CMT107 Visual Computing

I.3 Vectors and Matrices

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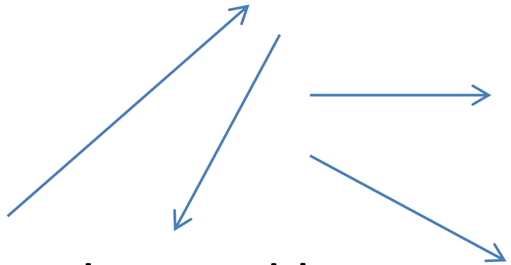
Overview

- Vectors
  - Vector Operations
  - Vector Geometry
  - Vector Projection
- 3D Vectors
  - Cross Product
  - 3D Vector Geometry
- Matrices
  - Special Matrices
  - Matrix Operations
  - Determinant

Vectors

- A vector is a directed line segment, characterised by:
  - Length
  - Direction
  - But NOT Position



- A vector with length 0 is a zero vector, denoted by **0**
  - Zero vectors doesn't have direction.

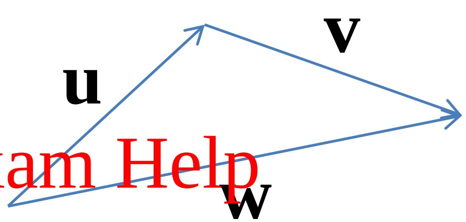
- A vector with length 1 is a unit vector.

- A vector **u** with the same length but opposite direction of vector **v** is the negative vector of **v**, denoted by **u** = -**v.**

- Two vectors are equal iff they have the same length and the same direction.
  - Two zero vectors are always equal, though their directions are undefined.

Vector Operations

- A vector **u** multiplied by a scalar α denoted by α**u** has the same direction of **u** if α >0 and the opposite direction if α <0. The length of α**u** is $|\alpha|$ times of the length of **u**.
- The sum **w** of two vectors **u** and **v**:



w = u + v

follows the head-to-tail rule. That is,

if the head of **u** is connected to the tail of **v**, then **w** is the directed line segment from the tail of **u** to the head of **v**.

- The subtraction of vector **v** from vector **w** is the addition of vector **w** and vector -**v**

u = w – v = w + (-v)

Vector Operations

- A nD vector is represented by:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{or} \quad \mathbf{v} = [v_1\ v_2 \cdots v_n]^T$$

- The sum and subtraction of two vectors are:

$$\mathbf{u} + \mathbf{v} = [u_1 + v_1\ u_2 + v_2 \cdots u_n + v_n]^T$$

$$\mathbf{u} - \mathbf{v} = [u_1 - v_1\ u_2 - v_2 \cdots u_n - v_n]^T$$

- The multiplication of a vector **v** by a scalar λ is defined by

$$\lambda \mathbf{v} = [\lambda v_1\ \lambda v_2 \cdots \lambda v_n]^T$$

- The inner product (dot product, scalar product) of two vectors is:

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Vector Geometry

- A vector has direction and length.
  - The length is defined by

$$|\mathbf{v}| = \sqrt{\mathbf{v}\cdot\mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

  - The direction is parallel to the direction from the origin to the point $(v_1, v_2, \cdots, v_n)$ in *n*D Euclidean space.
  - The angle θ between two vectors **u** and **v** is calculated by

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

- Normalisation of a vector **v** gives a unit vector **v'**, which has length 1:

$$\mathbf{v}' = \mathbf{v}/|\mathbf{v}|$$

- Vectors **u** and **v** are orthogonal if $\mathbf{u}\cdot\mathbf{v} = 0$, which means that they are perpendicular to each other, and the angle between these two vectors are 90°.

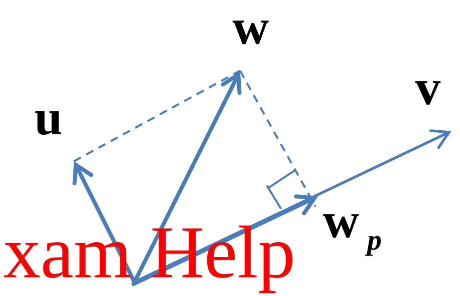
Vector Projection

- The vector projection $\mathbf{w}_p$ of a vector **w** on a nonzero vector **v** is a vector parallel to **v**, defined by

$$\mathbf{w}_p = \alpha \mathbf{v}$$

where α is a scalar calculated by

$$\alpha = \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}$$



- The vector **w** then can be represented by the sum of $\mathbf{w}_p$ and vector **u**, which is perpendicular to **v** (and $\mathbf{w}_p$).

$$\mathbf{w} = \mathbf{w}_p + \mathbf{u}$$

$$\mathbf{u} \cdot \mathbf{v} = 0, \quad \mathbf{u} \cdot \mathbf{w}_p = 0$$

Cross Product

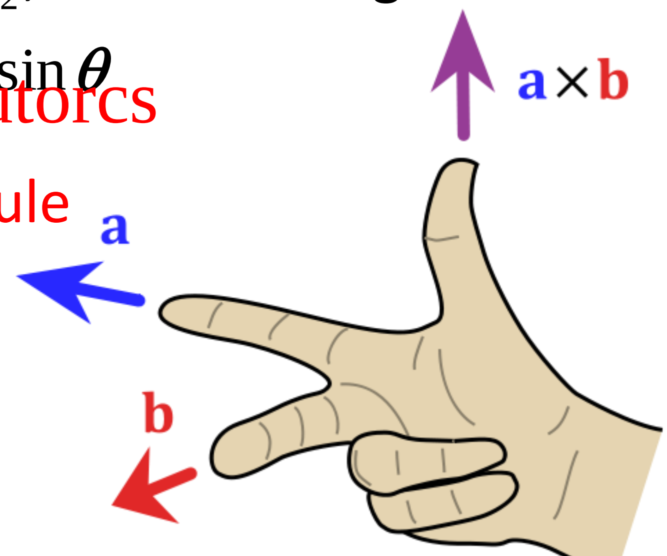
- Denote two 3D vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ by $\mathbf{v}_i = [x_i, y_i, z_i]^T$, the vector product (also called cross product, outer product) of $\mathbf{v}_1$ and $\mathbf{v}_2$ is defined by

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ z_1 x_2 - x_1 z_2 \\ x_1 y_2 - x_2 y_1 \end{bmatrix}$$

- If θ is the angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, then the length is:

$$|\mathbf{v}_1 \times \mathbf{v}_2| = |\mathbf{v}_1||\mathbf{v}_2| \sin\theta$$

- The direction satisfies right-hand rule
  - $\mathbf{v}_1 \times \mathbf{v}_2$ is perpendicular to both $\mathbf{v}_1$ and $\mathbf{v}_2$.



http://en.wikipedia.org/wiki/File:Right_hand_rule_cross_product.svg

3D Vector Geometry

- A point in 3D space can be represented by a 3D vector:

$$\boldsymbol{p} = [\boldsymbol{x}\ \boldsymbol{y}\ \boldsymbol{z}]^T$$

- A directed line segment from $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ can be represented by vector:

$$\boldsymbol{v} = \boldsymbol{p}_2 - \boldsymbol{p}_1$$

- Let $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are two directed line segments on a plane, then the normal direction is determined by the cross product of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$:

$$\boldsymbol{n} = \boldsymbol{v}_1 \times \boldsymbol{v}_2$$

Matrices

- A matrix is a rectangular array of scalars, arranged in rows and columns. The individual items in a matrix are called its elements or entries. The number of rows and columns are referred to as the row and column dimensions.
- The following matrix **A** has row dimension *m* and column dimension *n*, or simply, *m* x *n* dimension. a_{ij} is an element of the matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- The transpose of an *m* x *n* matrix **A**, denoted by $\mathbf{A}^T$, is the *n* x *m* matrix obtained by interchanging the rows and columns of A.
- To save the space, the matrix is often written as A = $[a_{ij}]_{mxn}$, or simply, A = $[a_{ij}]$, if the dimension of the matrix is implicitly known.

Special Matrices

- A square matrix is a matrix which has the same row and column dimension.
- A symmetric matrix is a square matrix that is equal to its transpose. Let **A**=[a_{ij}] be a symmetric matrix, then **A** = **A**^{T}. Its elements satisfy

$$\boldsymbol{a}_{ij} = \boldsymbol{a}_{ji}$$

- A diagonal matrix is a matrix (usually square matrix) in which the elements outside the main diagonal are all zero, *i.e.*, **A**=[a_{ij}],

$$\boldsymbol{a}_{ij} = 0, \quad \text{if } \boldsymbol{i} \neq \boldsymbol{j}$$

- An identity matrix, denoted by **I**, is a square diagonal matrix with 1's on the diagonal and 0's elsewhere

$$\boldsymbol{I} = [\boldsymbol{a}_{ij}], \quad \boldsymbol{a}_{ij} = \begin{cases} 1 & \text{if } \boldsymbol{i} = \boldsymbol{j}; \\ 0 & \text{otherwise.} \end{cases}$$

Special Matrices

- A row matrix is a matrix of dimension 1 x n. It is also called a row vector.

$$\mathbf{a} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_n \end{bmatrix}$$

- A column matrix is a matrix of dimension m x 1, also called a column vector.

$$\mathbf{a} = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_m \end{bmatrix} \qquad \mathbf{a} = \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \cdots & \boldsymbol{a}_m \end{bmatrix}^T$$

Matrix Operations

- Scalar-Matrix multiplication is defined by multiplying each element by the scalar
  - $\alpha \mathbf{A} = [\alpha a_{ij}]$
- Matrix-Matrix Addition of two matrices of the same dimension is defined by adding corresponding elements of the two matrices
  - $\mathbf{C} = \mathbf{A} + \mathbf{B} = [a_{ij} + b_{ij}]$

- Matrix-Matrix Multiplication of an m x l dimensional matrix $\mathbf{A}$ and an l x n dimensional matrix $\mathbf{B}$ is defined by
  - $\mathbf{C} = \mathbf{AB} = [c_{ij}]$
  - Where $c_{ij} = \sum_{k=1}^{l} a_{ik} b_{kj}$

- Inverse of a Square Matrix $\mathbf{A}$ is a square matrix $\mathbf{B}$, such that

$$\mathbf{AB} = \mathbf{I}$$

  - Denote by $\mathbf{B} = \mathbf{A}^{-1}$

Orthogonal Matrix

- An orthogonal matrix is a square matrix with real entries whose columns and rows are orthogonal unit vectors (i.e., orthonormal vectors).
- Equivalently, a matrix **Q** is orthogonal if its transpose is equal to its inverse:

$$\mathbf{Q}^{-1} = \mathbf{Q}^T$$

which entails

$$\mathbf{Q}\mathbf{Q}^T = \mathbf{Q}^T\mathbf{Q} = \mathbf{I}$$

Determinant

- The determinant is a value associated with a square matrix, denoted by det(**A**), det **A**, or |**A**|. It is defined as

$$|\mathbf{A}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |\mathbf{A}_{ij}|$$

  - where $\mathbf{A}_{ij}$ is the (i, j) minor matrix of **A**, which is obtained by deleting the *i*th row and the *j*th column of **A**.

- The determinant of a 2 x 2 matrix is calculated by

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- The determinant of a 3 x 3 matrix is calculated by

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}|\mathbf{A}_{11}| - a_{12}|\mathbf{A}_{12}| + a_{13}|\mathbf{A}_{13}| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Cross Product Using Determinant

- The cross product of two 3D vectors can be calculated using determinant as follows:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

$$= (y_1 z_2 - y_2 z_1)\mathbf{i} + (z_1 x_2 - z_2 x_1)\mathbf{j} + (x_1 y_2 - x_2 y_1)\mathbf{k}$$

Summary

- What are the characteristics of a vector?
- What operations are defined for vectors.
- How to calculate the vector projection onto another vector?
- How to calculate cross product? What is the geometric meaning of cross product?
- How to do matrix operations?
- What is a orthogonal matrix?
- How to calculate determinant?