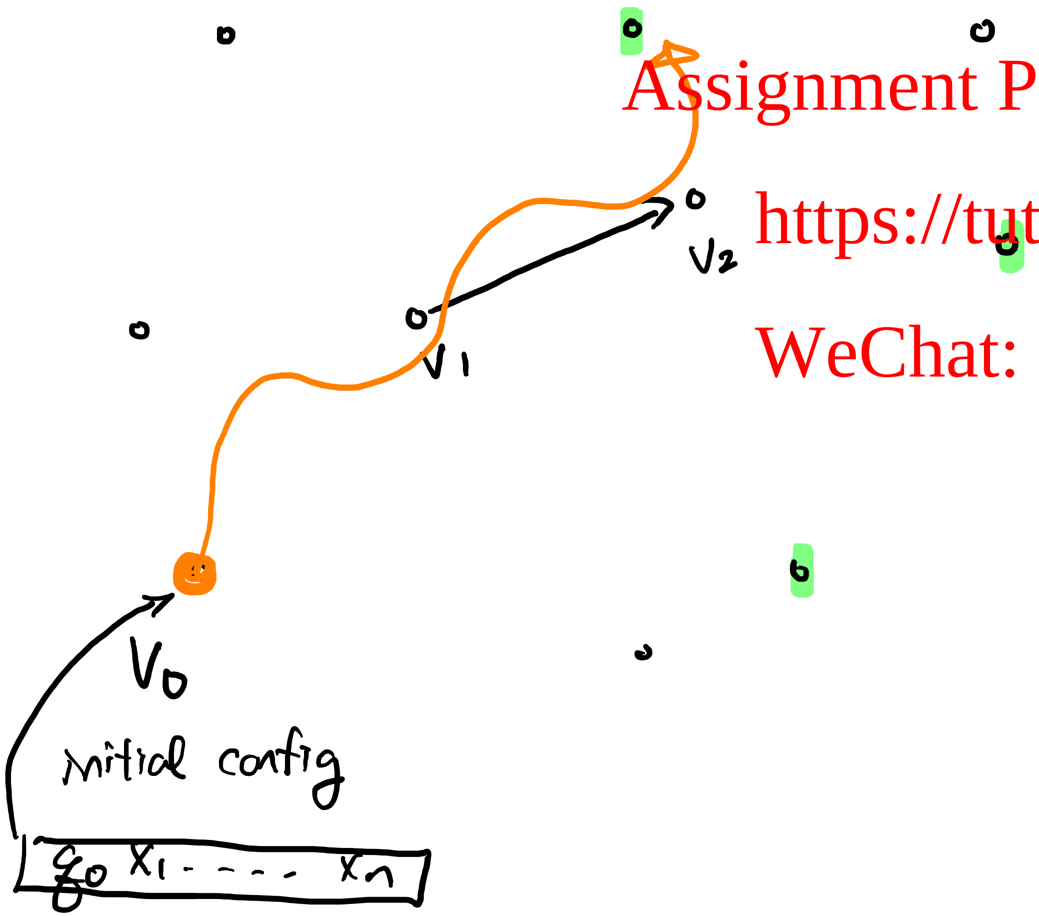
# Lecture 20

# Savitch's Theorem

$\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DSPACE}(S(n)^2)$

# Configuration Graph

N.T.M. $\underline{N}$ with input $\underline{x}$ space usage $s(n)$.




$s(n)$

$|V| \le 2^{O(s(n))}$

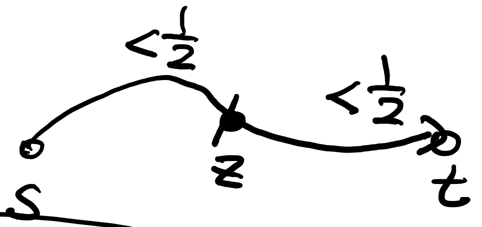
# Graph (s,t) Reachability

- Space efficiently determine reachability

# Reachability (s,z,i)

reach(s,t,i)

$\neq$ $\Rightarrow$ whether or not t is reachable from s in $\le 2^i$ steps.




- reach(s,t,0) $\longrightarrow$ t is a neighbor of s



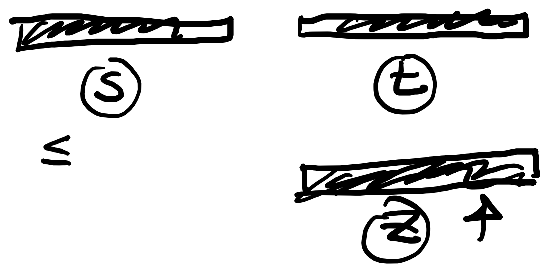
- reach(s,t,i) $\Longleftrightarrow$ $(\exists z \in V)$ (reach(s,z,i−1) and reach(z,t,i−1))

pick z



previous $x \in L$ reduces to reach(V, [illegible], $O(S(n))$)






reach(s,z,i−1) reach(z,t,i−1)

for every z

recycle.

space(i)

$\le O(s(n)) + \cancel{2}\,\text{space}(i-1)$ ff

$= \cancel{2}\cdot \text{space}(i-1) + O(s(n))$

# Space usage calculation

$$\text{Space}(i) \le \text{Space}(i-1) + O(S(n))$$

$$= O(S(n))$$

$$\le \text{space}(i-2) + O(S(n)) + O(S(n))$$

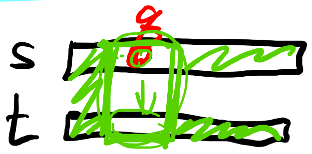


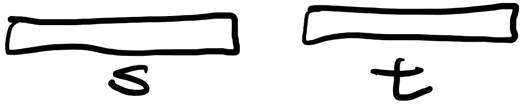
$$\vdots$$

$$\le \text{space}(0) + \underbrace{O(i \cdot S(n))}$$

$$= \boxed{\text{space}(0)} + O(S(n)^2) \le O(S(n)^2)$$

$$\text{space}(0) \rightarrow O(S(n))$$



# Recycling

$L \in NSPACE(S(n)) \longrightarrow$ create a config. graph.

reach( $V_0$, green nodes ).

can be done deterministically using $O(S(n)^2)$ space



green node

$O(S(n))$

workspace

to calculate

reach ($v_0$, green node) $\longrightarrow O(S(n)^2)$

Total space

$= O(S(n)^2)$

# Open Problem

$$\mathrm{NSPACE}(S(n)) \subseteq \mathrm{DSPACE}(S(n)^{2})$$



1.99





??

# PSPACE

# PSPACE

# PSPACE definition

$$\bigcup_{c>0} \underline{\text{DSPACE}}(\underline{\underline{n}}^c)$$

NSPACE

polynomial space.

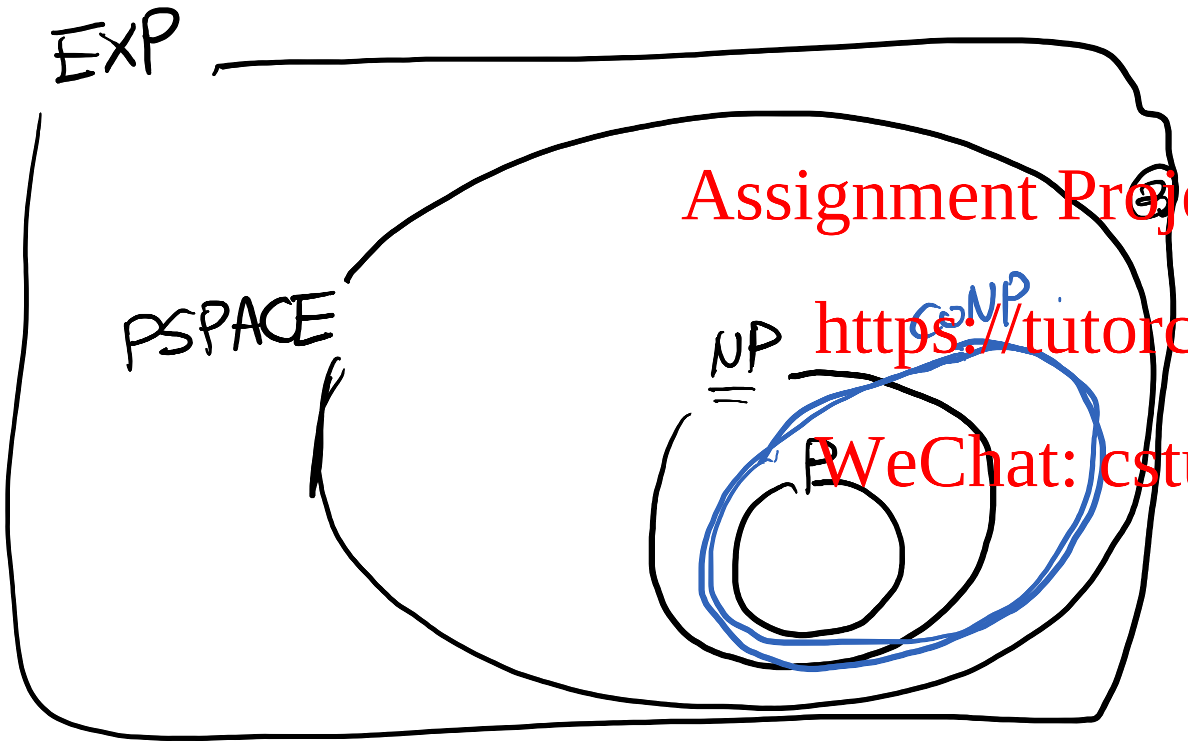
# Corollary of Savitch’s Theorem

- NPSPACE = PSPACE

$\mathrm{DSPACE}(n^{2c}) \supseteq \mathrm{NSPACE}(n^{c})$

Savitch's thm.

# Venn Diagram




① $P \subseteq PSPACE$ ?

$DSPACE(f(n)) \supseteq DTIME(f(n))$

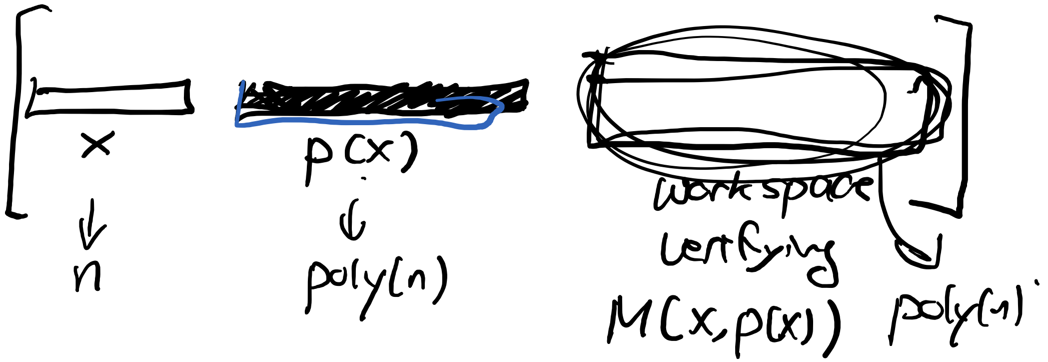
② $\boxed{NP \subseteq PSPACE\ ?}$

$NTIME(f(n)) \subseteq NSPACE(f(n))$

$x \in L$ iff $\exists p(x)$ $\exists$ D.T.M. $M(x, p(x)) = 1$.

$\forall p(x)$ $M(x, p(x)) = 0$ polynomial.

$|p(x)| \leq poly(n)$



# PSPACE-completeness

- L is PSPACE-complete if

1. L is in PSPACE ① $L \in PSPACE$
2. Every A in PSPACE is polytime reducible to L (PSPACE-hard)

② $\forall A \in PSPACE \quad A \leq_p L$.

# TQBF (Generalization of SAT)

- $Q_1 x_1 Q_2 x_2 \dots Q_n x_n \varphi(x_1, \dots, x_n)$

$(\exists x_1 \dots x_n \in \{0,1\} \; \varphi(x_1 \dots x_n)) \in SAT$

$(\forall x_1 \dots x_n \in \{0,1\} \; \neg\varphi(x_1 \dots x_n)) \in \neg SAT.$



PSPACE-complete.





$Q_1 \dots Q_n = \exists$ → NP-complete, this is SAT

$Q_1 \dots Q_n \Rightarrow \forall$ → coNP-complete, $\neg$SAT.

# TQBF is PSPACE-complete

- First TQBF is in PSPACE (recursive procedure)

$\mathcal{M}(\phi \in TQBF)$ returns 1, o/w 0

space$(n)$

$Q_1 x_1 \, Q_2 x_2 \ldots Q_n x_n \, \varphi(x_1 \ldots x_n)$

$Q_1 \to \exists$, $x_1 = 1$: $Q_2 x_2 \ldots Q_n x_n \, \varphi(1, \ldots x_n)$ $\to \mathcal{M}(\phi) = 1$

$x_1 = 0$: $Q_2 x_2 \ldots Q_n x_n \, \varphi(0, \ldots x_n)$

$\forall$ $x_1 = 1$: $Q_2 x_2 \ldots Q_n x_n \, \varphi(1, \ldots x_n)$

$x_1 = 0$: $\varphi(0, \ldots x_n)$ $\wedge$

1

$\mathcal{M}(\_|_{x_1=1}) \wedge \mathcal{M}(\_|_{x_1=0})$ $Q_1 = \forall$

$\vee$ $Q_1 = \exists$

space$(n) \leq \cancel{2}$ space$(n-1) + O(n)$

store the formula.

space$(n) \leq$ space$(n-1) + O(n)$

$\Longrightarrow$ space$(n) \leq$ poly$(n)$