CMT107 Visual Computing

VII.1 Curves

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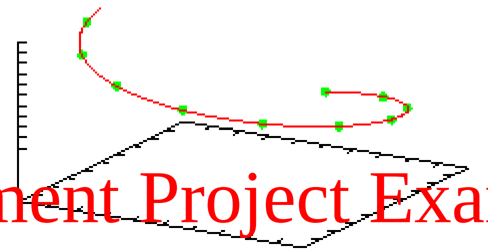
Overview

- Curve representations
  - Explicit representation
  - Implicit representation
- Parametric representation of curves
  - Piecewise polynomial curves (spline curves)
  - Bézier curves

Curves

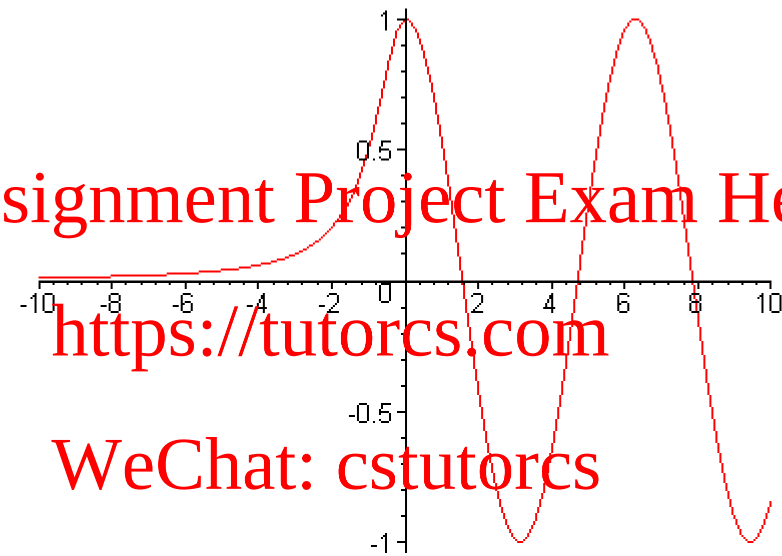
- A curve is a set of positions of a point moving with one degree of freedom



- Useful to describe shapes on a *higher level*
  - Not only straight lines or curved shapes approximated by short line segments
  - Simpler to create, edit and analyse
  - More accurate rendering and less storage (compared to linear approximation)

Explicit Representation

- *Explicit curve*: $y = f(x)$
  - Essentially a *function plot* over some interval $x \in [a, b]$



- Properties:
  - Simple to compute points and plot them
  - Simple to check whether a point lies on curve
  - Cannot represent closed or multi-valued curves:

    Only one y value for each x value (a function)

Implicit Representation

- Define curves *implicitly* as *solution of an equation system*
  - Straight line in 2D: $Ax + By + C = 0$
  - Circle of radius R in 2D: $x^2 + y^2 - R^2 = 0$
  - Conic section: $Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$
  - Matrix/vector representation up to order two:

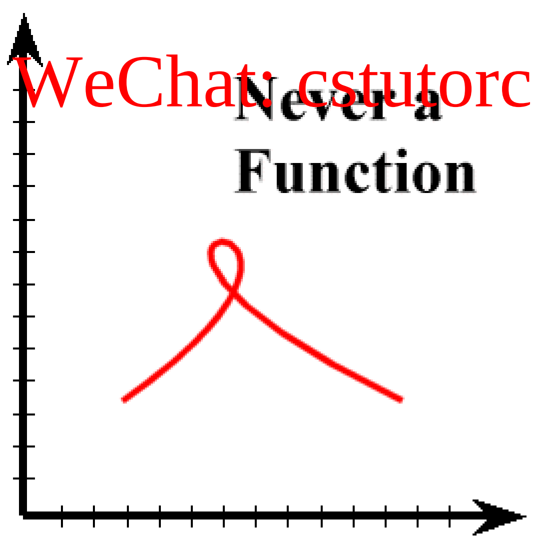
$$\mathbf{x}^T M\mathbf{x} + v^T\mathbf{x} + s = 0 \quad (\mathbf{x} = [x\ y]^T)$$

- In 3D, *two* equations are needed (*1 equation restricts 1 variable*, but there are 3 variables)

  - Straight line: $Ax + By + Cz + D = 0,$
    $Ex + Fy + Gz + H = 0$
  - A circle in x-y plane: $x^2 + y^2 = r^2,$
    $z = 0$

Properties of Implicit Curves

- Mainly use polynomial or rational functions
- Coefficients determine geometric properties
- *Properties:*
  - Hard to render (have to solve non-linear equation system)
  - Simple to check whether a point lies on curve
  - Can represent closed or multi-valued curves



Parametric Curves

- Describe the position on the curve by a parameter $u \in R$

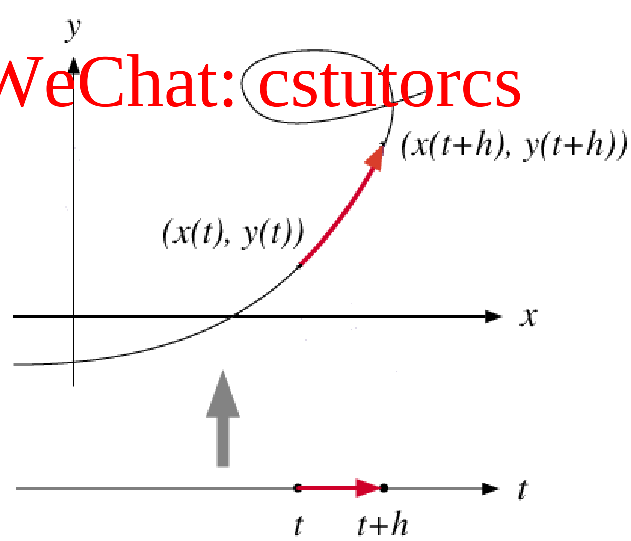
$$c(u) = \begin{pmatrix} x(u) \\ y(u) \\ z(u) \end{pmatrix}$$

- $x(u)$, $y(u)$, $z(u)$ are usually polynomial or rational functions in u
- $u \in [a, b]$, usually $u \in [0, 1]$

- Parameter function maps parameter to model coordinates
  - Parameter space: u (parameter domain)
  - Model space: x, y, z (Cartesian coordinates)

Properties of Parametric Curves

- *Properties*:
  - Simple to render (evaluate parameter function)
  - Hard to check whether a point lies on curve (must compute inverse mapping from (x, y, z) to u; involves solving non-linear equations)
  - Can represent closed or multi-valued curves

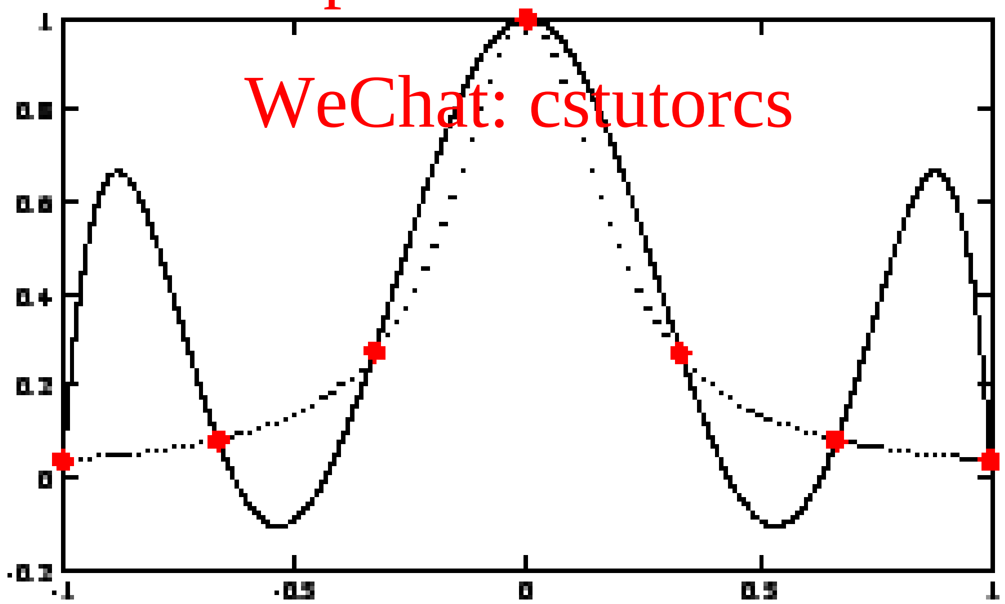


Parametric Polynomial Curves

- Describe coordinates by *polynomials*:

$$x(u) = \sum_{l=0}^{d} A_l u^l, \quad y(u) = \sum_{l=0}^{d} B_l u^l, \quad z(u) = \sum_{l=0}^{d} C_l u^l$$

- *Smooth* (infinitely differentiable)
- Higher order curves (say > 4) cause *numerical problems*
- Hard to control shape by *interpolation*



Bernstein Polynomials

- Bernstein *basis* polynomials

$$b_l^d(u) = \binom{d}{l} u^l (1-u)^{d-l}, \; l = 0, 1, \cdots, d.$$

  - $\binom{d}{l} = \dfrac{d!}{l!(d-l)!}$ is binomial coefficient.

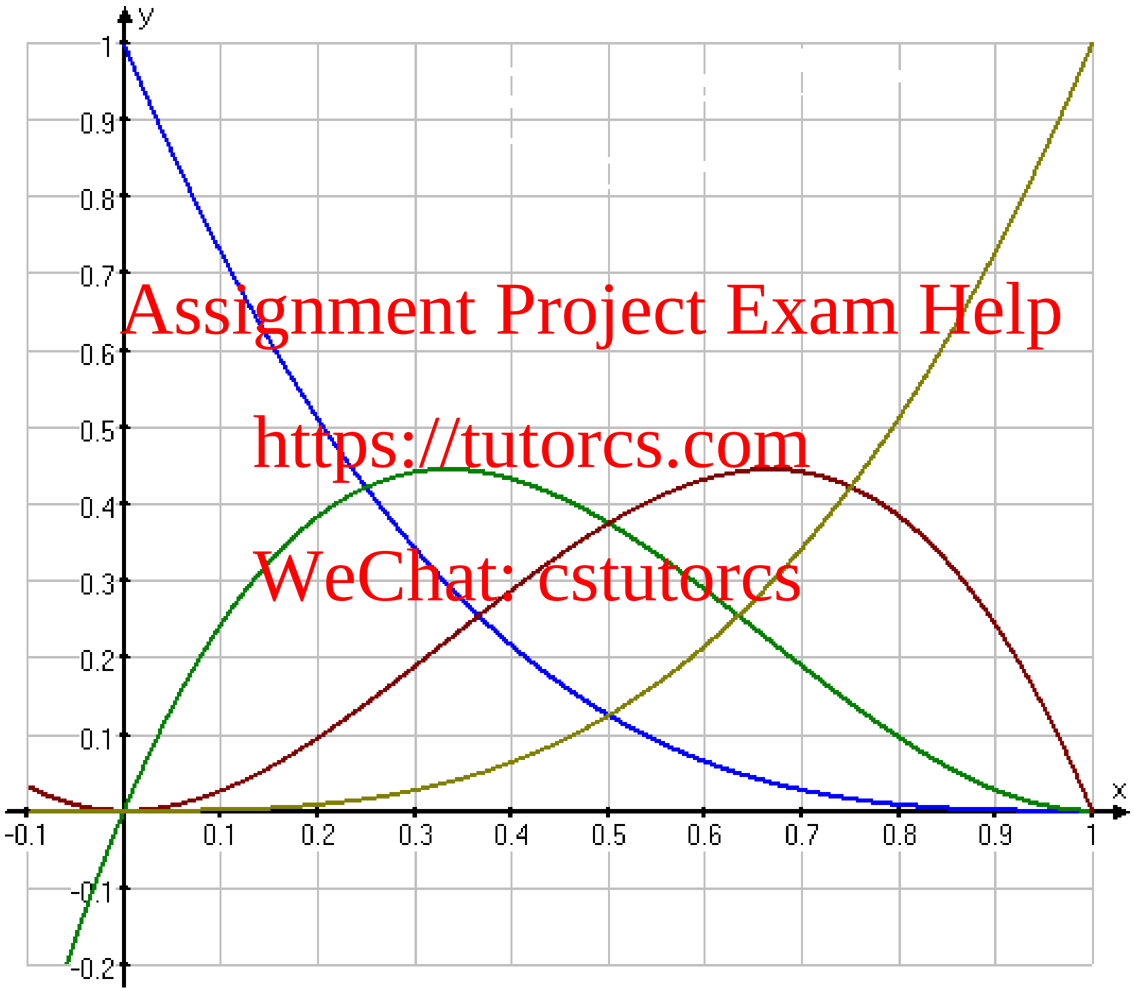
  - Property: $\sum_{l=0}^{d} b_l^d(u) = 1$ for $u \in [0,1]$

- A Bernstein polynomial is a linear combination of Bernstain basis polynomials

$$B(u) = \sum_{l=0}^{d} \beta_l b_l^d(u), \; u \in [0, 1].$$

Cubic Bernstein Basis Polynomials

- There are 4 cubic Bernstein basis polynomials



Piecewise Polynomial Curves

- Cut curve into *segments* and represent each segment as *polynomial* curve
- Can use *low-order polynomial* curves, e.g. cubic (order 3)
- But how to guarantee *smoothness at the joints*?
  - Continuity problem

Spline Curves

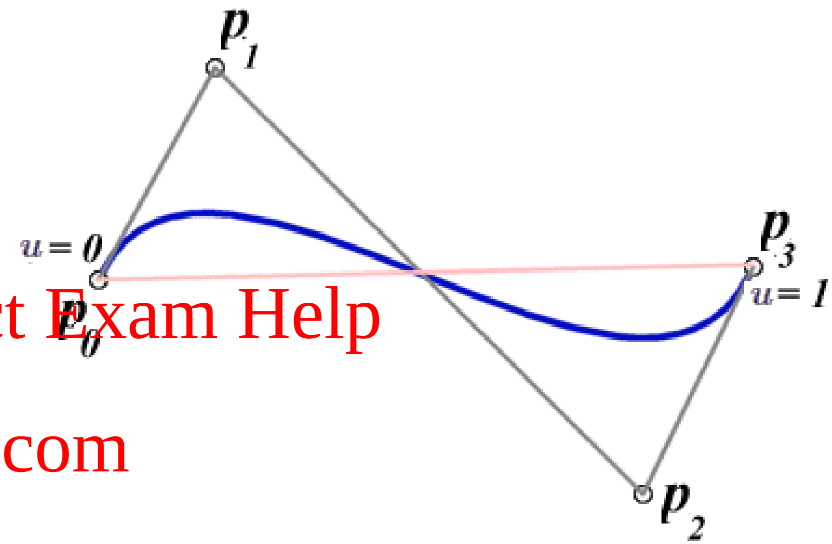
- In general, piecewise polynomial curves are called splines
  - Motivated by loftsman's spline
    - Long narrow strip of wood or plastic
    - Shaped by lead weights (called ducks)
  - Gives curves that are smooth or fair

Bézier Curves

- Represent a polynomial segment as

$$Q(u)=\sum_{l=0}^{d} p_l b_l^d(u),\ u\in[0,1]$$

$$Q(u)=\sum_{l=0}^{d} p_l \binom{d}{l} u^l (1-u)^{d-l},\ u\in[0,1].$$



  - Control points $p_l \in R^3$ or R^2 determine segment's shape
  - $b_l^d(u)$: l^{th} *Bernstein basis polynomial* of degree d.
- Cubic Bézier curve ($d = 3$) has four control points
  - Note that $\sum_{l=0}^{d} b_l^d(u)=1$ for $u\in[0,1]$

    ➡ *Convex combination* of control points

Properties of Bézier Curves

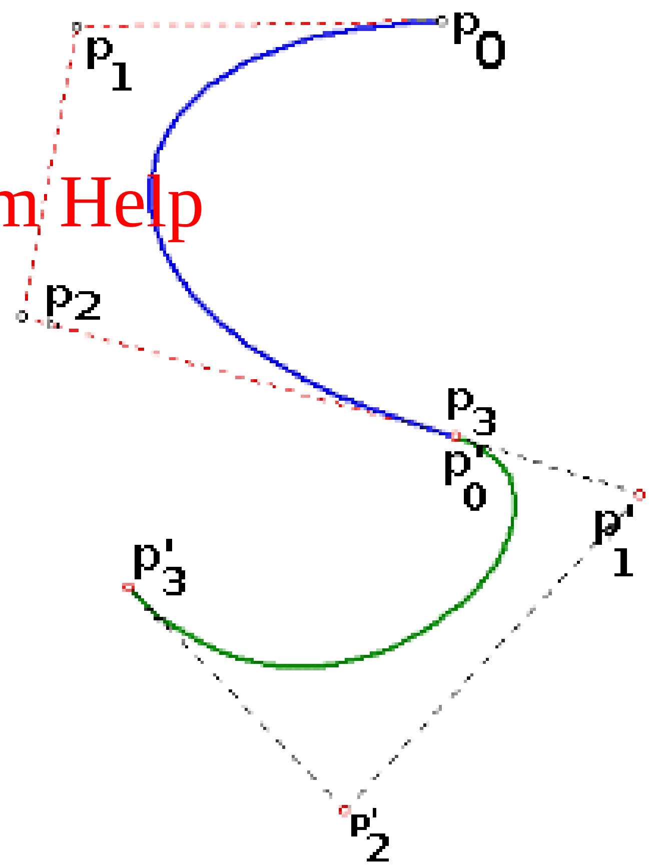
- *Convex hull*:
  - curve lies inside the convex hull of its control points
- *Endpoint interpolation:*

$$Q(0) = p_0$$
$$Q(1) = p_d$$

- *Tangents*

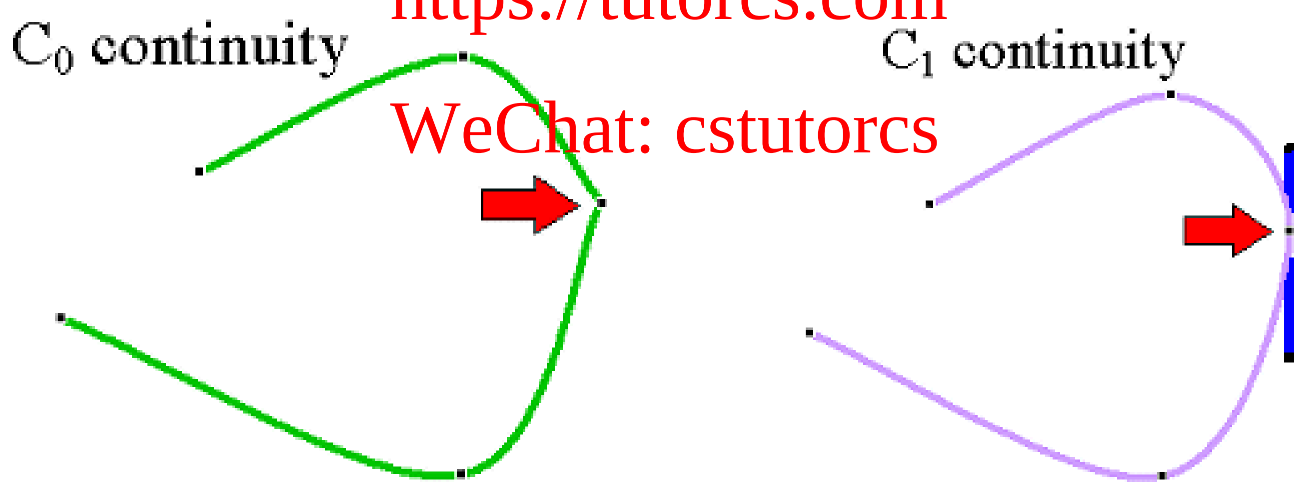
$$Q'(0) = d(p_1 - p_0)$$
$$Q'(1) = d(p_d - p_{d-1})$$

- *Symmetry*
  - Q(u) defined by $p_0, \ldots, p_d$ is equal to Q($1 - u$) defined by $p_d, \ldots, p_0$



Smooth Bézier Curves

- *Smooth joint* between two Bézier curves of order *d* with control points $\{p_0, \ldots, p_d\}$, $\{p'_0, \ldots, p'_d\}$ respectively
  - C$_0$: same end-control-points at joints: $p_d = p'_0$ (due to end-point interpolation)
  - C$_1$: control points $p_{d-1}, p_d = p'_0, p'_1$ must be collinear (due to tangent property)



➡ Continuity conditions create restrictions on control points

Parametric/Geometric Continuity

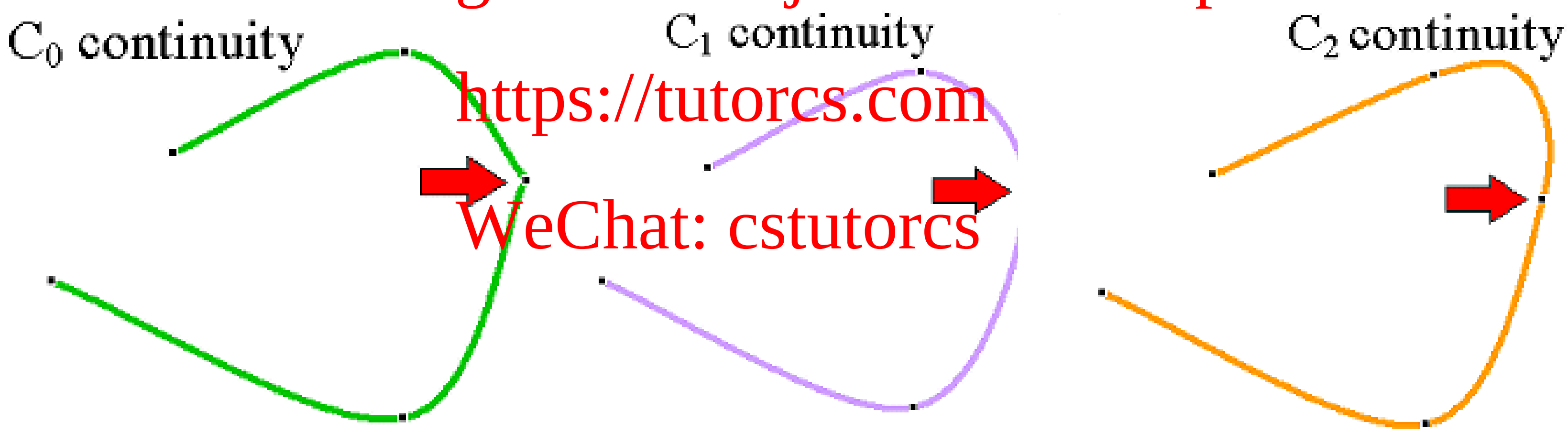
- Parametric continuity:
  - C^0: curves are joined
  - C^1: first derivatives are equal at the joint points
  - C^2: first *and* second derivatives are equal

    …

  - C^n: first *through* n^{th} derivatives are equal
- Geometric continuity:
  - G^0: The curves touch at the joint points
  - G^1: The curves also share a common tangent direction at the joint points (first derivatives are *proportional*)
  - G^2: The curves also share a common centre of curvature at the joint points (first and second derivatives are *proportional*)

Smoothness / Continuity

- Curve should be *smooth* to some order at joints
- Different types of *continuity at joints*
- *Geometric continuity*: from the geometric viewpoint
- *Parametric continuity*: for parametric curves



- Parametric continuity of order *n* implies geometric continuity of order n, but not vice versa.

Summary

- What is the implicit and explicit representation of a curve? What are the advantages and disadvantages of these representations?
- What are piecewise parametric polynomial curves (splines)? What is the advantage of this representation? What is the main problem?
- What are Bézier Curves and how are they defined? What properties do they have?
- What is the major problem when using piecewise polynomial curves? What conditions do the control points of a Bézier Curve have to fulfil in order to get C_0/C_1 continuous curves?