Math 191 Sample Midterm 2B
Spring 2023, Wilkening

Your Name:

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1a. (4 points) Let $A = USV^T$ be a full SVD of an arbitrary $m \times n$ real matrix $A$. Show that the singular values $\sigma_i = S_{ii}$ do not depend on which orthogonal matrices $U$ and $V$ are used.



1b. (4 points) Suppose $A$ is an $m \times n$ matrix with real entries. Prove that $\|A\|_2 = \sigma_1$, where $\sigma_1$ is the largest singular value of $A$ and we recall the definition

$$\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

2. (8 points) Compute the $LDL^T$ factorization of $A = \begin{pmatrix} 3 & 3 & 1 \\ 3 & a & 2 \\ 1 & 2 & b \end{pmatrix}$, assuming it exists. Which values of $a$ and $b$ cause (i) $A$ to be singular? (ii) $A$ to be positive definite?

3. A real $3 \times 3$ matrix $A$, not necessarily symmetric, has eigenvalues 5, 3 and 0 with corresponding eigenvectors $u$, $v$ and $w$, all unit vectors. Suppose $z$ is a unit vector orthogonal to $u$ and to $v$.

(a) (3 points) Find any solution of $Ax = b$ if $b = 2u + 3v$.
(b) (5 points) Find the minimum norm least squares solution of $Ax = b$ if $b = 2u + 3v + 4z$. (The answer will contain $ww^T$ somewhere in the formula. Justify your answer.)

4. (8 points) Compute the polar decompositon $A = Q|A|$ of

$$A = \begin{pmatrix} 9 & 6 & 6 \\ 6 & 0 & 12 \\ 6 & 6 & 0 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 2/3 & -2/3 \\ 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 18 & \\ & 9 \end{pmatrix} \begin{pmatrix} 2/3 & 1/3 & 2/3 \\ 1/3 & 2/3 & -2/3 \end{pmatrix},$$

i.e., compute the matrix entries of the partial isometry $Q$ with the same nullspace as $A$ and the symmetric, positive semi-definite matrix $|A|$.

5. (8 points) Draw the tilted ellipse $(5/2)x^2+2xy+y^2 = 1$ and find the half-lengths of its major and minor axes from the eigenvalues of the corresponding matrix $S$ for which $(5/2)x_1^2 + 2x_1x_2 + x_2^2 = x^TSx$.