Anomaly Detection Using Support Vector Machines

COMP90073
Security Analytics

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Semester 2, 2021

Outline

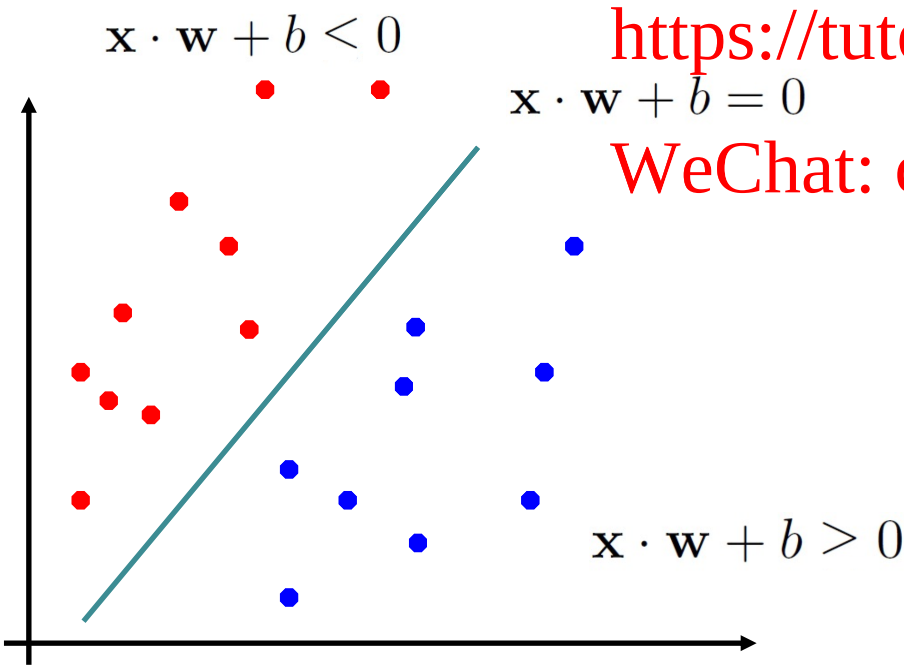
- Review of SVM
- Support Vector Data Description (SVDD)
- One-class Support Vector Machine (OCSVM)
- Recent developments of OCSVM/SVDD

SVM – Revision

$$\{\mathbf{x}_i, y_i\} \text{ where } i = 1 \dots L, y_i \in \{-1, 1\}, \mathbf{x}_i \in \mathbb{R}^D$$

This hyperplane can be described by $\mathbf{x} \cdot \mathbf{w} + b = 0$ where:

- $\mathbf{w}$ is normal to the hyperplane.
- $\frac{b}{\|\mathbf{w}\|}$ is the perpendicular distance from the hyperplane to the origin.



Classification rule

$$f(\mathbf{x}) = \text{sign}(\mathbf{x} \cdot \mathbf{w} + b) = \begin{cases} +1 \text{ if } \mathbf{x} \cdot \mathbf{w} + b \geq 0 \\ -1 \text{ if } \mathbf{x} \cdot \mathbf{w} + b < 0 \end{cases}$$

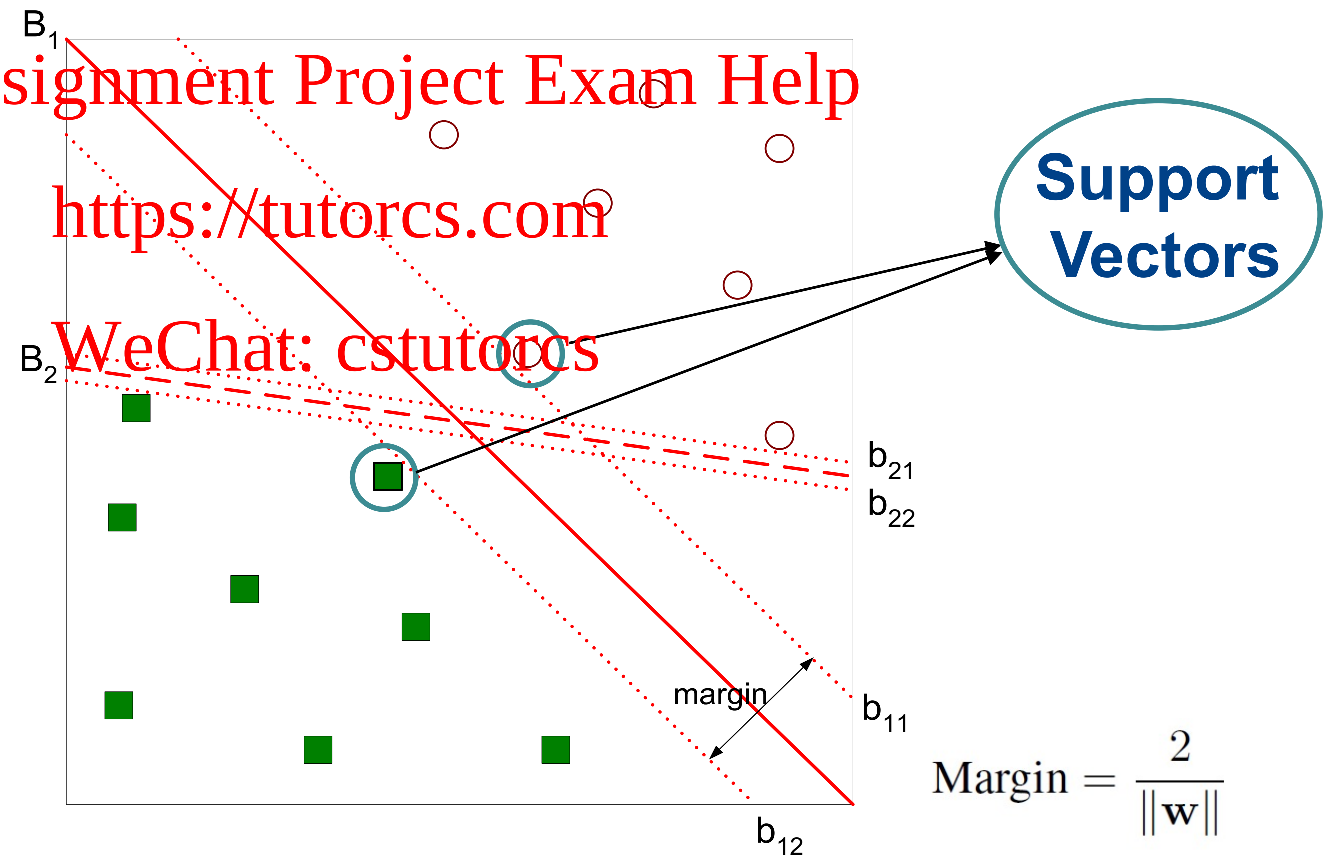
Find $\mathbf{w}$ and b such that:

$$\mathbf{x}_i \cdot \mathbf{w} + b \geq 0 \text{ for } y_i = +1$$
$$\mathbf{x}_i \cdot \mathbf{w} + b < 0 \text{ for } y_i = -1$$
$$\text{for all } i = 1 \dots L$$

Training objective

Large Margin Classifiers – Revision

- Find hyperplane maximises the margin => B1 is better than B2
- Margin: sum of shortest distances from the planes to the positive/negative samples



$$\text{Margin} = \frac{2}{\|\mathbf{w}\|}$$

Solving the Optimization Problem – Revision

***Primal problem**: solve for **w** and b*

$$\min \frac{1}{2}\|w\|^2 \qquad \text{s.t.} \qquad y_i(w^T\phi(x_i) + b) \geq 1,\ \forall\, i = 1, \ldots, n$$

Equivalent **dual problem** formulation: solve for $\alpha_1 \ldots \alpha_L$: Lagrange multipliers for each data point

$$\max_{\alpha} \sum_{i=1}^{L} \alpha_i - \frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{L} \alpha_i \alpha_j\, y_i y_j\, K(x_i, x_j)$$

s.t.

$$\alpha_i \geq 0$$

$$\sum_{i=1}^{L} \alpha_i y_i = 0$$

More convenient to solve

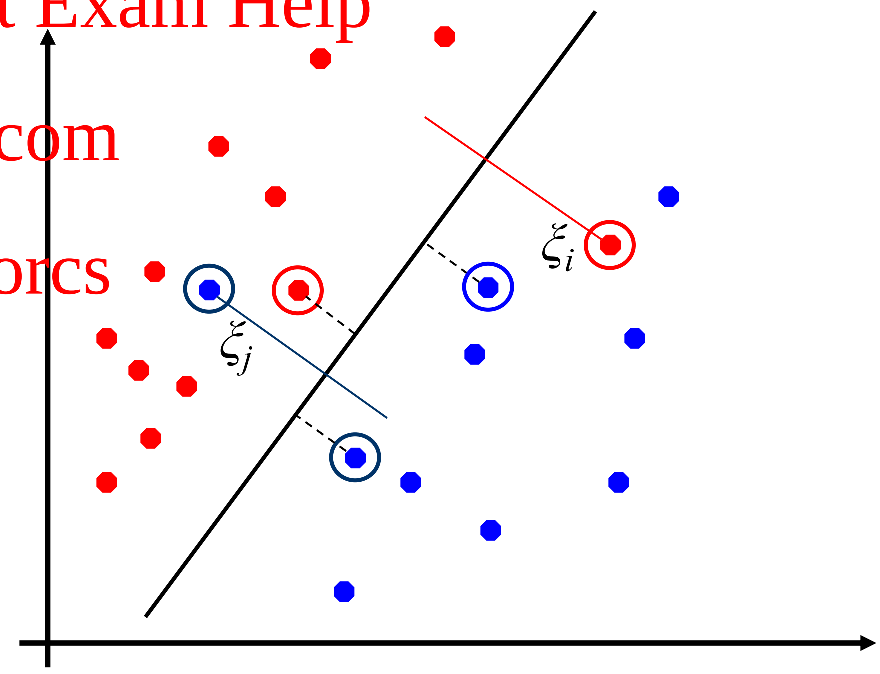
Soft Margin Classification – Revision

- ***Slack variables*** ξ_i can be added to allow misclassification of difficult or noisy examples, resulting margin called *soft*.

$$\min_{w,b,\xi_i} \frac{1}{2}\|w\|^2 + C\sum_{i=1}^{n}\xi_i$$

s.t.

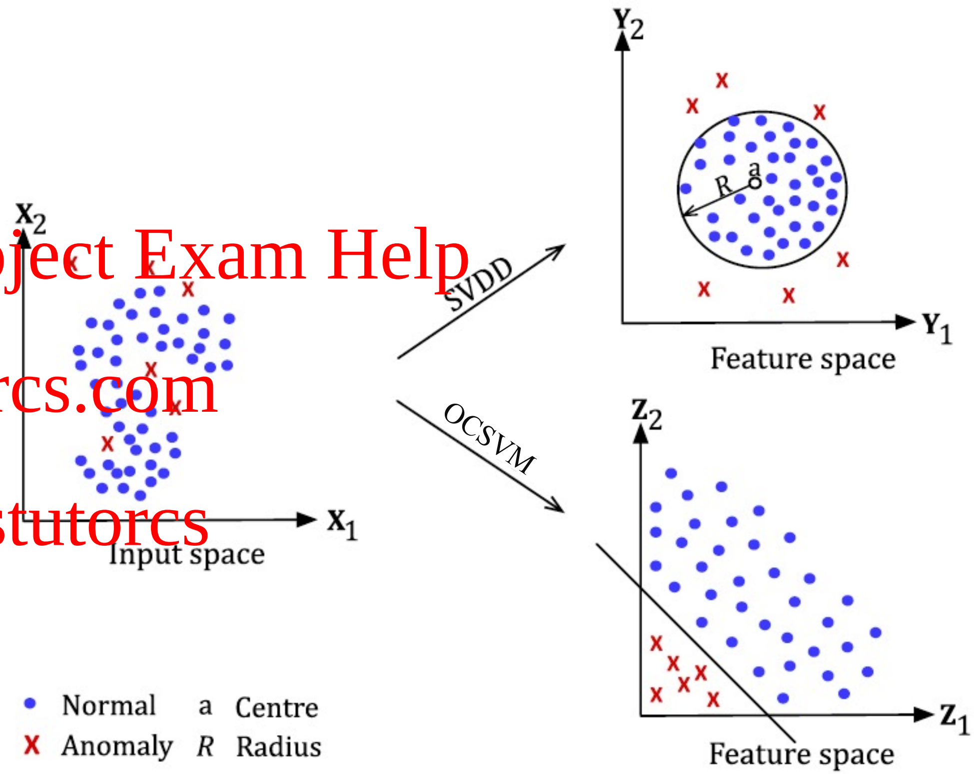
$$y_i(w^T\phi(x_i)+b) \geq 1-\xi_i,\ \forall\, i=1,\dots,n$$
$$\xi_i \geq 0, \forall\, i=1,\dots,n$$



Anomaly Detection using SVM

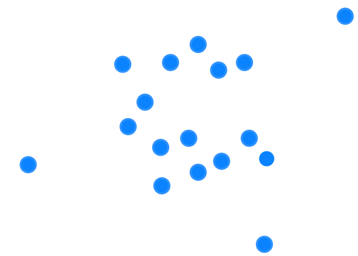
- **Assumption:** All (or majority of) training examples belong to the *normal* (positive) class.

- **Objective:** identify anomalies by modeling normal pattern.
  - Support Vector Data Description (SVDD) [1]
  - One-class Support Vector Machine (OCSVM) [2]



Support Vector Data Description (SVDD) [1]

- Find the minimal circumscribing hyperball in high-dimensional space encompass (almost) all the observations.



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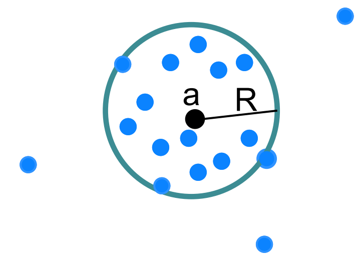

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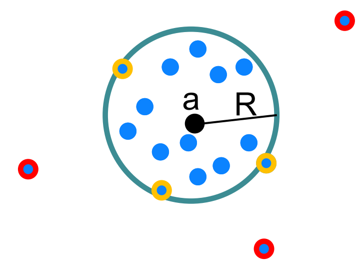

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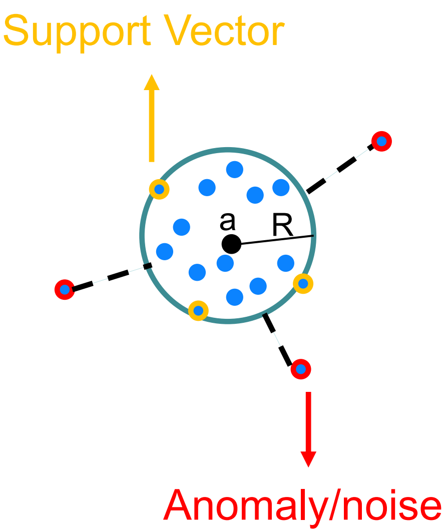

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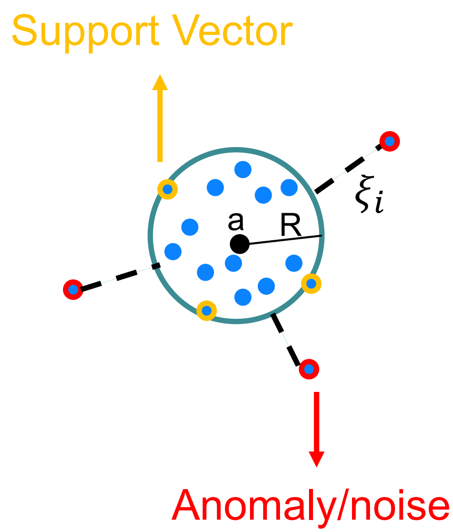

$$\min_{R,\xi,a} \; R^2 + C\sum_{i=1}^{n} \xi_i$$

s.t.

$$\|\phi(x_i) - a\|^2 \le R^2 + \xi_i, \forall\, i = 1, \ldots, n$$

$$\xi_i \ge 0, \forall\, i = 1, \ldots, n$$

where,

- R: Reduce of the ball
- ξ: Slack variable
- a: Center of the ball
- $\phi(.)$: non-linear function

Support Vector Data Description (SVDD)

Lagrangian form:

$$L(a, R, \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\gamma}) = R^2 + C\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\alpha_i\left(R^2 + \xi_i - (\phi(x_i) - a)^T(\phi(x_i) - a)\right) - \sum_{i=1}^{n}\gamma_i\xi_i$$

where, $\gamma_i \geq 0$ and $\alpha_i \geq 0$ are Lagrange multipliers.

Support Vector Data Description (SVDD)

Lagrangian form:

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where, $\gamma_i \geq 0$ and $\alpha_i \geq 0$ are Lagrange multipliers.

Set the derivatives with respect to the primal variables $R, \mathbf{a}, \boldsymbol{\xi}$ equal to zero, we get

- $\frac{\partial L}{\partial R} = ?$
- $\frac{\partial L}{\partial \mathbf{a}} = ?$
- $\frac{\partial L}{\partial \xi_i} = ?$

Support Vector Data Description (SVDD)

Lagrangian form:

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Set the derivatives with respect to the primal variables $R, a, \boldsymbol{\xi}$ equal to zero, we get

- $\frac{\partial L}{\partial R} = 2R - 2R\sum_{i=1}^{n}\alpha_i = 0$ $\qquad \sum_{i=1}^{n}\alpha_i = 1$
- $\frac{\partial L}{\partial a} = ?$
- $\frac{\partial L}{\partial \xi_i} = ?$

Support Vector Data Description (SVDD)

Lagrangian form:

$$L(a, R, \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\gamma}) = R^2 + C\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\alpha_i\left(R^2 + \xi_i - (\phi(x_i) - a)^T(\phi(x_i) - a)\right) - \sum_{i=1}^{n}\gamma_i\xi_i$$

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Set the derivatives with respect to the primal variables $R, a, \boldsymbol{\xi}$ equal to zero, we get

- $\frac{\partial L}{\partial R} = 2R - 2R\sum_{i=1}^{n}\alpha_i = 0$ $\qquad \sum_{i=1}^{n}\alpha_i = 1$
- $\frac{\partial L}{\partial a} = 2a\sum_{i=1}^{n}\alpha_i - 2\sum_{i=1}^{n}\alpha_i\,\phi(x_i) = 0$ $\qquad a = \sum_{i=1}^{n}\alpha_i\phi(x_i)$
- $\frac{\partial L}{\partial \xi_i} =?$

Support Vector Data Description (SVDD)

Lagrangian form:

$$L(\boldsymbol{a}, R, \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\gamma}) = R^2 + C\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\alpha_i\left(R^2 + \xi_i - (\phi(x_i) - \boldsymbol{a})^T(\phi(x_i) - \boldsymbol{a})\right) - \sum_{i=1}^{n}\gamma_i\xi_i$$

where, $\gamma_i \geq 0$ and $\alpha_i \geq 0$ are Lagrange multipliers.

Set the derivatives with respect to the primal variables $R, \boldsymbol{a}, \boldsymbol{\xi}$ equal to zero, we get

- $\frac{\partial L}{\partial R} = 2R - 2R\sum_{i=1}^{n}\alpha_i = 0$ $\quad\quad$ $\sum_{i=1}^{n}\alpha_i = 1$
- $\frac{\partial L}{\partial \boldsymbol{a}} = 2\boldsymbol{a}\sum_{i=1}^{n}\alpha_i - 2\sum_{i=1}^{n}\alpha_i\,\phi(x_i) = 0$ $\quad\quad$ $\boldsymbol{a} = \sum_{i=1}^{n}\alpha_i\phi(x_i)$
- $\frac{\partial L}{\partial \xi_i} = C - \alpha_i - \gamma_i = 0$ $\quad\quad$ $C = \alpha_i + \gamma_i$

Support Vector Data Description (SVDD)

$$L(a, R, \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\gamma})$$
$$= R^2 + C\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\alpha_i\big(R^2 + \xi_i - (\phi(x_i) - a)^T(\phi(x_i) - a)\big) - \sum_{i=1}^{n}\gamma_i\xi_i$$

$\sum_{i=1}^{n}\alpha_i = 1$
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$$= R^2 + C\sum_{i=1}^{n}\xi_i$$

$$+ \sum_{i=1}^{n}\alpha_i k(x_i, x_i) - 2\sum_{i=1}^{n}\alpha_i\phi(x_i)^T a + a^T a\sum_{i=1}^{n}\alpha_i - R^2\sum_{i=1}^{n}\alpha_i - \sum_{i=1}^{n}(\alpha_i + \gamma_i)\xi_i$$

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Support Vector Data Description (SVDD)

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\begin{aligned}
L(a, R, \boldsymbol{\xi}, \boldsymbol{\alpha}, \boldsymbol{\gamma})
&= R^2 + C\sum_{i=1}^{n}\xi_i - \sum_{i=1}^{n}\alpha_i\left(R^2 + \xi_i - (\phi(x_i) - a)^T(\phi(x_i) - a)\right) - \sum_{i=1}^{n}\gamma_i\xi_i \\
&= R^2 + C\sum_{i=1}^{n}\xi_i \\
&\quad + \sum_{i=1}^{n}\alpha_i k(x_i, x_i) - 2\sum_{i=1}^{n}\alpha_i\phi(x_i)^T a + a^T a\sum_{i=1}^{n}\alpha_i - R^2\sum_{i=1}^{n}\alpha_i - \sum_{i=1}^{n}(\alpha_i + \gamma_i)\xi_i
\end{aligned}
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$$= R^2 + C\sum_{i=1}^{n}\xi_i$$

$$+ \sum_{i=1}^{n}\alpha_i k(x_i, x_i) - 2\sum_{i=1}^{n}\alpha_i\phi(x_i)^T\mathsf{a} + \mathsf{a}^T\mathsf{a}\sum_{i=1}^{n}\alpha_i - R^2\sum_{i=1}^{n}\alpha_i - \sum_{i=1}^{n}(\alpha_i+\gamma_i)\xi_i$$

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$$C = \alpha_i + \gamma_i$$

Support Vector Data Description (SVDD)

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&= \cancel{R^2} + C\sum_{i=1}^{n}\xi_i \\
&+ \sum_{i=1}^{n}\alpha_i k(x_i, x_i) - 2\sum_{i=1}^{n}\alpha_i\phi(x_i)^T a + a^T a\underbrace{\sum_{i=1}^{n}\alpha_i}_{=1} - \cancel{R^2\sum_{i=1}^{n}}\underbrace{\alpha_i}_{=1} - \sum_{i=1}^{n}(\alpha_i + \gamma_i)\xi_i
\end{aligned}
$$

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$$+ \sum_{i=1}^{n} \alpha_i k(x_i, x_i) - 2\underbrace{\sum_{i=1}^{n} \alpha_i \phi(x_i)^T}_{=a^T} a + a^T a \underbrace{\sum_{i=1}^{n} \alpha_i}_{=1} - \cancel{R^2} \underbrace{\sum_{i=1}^{n} \alpha_i}_{=1} - \sum_{i=1}^{n} \underbrace{(\cancel{\alpha_i + \gamma_i})}_{=C} \xi_i$$

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$$= \sum_{i=1}^{n}\alpha_i k(x_i, x_i) - a^T a$$

$$= \sum_{i=1}^{n}\alpha_i k(x_i, x_i) - \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_i\alpha_j k(x_i, x_j)$$

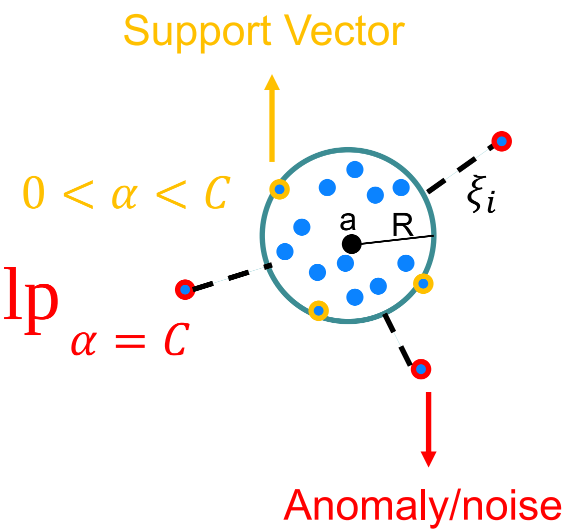
$\sum_{i=1}^{n}\alpha_i = 1$
$a = \sum_{i=1}^{n}\alpha_i\phi(x_i)$
$C = \alpha_i + \gamma_i$

Support Vector Data Description (SVDD)

$$\underset{\alpha}{\operatorname{argmax}} \sum_{i=1}^{n} \alpha_i k(x_i, x_i) - \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j k(x_i, x_j)$$

s.t.

$$0 \leq \alpha_i \leq C, \quad \sum_{i=1}^{n} \alpha_i = 1$$



Support Vector Data Description (SVDD)

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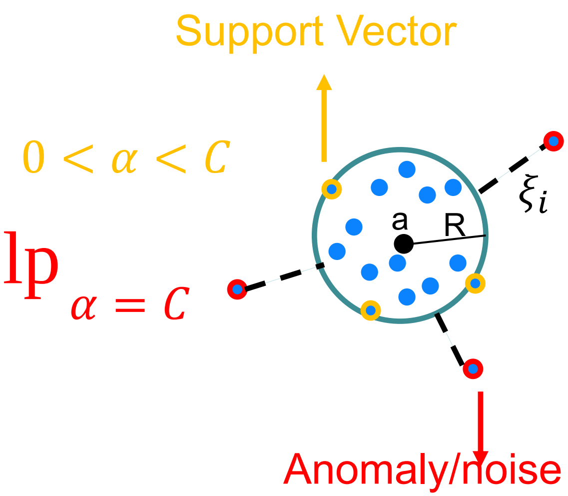
s.t.

$$0 \leq \alpha_i \leq C, \quad \sum_{i=1}^{n} \alpha_i = 1$$



- $\|\phi(x_i) - a\|^2 < R^2 \ \rightarrow \alpha_i = 0$
- $\|\phi(x_i) - a\|^2 = R^2 \ \rightarrow 0 < \alpha_i < C$
- $\|\phi(x_i) - a\|^2 > R^2 \ \rightarrow \alpha_i = C$

Support Vector Data Description (SVDD)

- $a = \sum_{i=1}^{n} \alpha_i \phi(x_i)$

Support Vector Data Description (SVDD)

- $a = \sum_{i=1}^{n} \alpha_i \phi(x_i)$

- $R^2 = \|\phi(x_i) - a\|^2$, where, x_i are support vectors with $0 < \alpha_i < C$

$$R^2 = k(x_i, x_i) - 2\sum_{i=1}^{n} \alpha_i k(x_i, x_j) + \sum_{i=1}^{n}\sum_{i=1}^{n} \alpha_i \alpha_j k(x_i, x_j)$$

Support Vector Data Description (SVDD)

- $a = \sum_{i=1}^{n} \alpha_i \phi(x_i)$
- $R^2 = \|\phi(x_i) - a\|^2$, where, x_i are support vectors with $0 < \alpha_i < C$

$$R^2 = k(x_i, x_i) - 2\sum_{i=1}^{n} \alpha_i k(x_i, x_j) + \sum_{i=1}^{n}\sum_{i=1}^{n} \alpha_i \alpha_j k(x_i, x_j)$$

- New sample z is identified as normal if

$$\|z - a\|^2 = (z.z) - 2\sum_{i=1}^{n} \alpha_j (z.x_i) + \sum_{i=1}^{n}\sum_{i=1}^{n} \alpha_i \alpha_j (x_i.x_j) \le R^2$$

One-class Support Vector Machine (OCSVM) [2]

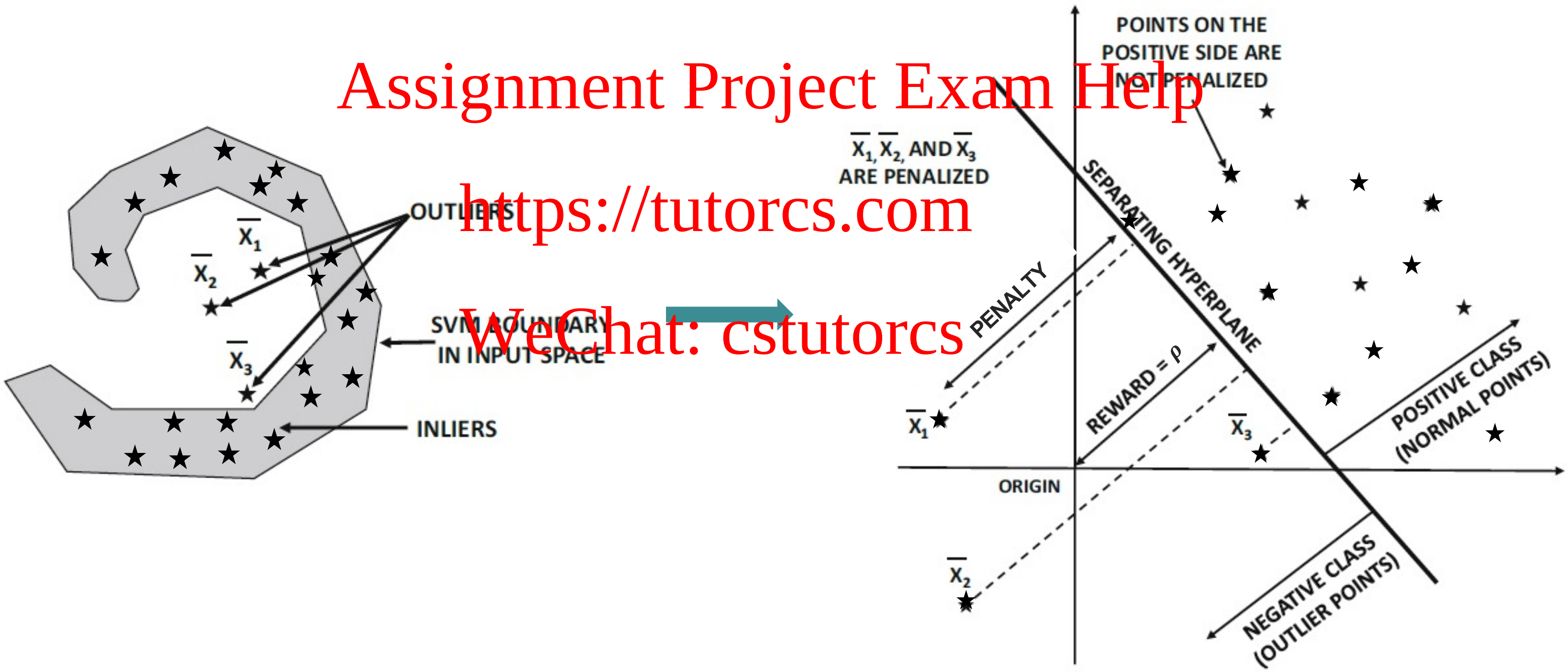
- Map input data into a high dimensional feature space
- Iteratively finds the maximal margin in the hyperplane which best separates the training *data from the origin*

- Decision boundary

$$w \cdot \phi(x_i) - \rho = 0$$

- Formulate the optimization problem so it returns positive for as many of the *N* training examples as possible.

OCSVM Intuition



OCSVM

- Solve quadratic problem

$$\min_{w,\xi_i,\rho} \frac{1}{2}\|w\|^2 + \frac{1}{\nu n}\sum_{i=1}^{n}\xi_i - \rho$$

s.t.

$$(w \cdot \phi(x_i)) \geq \rho - \xi_i, \forall\, i = 1, \dots, n$$

$$\xi_i \geq 0, \forall\, i = 1, \dots, n$$

- ν:
  - $\nu = \frac{1}{C}$
  - A prior probability that a data point in the training set is an anomaly.
  - Regulates the trade-off between false positives and false negatives in this model.
  - Due to the importance of ν, OCSVM is often referred to as ν-SVM

One-class Support Vector Machine (OCSVM)

- The problem can be simplified to

$$\underset{\alpha}{\operatorname{argmin}} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j k(x_i, x_j)$$

s.t. $0 \leq \alpha_i \leq \frac{1}{\nu n}$ $\quad \sum_{i=1}^{n} \alpha_i = 1$

One-class Support Vector Machine (OCSVM)

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$$\underset{\alpha}{\operatorname{argmin}} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j k(x_i, x_j)$$

$$\text{s.t.} \quad 0 \le \alpha_i \le \frac{1}{\nu n}, \quad \sum_{i=1}^{n} \alpha_i = 1$$

- Soft margin SVM for binary classification:

$$\underset{\alpha}{\operatorname{argmin}} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \, y_i y_j \, k(x_i, x_j) - \sum_{i=1}^{n} \alpha_i$$

$$\text{s.t.} \quad \alpha_i \ge 0\,, \quad \sum_{i=1}^{n} \alpha_i y_i = 0$$

One-class Support Vector Machine (OCSVM)

- The problem can be simplified to

$$\underset{\alpha}{\operatorname{argmin}} \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j k(x_i, x_j)$$

s.t. $0 \leq \alpha_i \leq \frac{1}{\nu n}, \quad \sum_{i=1}^{n} \alpha_i = 1$

- $w = \sum_{i=1}^{n} \alpha_i \phi(x_i)$
- $\rho = w. \phi(x_i)$

One-class Support Vector Machine (OCSVM)

- Anomaly score for new sample z:
  - $Score(z) = \sum_{i=1}^{n} \alpha_i k(x_i, z) - \rho$
  - $Score(z) < 0$: Anomaly
  - $Score(z) \geq 0$: Normal

- For especial case of (invariant) kernels, e.g., Gaussian, $\mathrm{SVDD} \cong \mathrm{OCSVM}$

Deep SVDD [4]

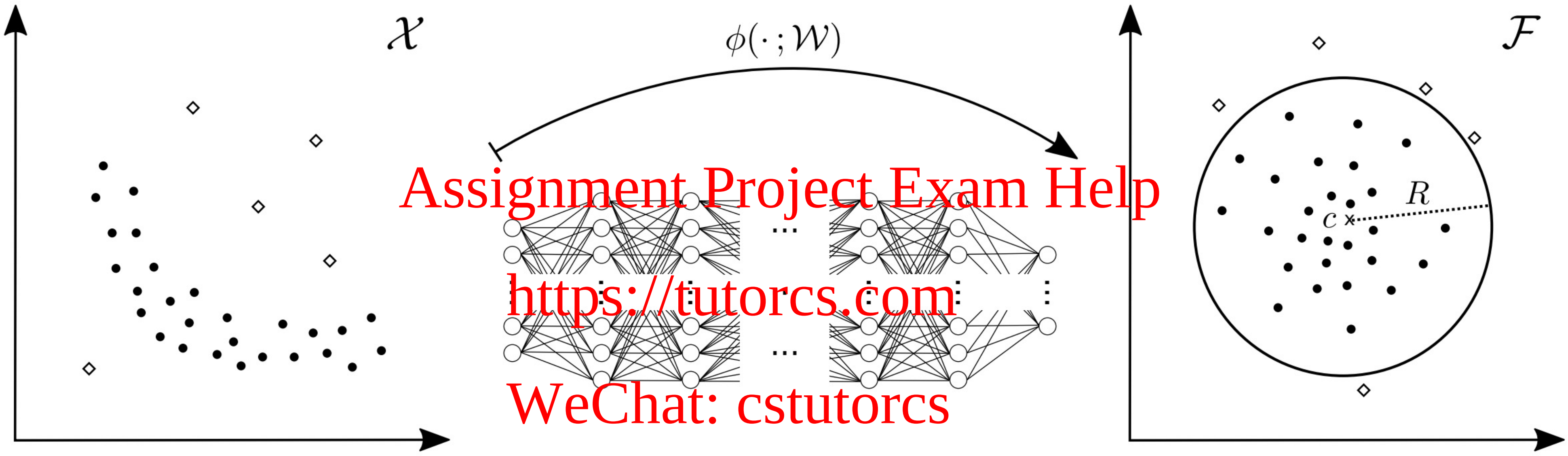

Figure: Deep SVDD learns a neural network transformation $\phi(\cdot;\mathcal{W})$ with weights $\mathcal{W}$ from input space $\mathcal{X} \subseteq \mathbb{R}^d$ to output space $\mathcal{F} \subseteq \mathbb{R}^p$ that attempts to map most of the data network representations into a hypersphere characterized by centre c and radius R of minimum volume. Mappings of normal examples fall within, whereas mappings of anomalies fall outside the hypersphere.

Hybrid Deep-1SVM [5]

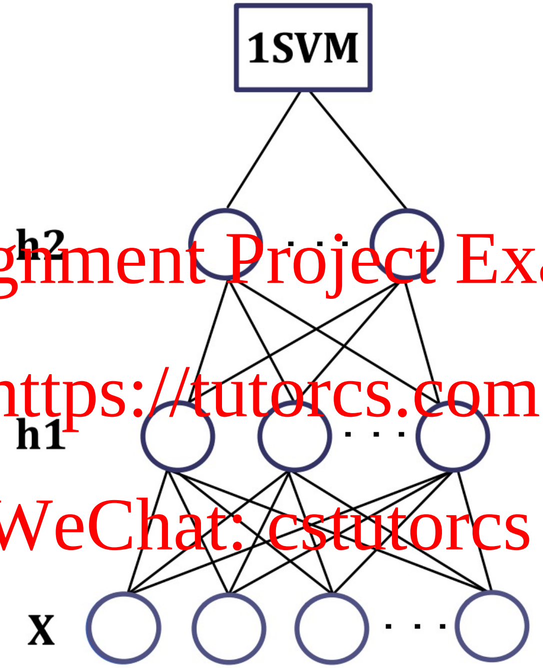

Figure: Deep model (AE) is trained to extract features that are relatively invariant to irrelevant variations in the input, so that the one-class SVM (1SVM) can effectively separate the normal data from anomalies in the learned feature space, using linear kernel.

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Deep Learning and One-class SVM based Anomalous Crowd Detection [6]

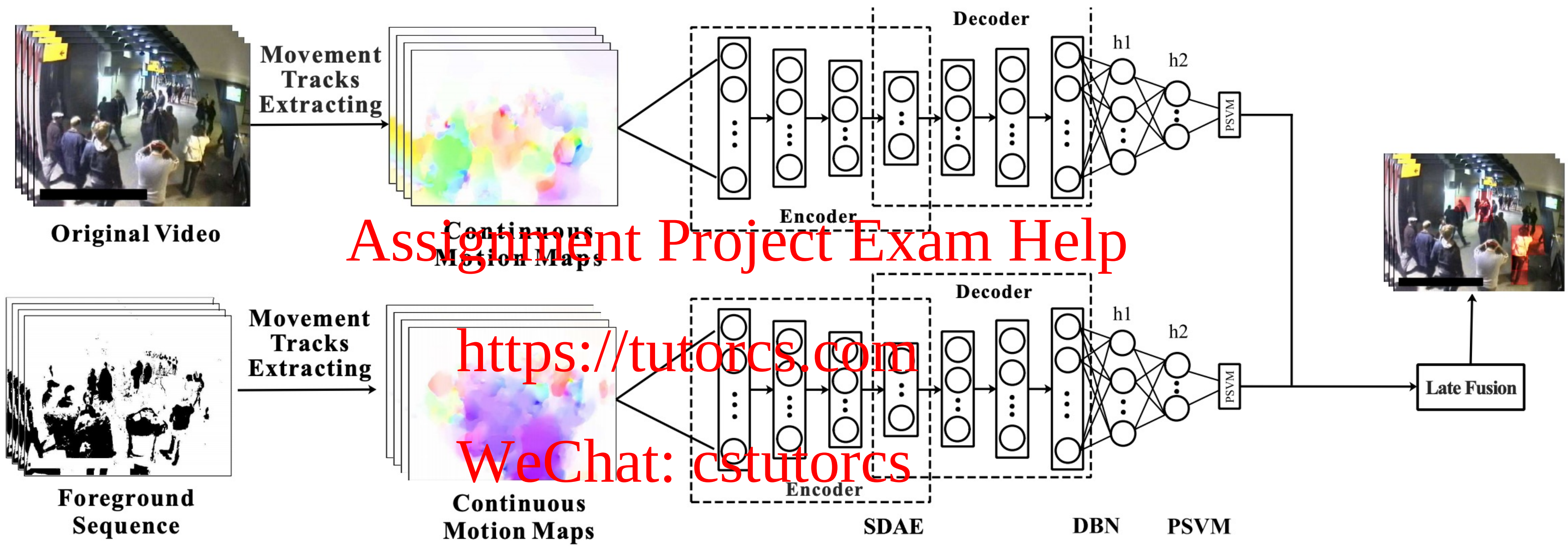

Figure: Original video and foreground video sequence are taken as the input of two branches of channel, then movement tracks are extracted to produce continuous motion maps. Training and testing on a hybrid deep learning model SDAE-DBN-PSVM, then follows to achieve anomalous event detection

Summary

- What is SVDD?
- How to derive dual formulation of SVDD?
- How to extend SVM to OCSVM?

Next: Autoencoders and their applications

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