 1 ohm resistor

1) ***<u>Coherent(matched filter) detectors:</u>***

i) Antipodal signals (e.g. optimum PSK or baseband).

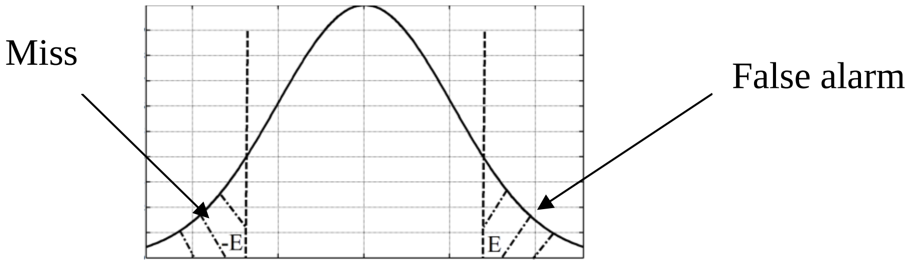


Note that $n_0(t)$ does not necessarily have to be white for all frequencies but for the frequency range of interest, but its statistics are still Gaussian.

Analysis approach:

- From previous analysis $g(T)=\underline{+}E$

- An error will occur if transmitted signal is a mark and $n_0(t)<-E$, this is referred to as a miss; or if the transmitted signal is a space and $n_0(T) > +E$, this is referred to as false alarm. From the Gaussian PDF shown below, the shaded areas indicate the areas where an error occurs. Due to the symmetry of the curve we can see that the two areas are equal.



From the symmetry of the PDF of noise we see that $p_{FA} = p_{miss}$
$\Rightarrow$ total probability of error
$=$p(transmitting mark).p_{miss} + p(transmitting space).p_{FA}

$$\boxed{p_e = p_{FA}\,(p(m) + p(s))}$$

Since we are transmitting either a mark or a space signal, then

$$\boxed{p(m) + p(s) = 1}$$

$$p_e = p_{miss} = p_{false\ alarm} = \int_E^{\infty} \frac{1}{\sqrt{2\pi}\sigma_v} e^{-\frac{v^2}{2\sigma_v^2}} dv = \int_E^{\infty} \frac{1}{\sqrt{2\pi N}} e^{-\frac{v^2}{2\sigma_v^2}} dv$$

Let $u = \frac{v}{\sqrt{2\sigma^2{}_v}}$ then $du = \frac{dv}{\sqrt{2\sigma^2{}_v}}$

$$p_e = p_{miss} = p_{false\ alarm} = \frac{1}{\sqrt{\pi}} \int_{\frac{E}{\sqrt{2\sigma_v^2}}}^{\infty} e^{-u^2} du$$

This cannot be evaluated in closed form.

Instead it is tabulated and it is defined in terms of the error function and the complementary error function defined below.

Using the error function $erf(x)$ and the complementary error function, $erfc(x)$the probability of error can be expressed as

$$\boxed{P_e = \frac{1}{2} erfc \frac{E}{\sqrt{2N}} \ where\ N = \sigma_v^2}$$

$$where\ erfc(x) = 1 - erf(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-z^2} dz$$

From the matched filter analysis (equations 5 and 16) we get:

$$(SNR)_o = \frac{|g(T)|^2}{N} = \frac{E^2}{N} = \frac{2E}{N_o}$$

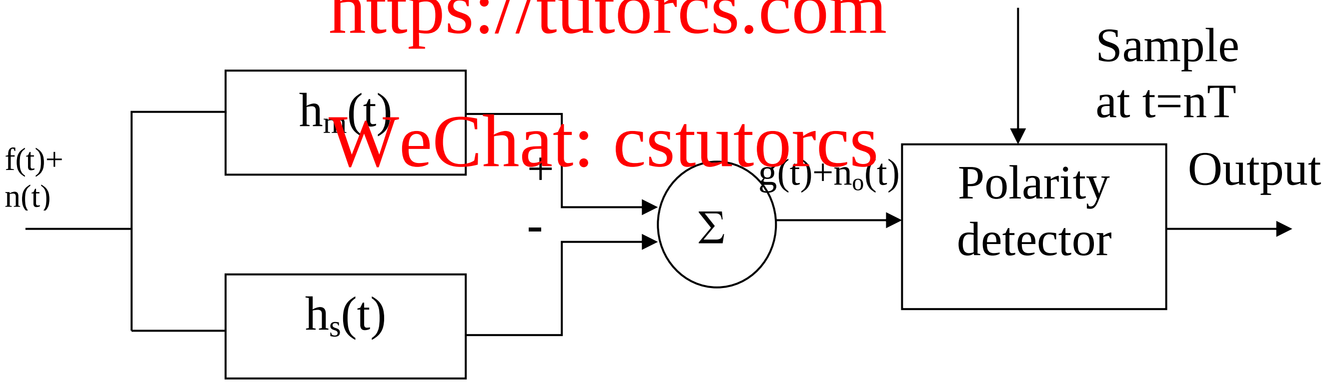
Then $\frac{E}{\sqrt{2N}} = \sqrt{\frac{E}{N_o}}$

Thus

$$\boxed{P_e = \frac{1}{2} erfc \sqrt{\frac{E}{N_o}}} \qquad (25)$$

ii) General matched filter:

In general we need two detectors: one for the mark and one for the space signal as previously discussed. This gives the generalised matched filter detector in the figure below.



Block diagram of the general matched filter detector

For the general matched filter:

g(T) = $\pm$ (1 - ρ)E

$$p_e = p_{miss} = p_{false\ alarm} = \int_{E(1-\rho)}^{\infty} \frac{1}{\sqrt{2\pi N}} e^{-v^2/2N} du$$

The output noise power N of the general matched filter can be shown to be equal to $N_o E(1-\rho)$

$$\boxed{P_e = \frac{1}{2} erfc \frac{E(1-\rho)}{\sqrt{2N}} = \frac{1}{2} erfc \sqrt{\frac{E(1-\rho)}{2N_o}}} \tag{26}$$

Taking the special case of orthogonal functions i.e. when $\rho = 0$ for example FSK signals or orthogonal Manchester code

$$\boxed{P_e = \frac{1}{2} erfc \sqrt{\frac{E}{2N_o}}} \tag{27}$$

Note:

1) The above result is general and could have been applied to the PSK case or bipolar baseband by substituting $\rho = -1$ which gives the same answer as in equation (25)

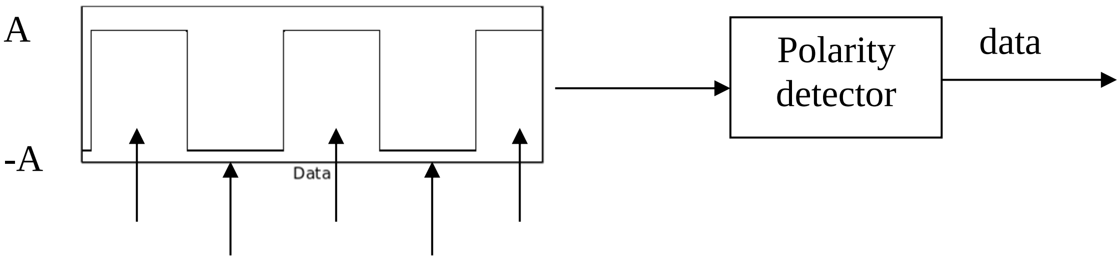
$$P_e = \frac{1}{2} erfc \sqrt{\frac{E}{N_o}}$$

2) From the tables of $erfc$ we see that P_e for PSK is ½ of FSK. That is we need to double the energy per bit for the same error rate.

ii) Non-coherent detectors:

In general the analysis of non-coherent detectors is more difficult due to the inclusion of non-linear elements. In some cases however, it is possible to make comparisons with the optimum ideal matched filter performance.

i) Baseband transmission without matched filter:



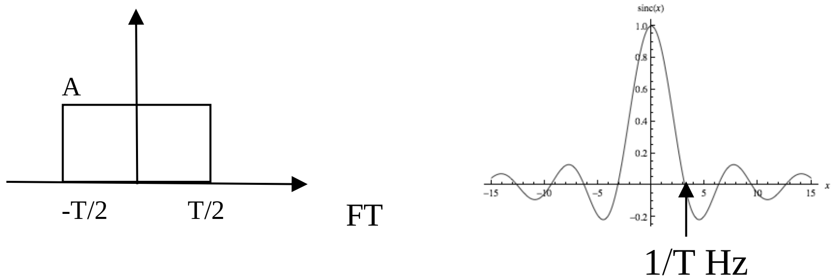
This is the same as before except that now E is replaced by A

$$P_e = \frac{1}{2} erfc \frac{A}{\sqrt{2N}}$$

To obtain the expression in terms of E and N_0 we determine N and E

$$E = \int_0^T A^2 dt = A^2 T \Rightarrow A = \sqrt{\frac{E}{T}}$$

The output noise power = $(N_0/2)\text{x}(2B) = N_0B$, where B is the BW of the channel. For a baseband signal, with bit duration equal to T, the required transmission bandwidth for minimum distortion is 1/T as illustrated below.

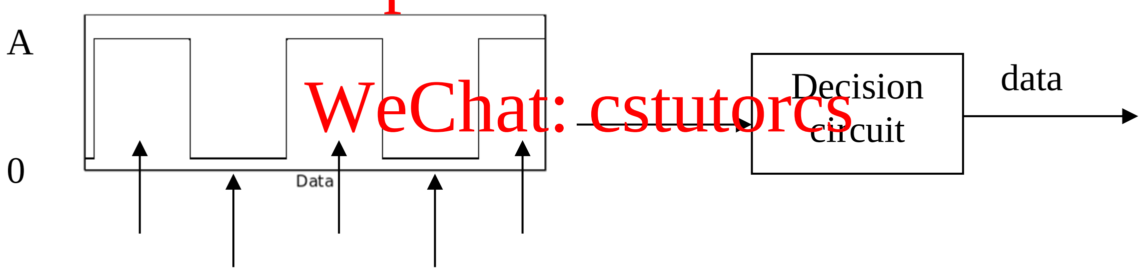


That is B~1/T for minimum distortion transmission

$$P_e = \frac{1}{2} erfc\sqrt{\frac{E}{2N_o BT}} \tag{28}$$

That is the probability of error is a function of the bandwidth and the duration of the bit.

ii) Orthogonal(on-off)



Compare with respect to A/2
Error if we transmit a mark and the noise > -A/2
Error if we transmit a space and the noise < A/2

This gives the probability of error as before but with lower threshold

$$\boxed{P_e = \frac{1}{2} erfc\sqrt{\frac{E}{4N_o BT}}} \tag{29}$$

<u>Note</u>:

We should expect a higher p_e since the distance between the mark and space =A in this case whereas in the previous case it is 2A.

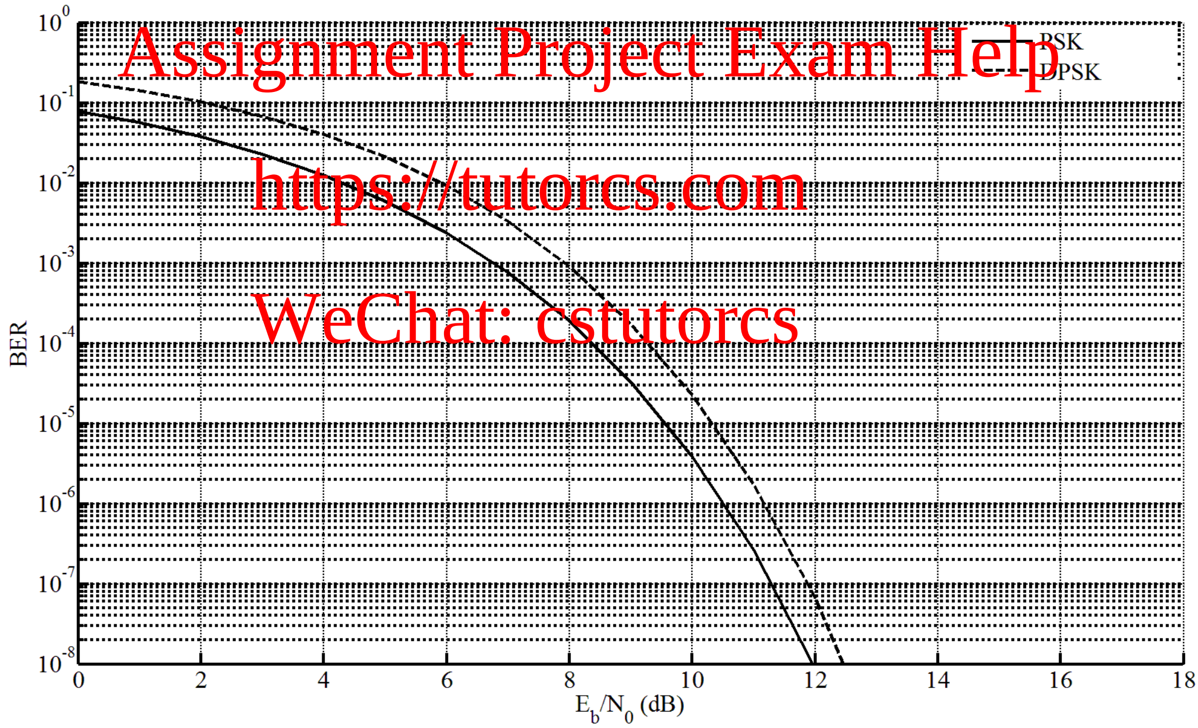
iii) DPSK

$P_e = \frac{1}{2} e^{\frac{-E}{N_o}}$ which is < 1 dB worse than PSK at useful error rates.

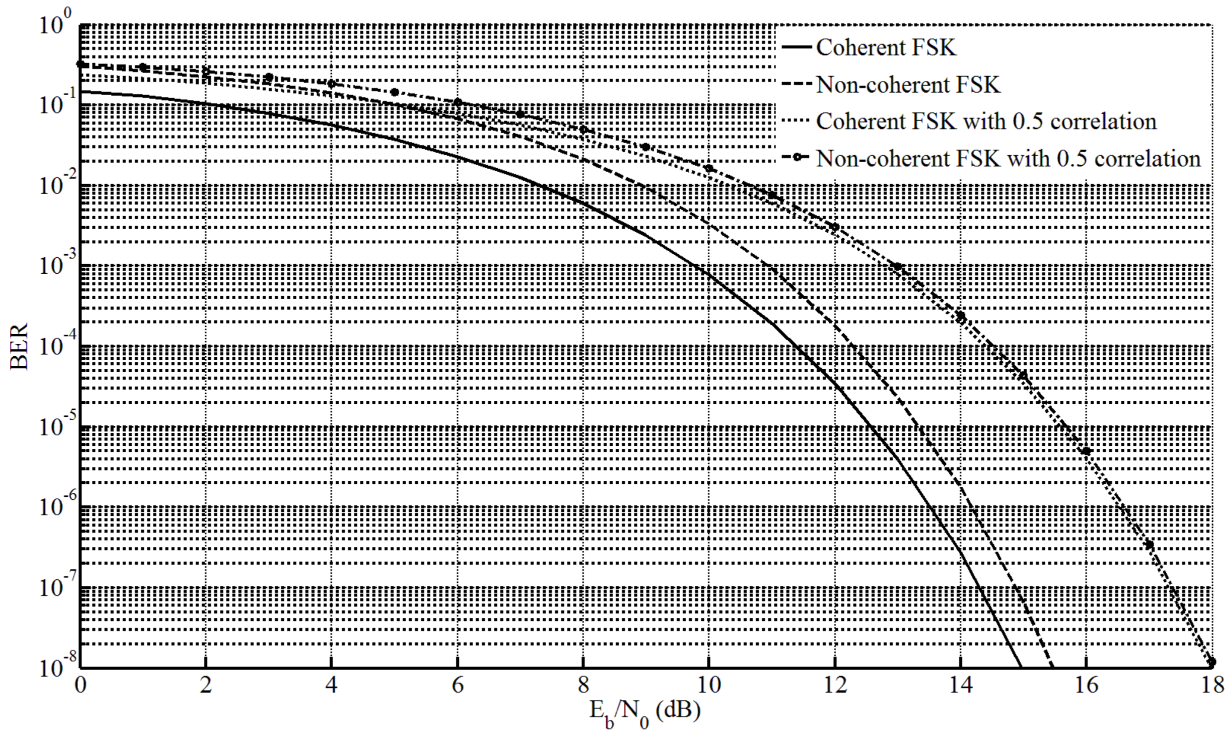
iv) Non-coherent FSK (BP filters and envelope detector)

$P_e = \frac{1}{2} e^{\frac{-E}{2N_o}}$ giving a degration of <1 dB compared with coherent FSK

** from above DPSK and non-coherent FSK are very popular due to their small inferiority to ideal detectors but straight forward implementations. Where in the case of FSK we make sure that *Δf* is large enough to ensure the filter BP's do not overlap. The figures below give predicted bit error rate performance in additive white Gaussian noise for PSK in comparison with DPSK and for coherent and non-coherent FSK with and without correlation between the mark and space signals.



BER curves for PSK and DPSK



BER curves for coherent and non-coherent FSK with and without correlation between the mark and space signals.

Table of values of the error function and the complementary error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

$$\operatorname{erf}c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du$$

x	erf(x)	erfc(x)	x	erf(x)	erfc(x)
0.00	0.0000000	1.0000000	1.30	0.9340079	0.0659921
0.05	0.0563720	0.9436280	1.40	0.9522851	0.0477149
0.10	0.1124629	0.8875371	1.50	0.9661051	0.0338949
0.15	0.1679960	0.8320040	1.60	0.9763484	0.0236516
0.20	0.2227026	0.7772974	1.70	0.9837905	0.0162095
0.25	0.2763264	0.7236736	1.80	0.9890905	0.0109095
0.30	0.3286268	0.6713732	1.90	0.9927904	0.0072096
0.35	0.3793821	0.6206179	2.00	0.9953223	0.0046777
0.40	0.4283924	0.5716076	2.10	0.9970205	0.0029795
0.45	0.4754817	0.5245183	2.20	0.9981372	0.0018628
0.50	0.5204999	0.4795001	2.30	0.9988568	0.0011432
0.55	0.5633234	0.4366766	2.40	0.9993115	0.0006885
0.60	0.6038561	0.3961439	2.50	0.9995930	0.0004070
0.65	0.6420293	0.3579707	2.60	0.9997640	0.0002360
0.70	0.6778012	0.3221988	2.70	0.9998657	0.0001343
0.75	0.7111556	0.2888444	2.80	0.9999250	0.0000750
0.80	0.7421010	0.2578990	2.90	0.9999589	0.0000411
0.85	0.7706681	0.2293319	3.00	0.9999779	0.0000221
0.90	0.7969082	0.2030918	3.10	0.9999884	0.0000116
0.95	0.8208908	0.1791092	3.20	0.9999940	0.0000060
1.00	0.8427008	0.1572992	3.30	0.9999969	0.0000031
1.10	0.8802051	0.1197949	3.40	0.9999985	0.0000015
1.20	0.9103140	0.0896860	3.50	0.9999993	0.0000007