CMPSC 464: Introduction to the Theory of Computation **Date: 10/26/2022**

Recitation #8

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Notice: Try to solve these problems before the recitation session by yourself. Sometimes we put up more problems than one is able to discuss in 50 minutes. We want to make sure that we do not run out of problems during each session.

Problem 1

To show $DOMINATING-SET$ is NP-complete, we show that

(a) $DOMINATING-SET \in NP$

(b) A poly-time reduction f from $VERTEX-COVER$ to $DOMINATING-SET$: $VERTEX-COVER \leq_P DOMINATING-SET$.

Define $f : (G, k) \to (G', k')$ as follows:

- Given $G = (V, E)$. Let $G' = (V', E')$:
  for each edge $e = (u, v)$ in the graph G, we add a new node n_{uv} and the edges (u, n_{uv}) and (v, n_{uv}) (like a "triangle" on each edge of G). Formally, $V' = V \cup \{n_{uv}\}_{(u,v)\in E}$ and $E' = E \cup \{(v, n_{uv}), (u, n_{uv})\}_{(u,v)\in E}$
- Let $k' = k$

We need to show that $(G, k) \in VERTEX - COVER$ (i.e., G has a vertex cover of size k) if and only if $(G', k) \in DOMINATING - SET$ (i.e., G' has a dominating set of size k).

We first show that if G has a vertex cover of size k, then G' has a dominating set of size k. Let C be a vertex cover in G with size k. It is easy to see that C is also a dominating set in G', since for each edge (u, v), either u or v (or both) is in C. This makes every node outside C (including new nodes n_{uv}) has a adjacent vertex in C.

We then proceed to the other direction proving that if G' has a dominating set of size at most k, then G has a vertex cover of size at most k. Let D be a dominating set of size k in G'. Consider the two cases:

(a) D only contains vertices from the original G. All the new vertices have an edge to a vertex in D. Since each new vertex n_{uv} corresponds to an edge $(u, v) \in E$, we get that for all edges $(u, v) \in E$ at least one of u, v must be in D and thus D is a valid vertex cover in G.

(b) D contains some new vertices n_{uv} from G'. However, if there are any new vertices in D, say n_{uv}. Construct C: initialize $C = D$; then for every $n_{uv} \in C$, remove it and add u or v in C. Observe that in the end the size of C still equals size of D, and we ensured that every edge (u, v) get covered by C. So there is a vertex cover C in G of size k.

Problem 2

(a) $DOUBLE-SAT$ is in NP.

(b) $SAT \leq_P DOUBLE-SAT$.

TM F computes the polynomial time reduction f.

$F =$ "On input ϕ, a Boolean formula with variables $x_1, \ldots, x_m$:

1. Let ϕ' be $\phi \wedge (x \vee \overline{x})$, where x is a new variable.
2. Output $\langle \phi' \rangle$."

If $\phi \in SAT$, then ϕ' has at least two satisfying assignments that can be obtained from the original satisfying assignment of ϕ by changing the value of x. If $\phi' \in DOUBLE-SAT$, the ϕ is also satisfiable, because x does not appear in ϕ. Therefore $\phi \in SAT$ iff $f(\phi) \in DOUBLE-SAT$