
# COMP9020

Foundations of Computer Science



Lecture 5: Relations

# Relations and Functions

**Relations** are an abstraction used to capture the idea that the objects from certain domains (often the same domain for several objects) are *related*. These objects may

- influence one another (each other for binary relations; self(?) for unary)
- share some common properties
- correspond to each other precisely when some constraints are satisfied



**Functions** capture the idea of transforming *inputs* into *outputs*.

In general, functions and relations formalise the concept of interaction among objects from various domains; however, there must be a specified domain for each type of objects.

# Applications in Computer Science



- Relations are the building blocks of nearly all Computer Science structures
- Databases are collections of relations
- Any ordering is a relation
- Common data structures (e.g. graphs) are relations
- Functions/procedures/programs compute relations between their input and output

# Applications in Computer Science

Many binary relations (i.e. relationships between two entities) that appear in CS fall into two broad categories:



Equivalence relations (generalizing "equality"):

- Programs that exhibit the same behaviour
- Logically equivalent statements
- The `.equals()` method in Java



Partial orders (generalizing "less than or equal to"):



- Object inheritance
- Simulation
- Requirement specifications
- The `.compareTo()` method in Java

# Outline

# Relations

**Definition**

An **n-ary relation** is a subset of the cartesian product of $n$ sets.

$$R \subseteq S_1 \times S_2 \times \ldots \times S_n$$

To show tuples related by $R$ we write:

$$(x_1, x_2, \ldots, x_n) \in R \quad \text{or} \quad R(x_1, x_2, \ldots, x_n)$$

If $n = 2$ we have a **binary** relation $R \subseteq S \times T$ and to show pairs related by $R$ we write:

$$(x, y) \in R \quad \text{or} \quad R(x, y) \quad \text{or} \quad xRy$$

$\mathcal{U} = S_1 \times S_2 \times \ldots \times S_n$ is the **domain** of $R$, and we say $R$ is a **relation on** $\mathcal{U}$ (or **on** $S$ if $S_1 = \cdots = S_n = S$ and $n$ is clear).

# Examples



**Examples**

- Equality: $=$
- Inequality: $\leq$, $\geq$, $<$, $>$, $\neq$
- Divides relation: $|$
- Element of: $\in$
- Subset, superset: $\subseteq$, $\subset$, $\supseteq$, $\supset$
- Congruence modulo $n$: $m \equiv_{(n)} p$

# Database Examples

**Example (Course enrolments)**

$S$ = set of CSE students
($S$ can be a subset of the set of all students)
$C$ = set of CSE courses
(likewise)
$E$ = enrolments $= \{(s, c) : s \text{ takes } c\}$

$$E \subseteq S \times C$$



In practice, almost always there are various 'onto' (nonemptiness) and 1–1 (uniqueness) constraints on database relations.

**Example (Class schedule)**

$C =$ CSE courses

$T =$ starting time (hour & day)

$R =$ lecture rooms

$S =$ schedule $=$

$$\{(c, t, r) : c \text{ is at } t \text{ in } r\} \subseteq C \times T \times R$$

**Example (sport stats)**

$$R \subseteq \text{competitions} \times \text{results} \times \text{years} \times \text{athletes}$$

# Defining Relations



Just as with sets $R$ can be defined by

- explicit enumeration of interrelated $k$-tuples (ordered pairs in case of binary relations);
- properties that identify relevant tuples within the entire $S_1 \times S_2 \times \ldots \times S_k$;
- construction from other relations (e.g. union, intersection, complement etc.).

# Outline

# Binary relations



A **binary relation between** $S$ **and** $T$ is a subset of $S \times T$: i.e. a set of ordered pairs.

Also: over $S$ and $T$; from $S$ to $T$; on $S$ (if $S = T$).



**Example (Special (Trivial) Relations)**

- **Identity**: (diagonal, equality) $I = \{ (x, x) : x \in S \}$
- **Empty**: $\emptyset$
- **Universal**: $U = S \times S$

# Defining binary relations: Set-based definitions



Defining a relation $R \subseteq S \times T$:

- Explicitly listing tuples: e.g. $\{(1,1),(2,3),(3,2)\}$
- Set comprehension: $\{(x,y) \in [1,3] \times [1,3] \mid xy - 1\}$
- Construction from other relations:
  $\{(1,1)\} \cup \{(2,3)\} \cup \{(2,3)\}^{\leftarrow}$

# Defining binary relations: Matrix representation

Defining a relation $R \subseteq S \times T$:
Rows enumerated by elements of $S$, columns by elements of $T$:

**Examples**

- The relation $\{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:

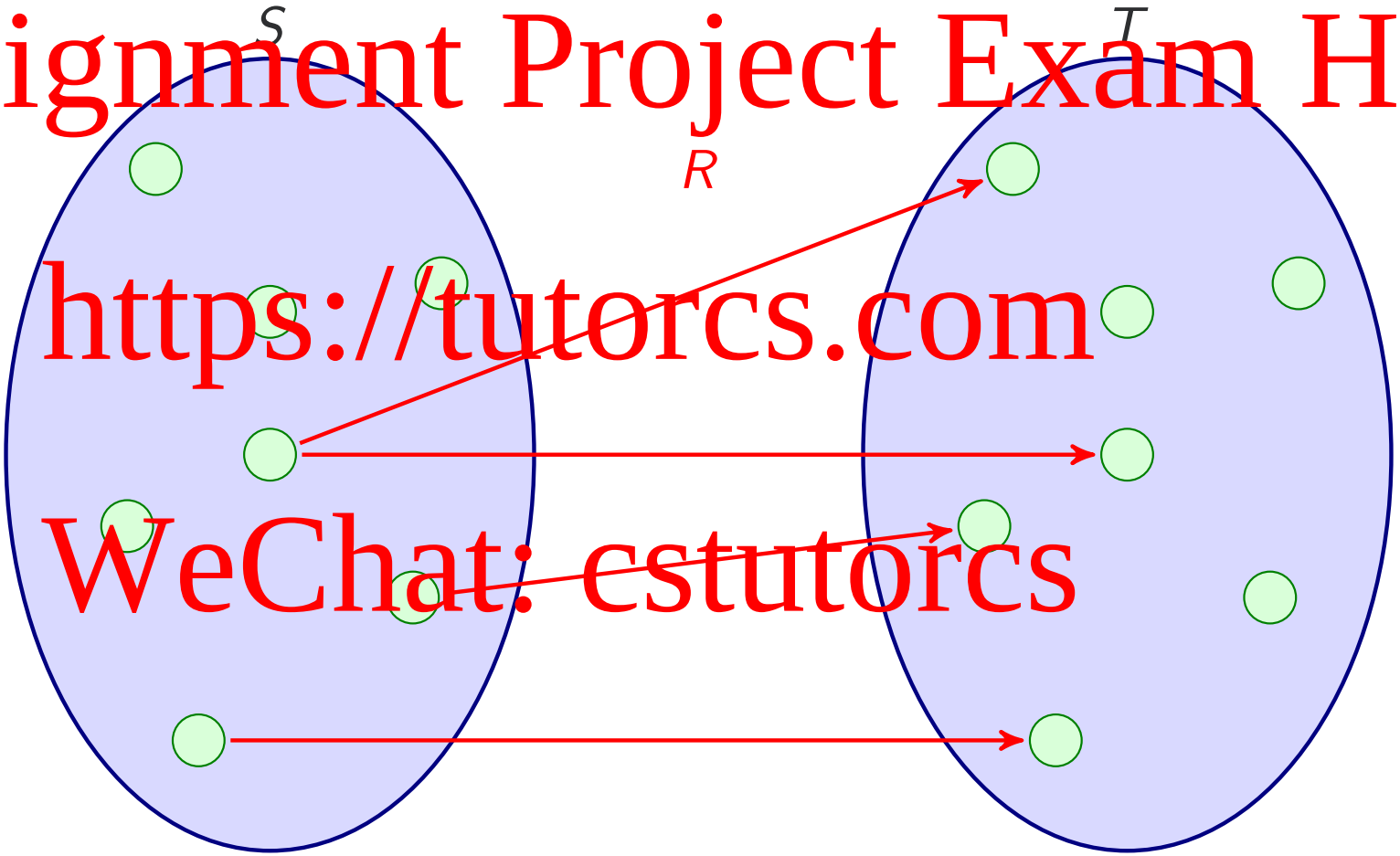
$$\begin{bmatrix} \bullet & \circ & \circ \\ \circ & \circ & \bullet \\ \circ & \bullet & \circ \end{bmatrix}$$

- The relation $\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ \circ & \bullet & \circ & \circ \\ \circ & \bullet & \circ & \circ \end{bmatrix}$$

# Defining binary relations: Graphical representation

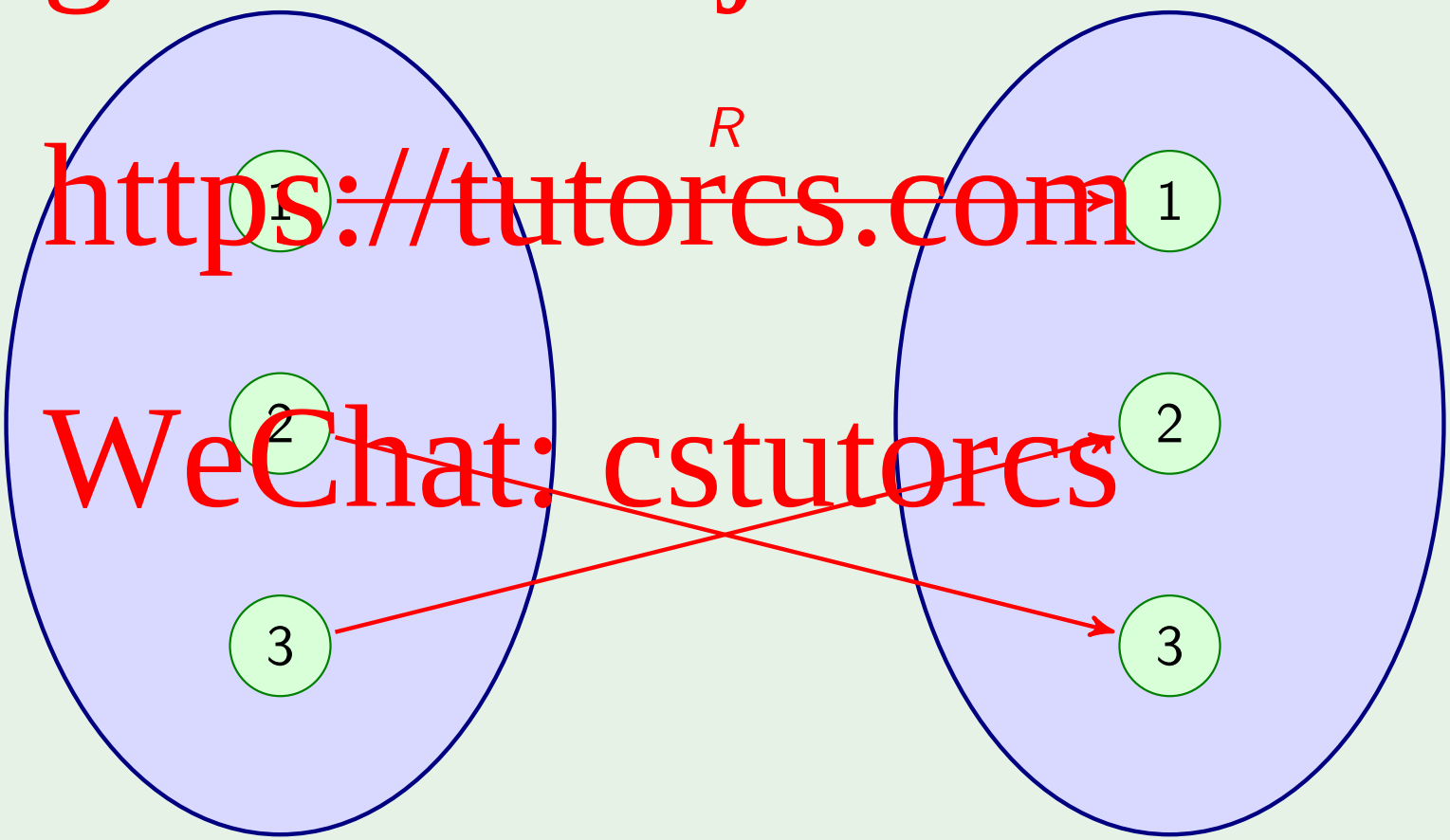
Defining a relation $R \subseteq S \times T$:



# Defining binary relations: Graphical representation

**Example**

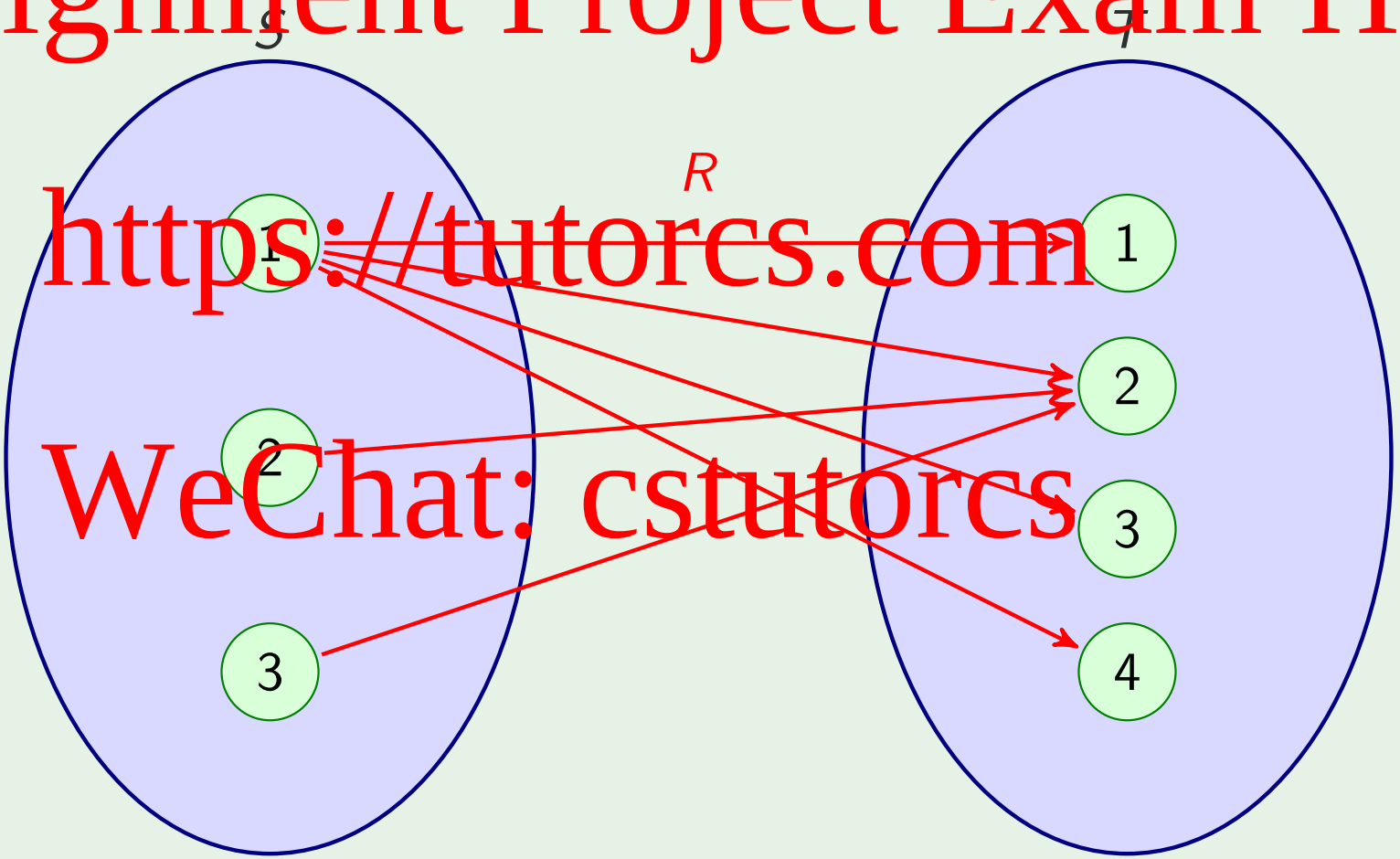
$R = \{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:



# Defining binary relations: Graphical representation

**Example**

$\{(1,1),(1,2),(1,3),(1,4),(2,2),(3,2)\} \subseteq [1,3] \times [1,4]$:



# Defining binary relations: Graph representation

If $S = T$ we can define $R \subseteq S \times S$ as a **directed graph** (week 5).

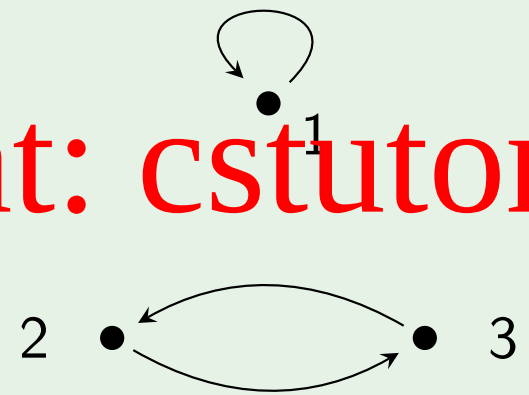
- Nodes: Elements of $S$
- Edges: Elements of $R$



**Example**

$R = \{(1,1),(2,3),(3,2)\} \subseteq [1,3] \times [1,3]$:





# Operations for binary relations

Relations are sets, so the standard set operations ($\cap$, $\cup$, $\setminus$, $\oplus$, etc) can be used to build new relations.

Two operations that apply to binary relations uniquely:

- **Converse**: If $R \subseteq S \times T$ is a relation, then $R^{\leftarrow} \subseteq T \times S$:

$$R^{\leftarrow} \stackrel{\text{def}}{=} \{(t, s) \in T \times S \ : \ (s, t) \in R\}$$

- **Composition**: If $R_1 \subseteq S \times T$ and $R_2 \subseteq T \times U$ then $R_1; R_2 \subseteq S \times U$:

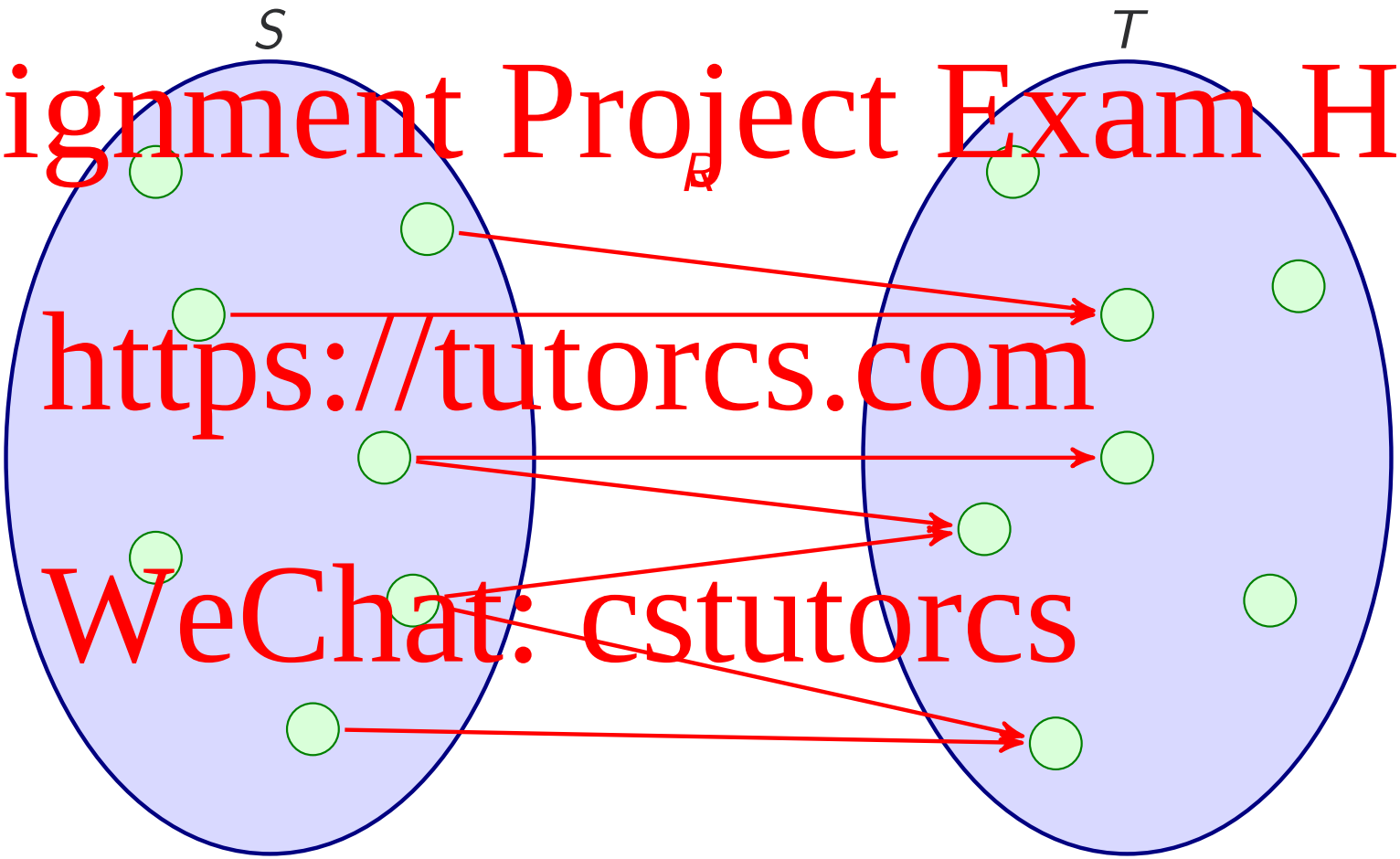
$$R_1; R_2 \stackrel{\text{def}}{=} \{(s, u) \in S \times U \ : \ \text{there exists } t \in T \text{ such that } (s, t) \in R_1 \text{ and } (t, u) \in R_2\}.$$



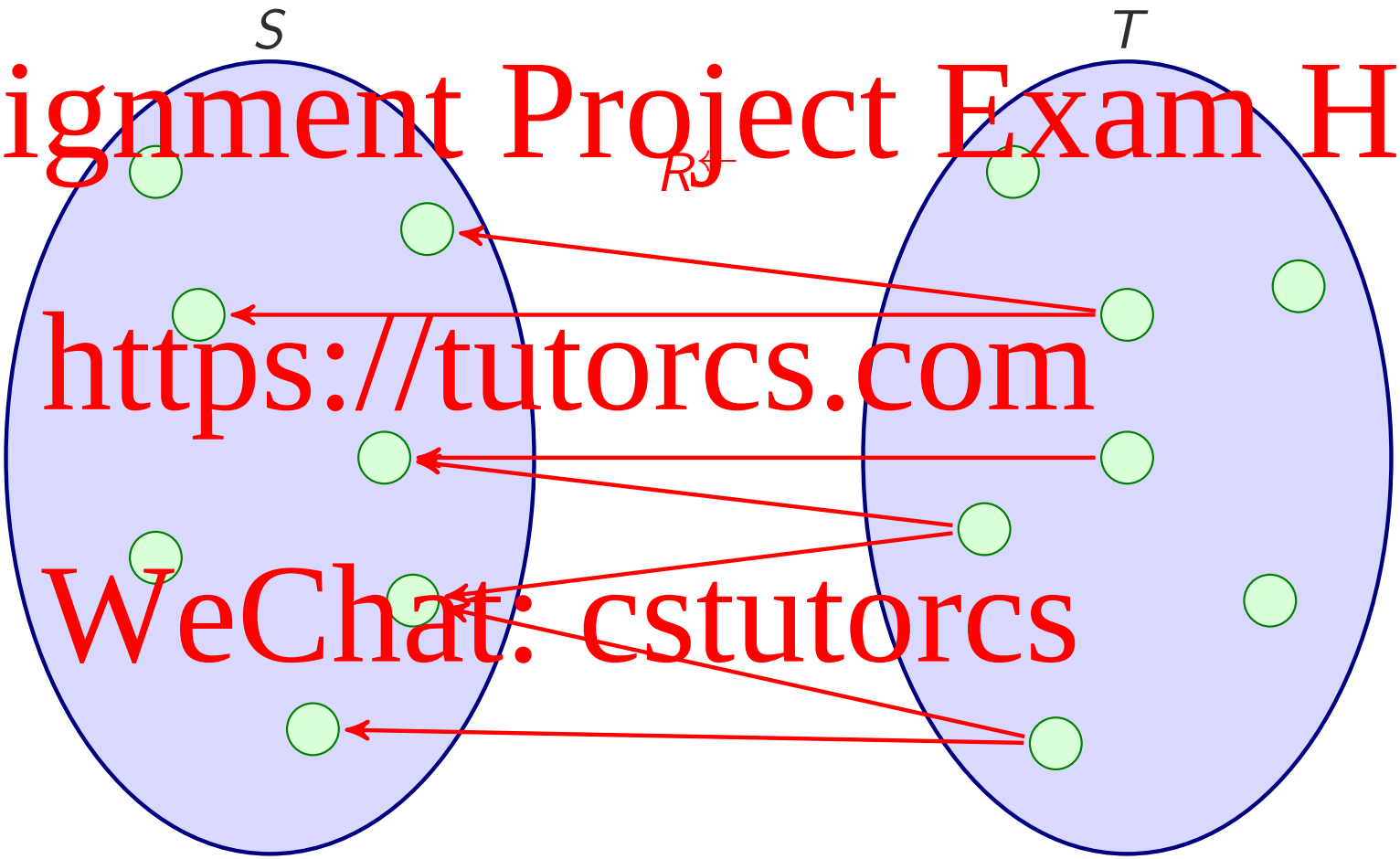
**Fact**

$(R^{\leftarrow})^{\leftarrow} = R$

# Binary relation: Graphical representation



# Binary relation: Graphical representation



# Relational images

Given $R \subseteq S \times T$, $A \subseteq S$, and $B \subseteq T$.

**Definition**

- Relational image of $A$, $R(A)$:

$$R(A) \stackrel{\text{def}}{=} \{t \in T \ : \ (s,t) \in R \text{ for some } s \in A\}$$

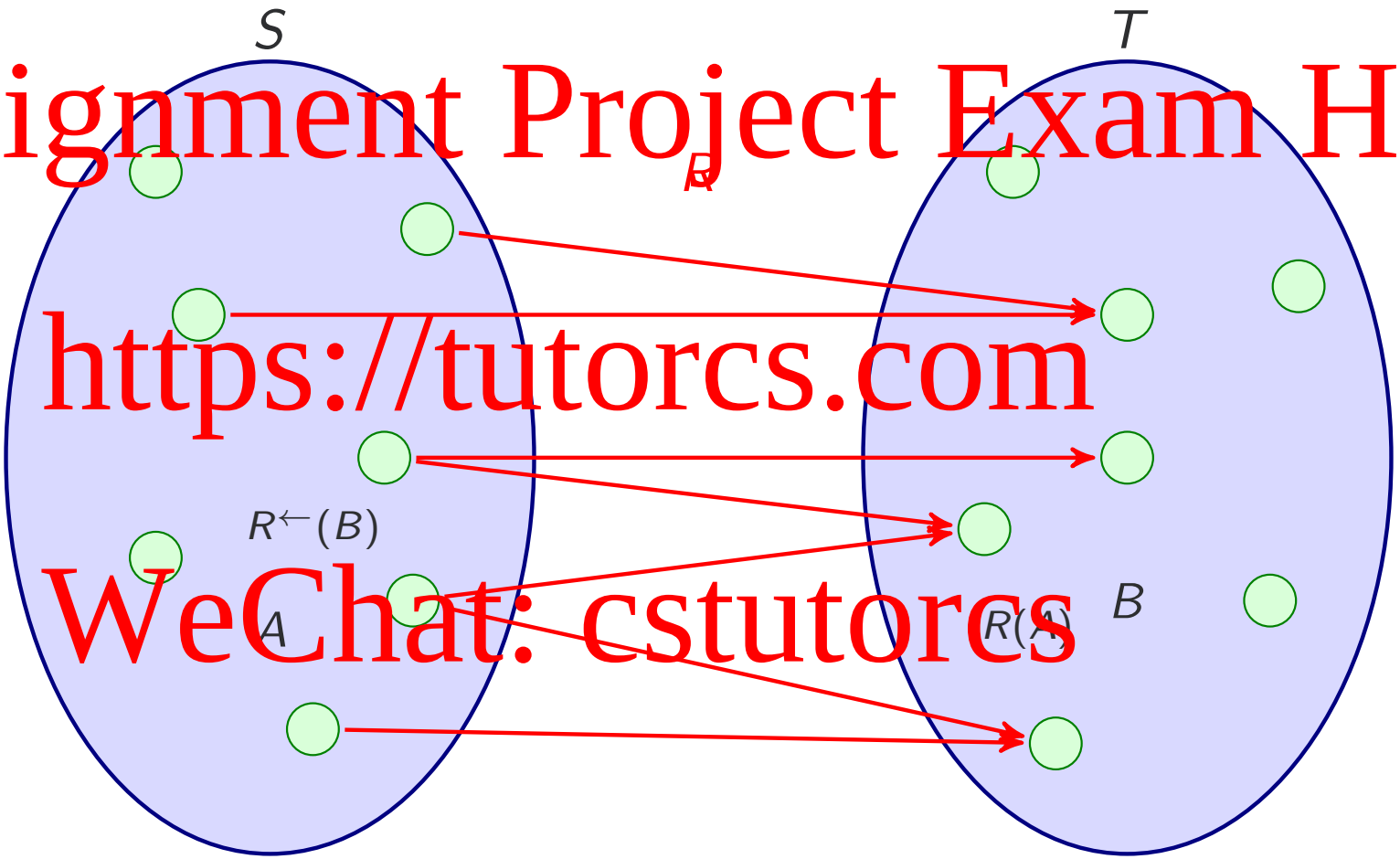
- Relational pre-image of $B$, $R^{\leftarrow}(B)$:

$$R^{\leftarrow}(B) \stackrel{\text{def}}{=} \{s \in S \ : \ (s,t) \in R \text{ for some } t \in B\}$$

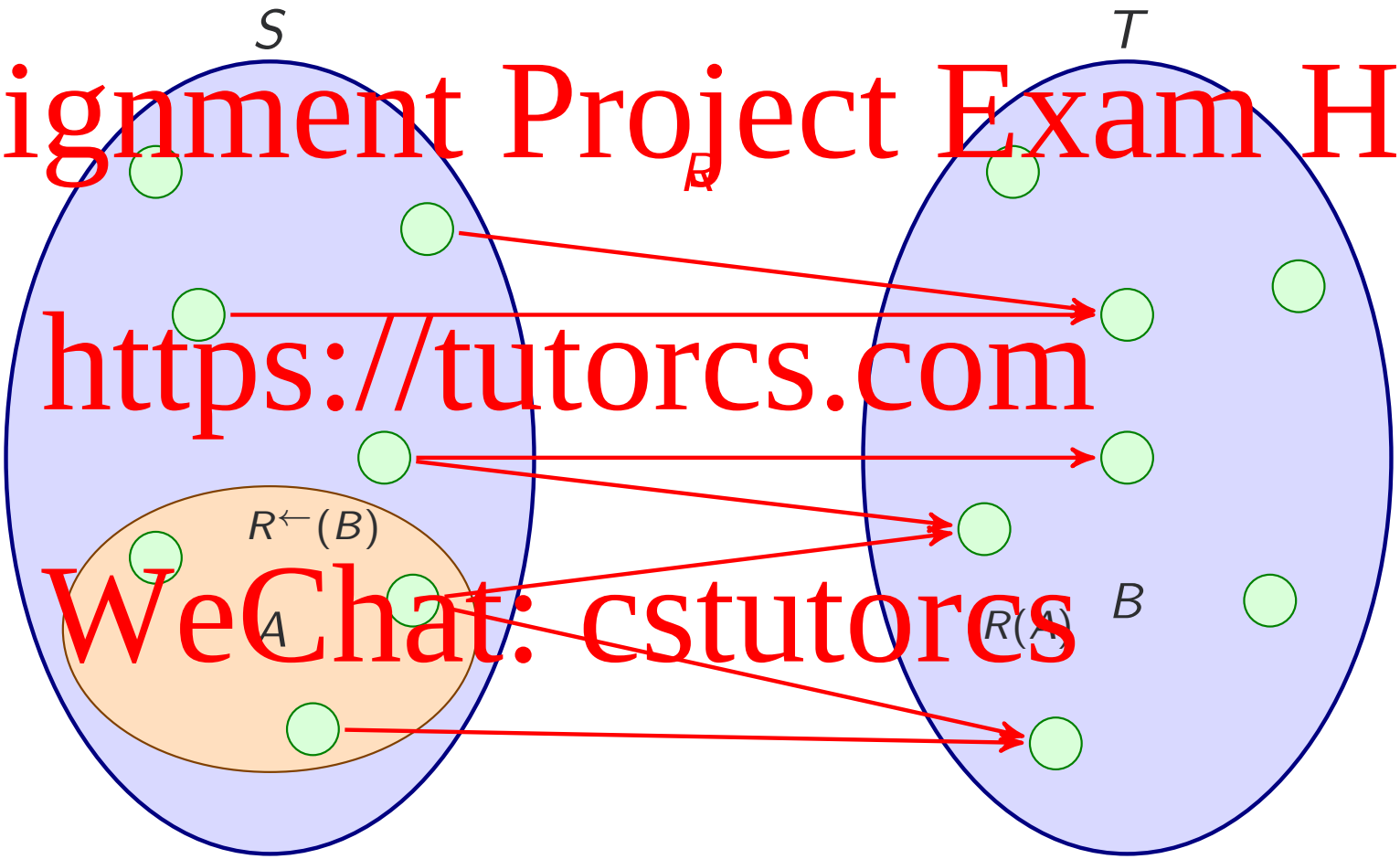


Observe that the relational pre-image is the relational image of the converse relation.

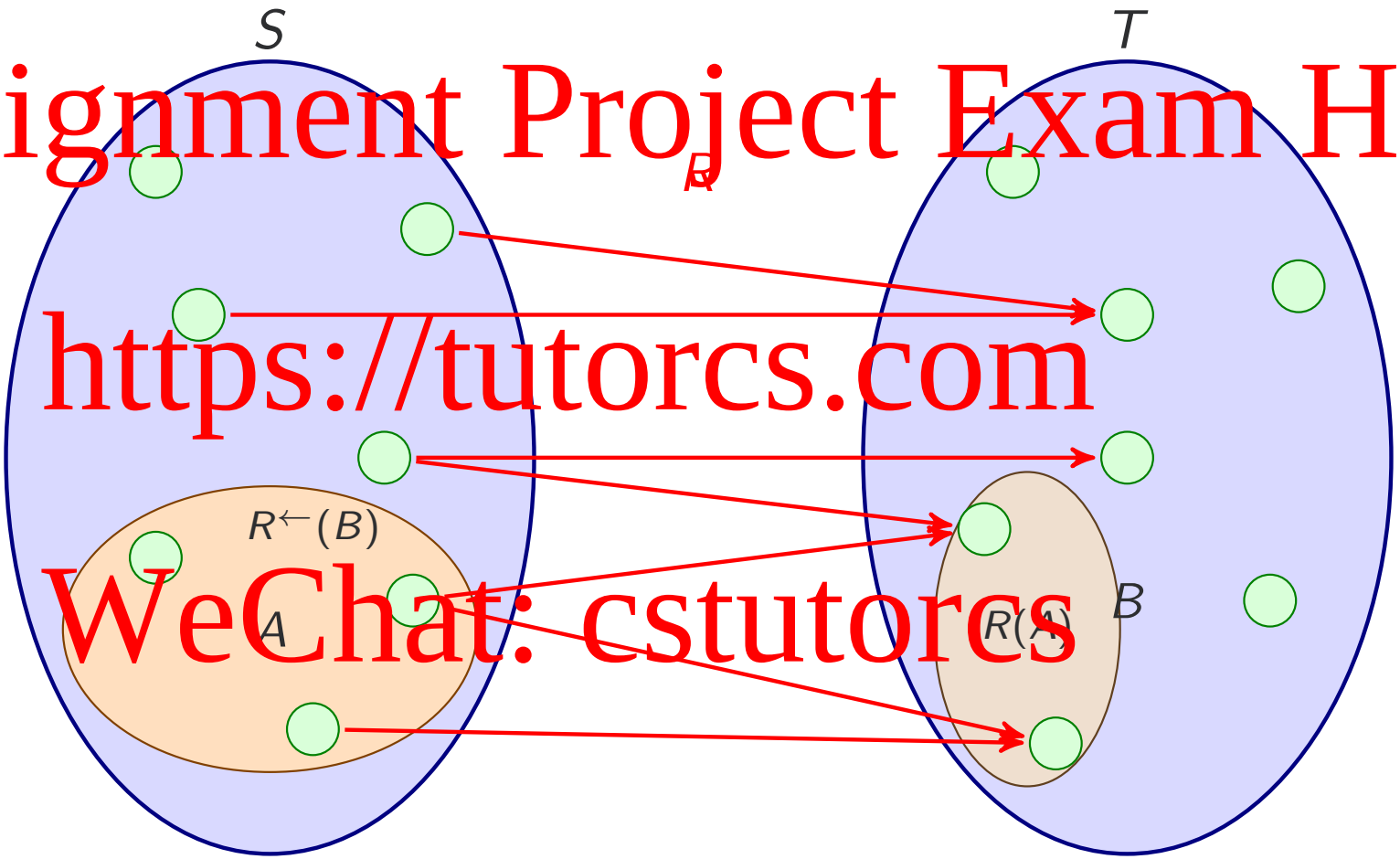
# Binary relation: Graphical representation



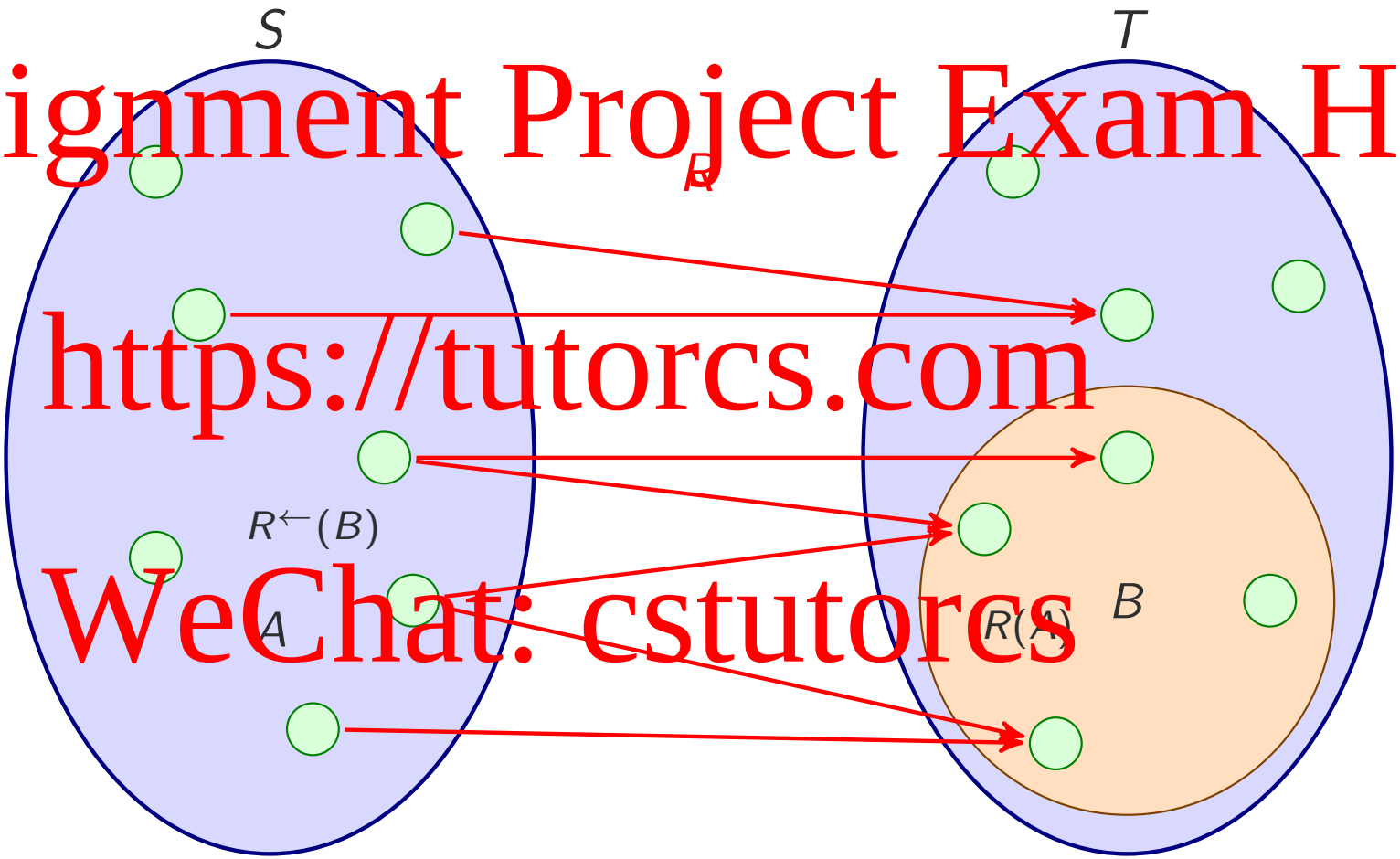
# Binary relation: Graphical representation



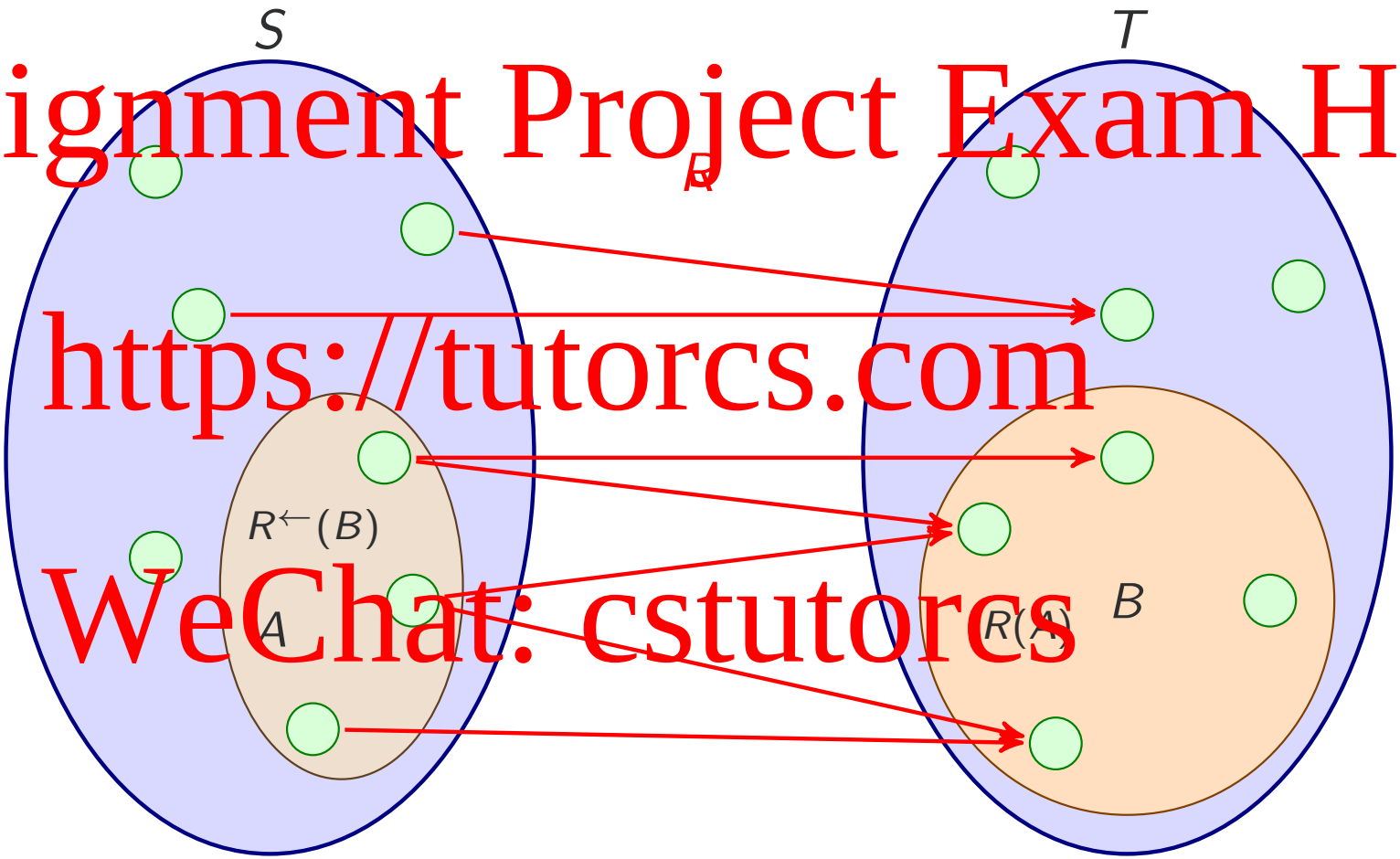
# Binary relation: Graphical representation



# Binary relation: Graphical representation



# Binary relation: Graphical representation



# Exercises



## Exercises

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, $X = [1, 4]$, $M = \{A, B, C\}$, $N = \{A, B, C, X\}$.



- $\mid$ on $X$:
- $\in$ on $X \times M$:
- $\subseteq^{\leftarrow}$ on $N$:
- $\ni$:



- $<$ $(\{2\})$ (on $X$):

# Exercises



**Exercises**

Let $A = \{1, 2\}$, $B = \{2, 3\}$, $C = \{3, 4\}$, $X = [1, 4]$, $M = \{A, B, C\}$, $N = \{A, B, C, X\}$.

- $\subseteq$ on $X$: $\{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$
- $\in$ on $X \times M$: $\{(1, A), (2, A), (2, B), (3, B), (3, C), (4, C)\}$
- $\subseteq^{\leftarrow}$ on $N$: $\{(A, A), (X, A), (B, B), (X, B), (C, C), (X, C), (X, X)\}$
- $\not\in$: $\{(1, A), (1, B), (1, C), (2, A), (2, B), (2, C), (3, B), (3, C), (4, C)\}$
- $<$ $(\{2\})$ (on $X$): $\{3, 4\}$

# Outline

# Properties of Binary Relations $R \subseteq S \times T$

A binary relation $R \subseteq S \times T$ is:

**Definition**

| | | |
|---|---|---|
| (Fun) | functional | For all $s \in S$ there is at most one $t \in T$ such that $(s,t) \in R$ |
| (Tot) | total | For all $s \in S$ there is at least one $t \in T$ such that $(s,t) \in R$ |
| (Inj) | injective | For all $t \in T$ there is at most one $s \in S$ such that $(s,t) \in R$ |
| (Sur) | surjective | For all $t \in T$ there is at least one $s \in S$ such that $(s,t) \in R$ |
| (Bij) | bijective | Injective and surjective |

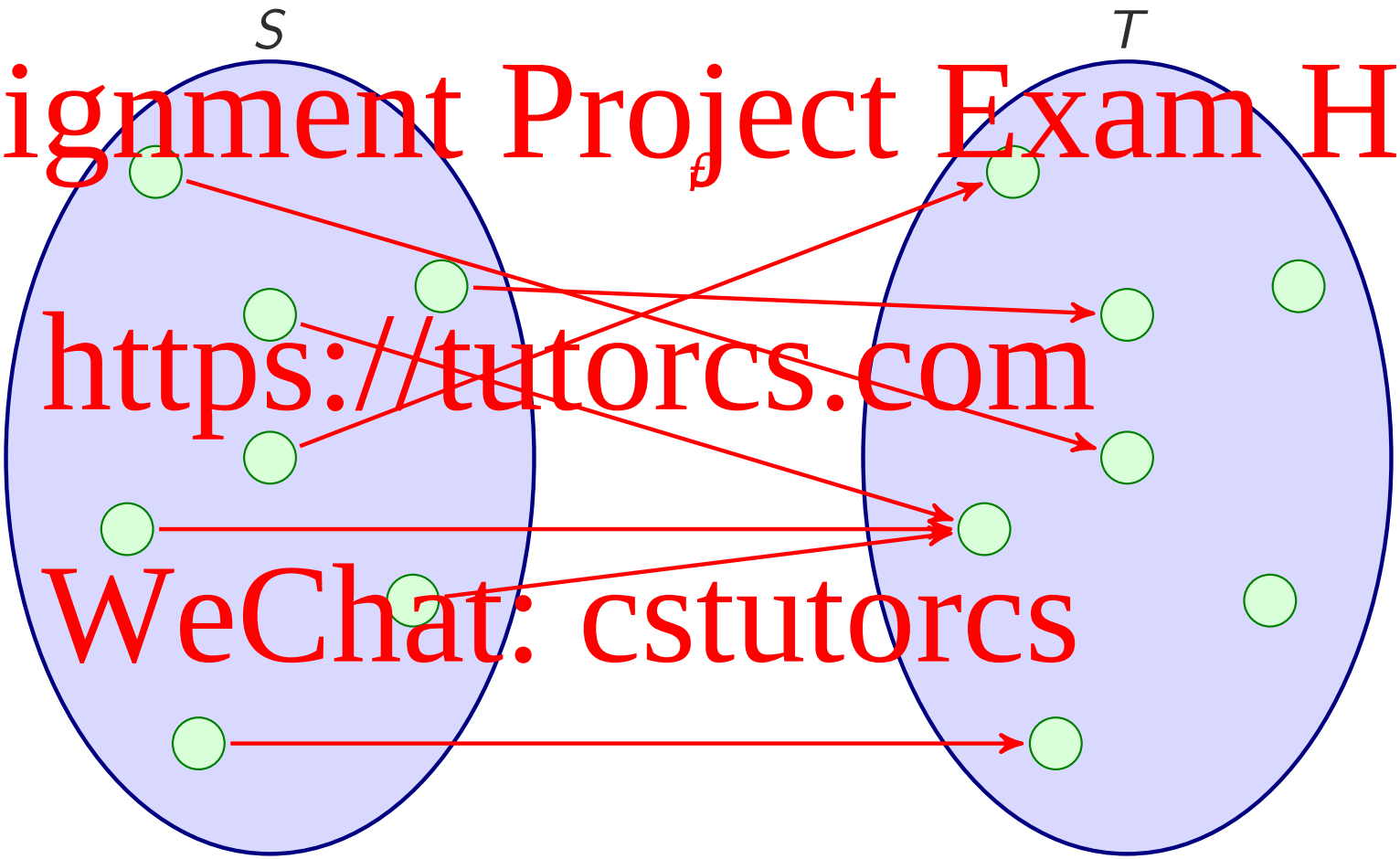
# Functions and function properties



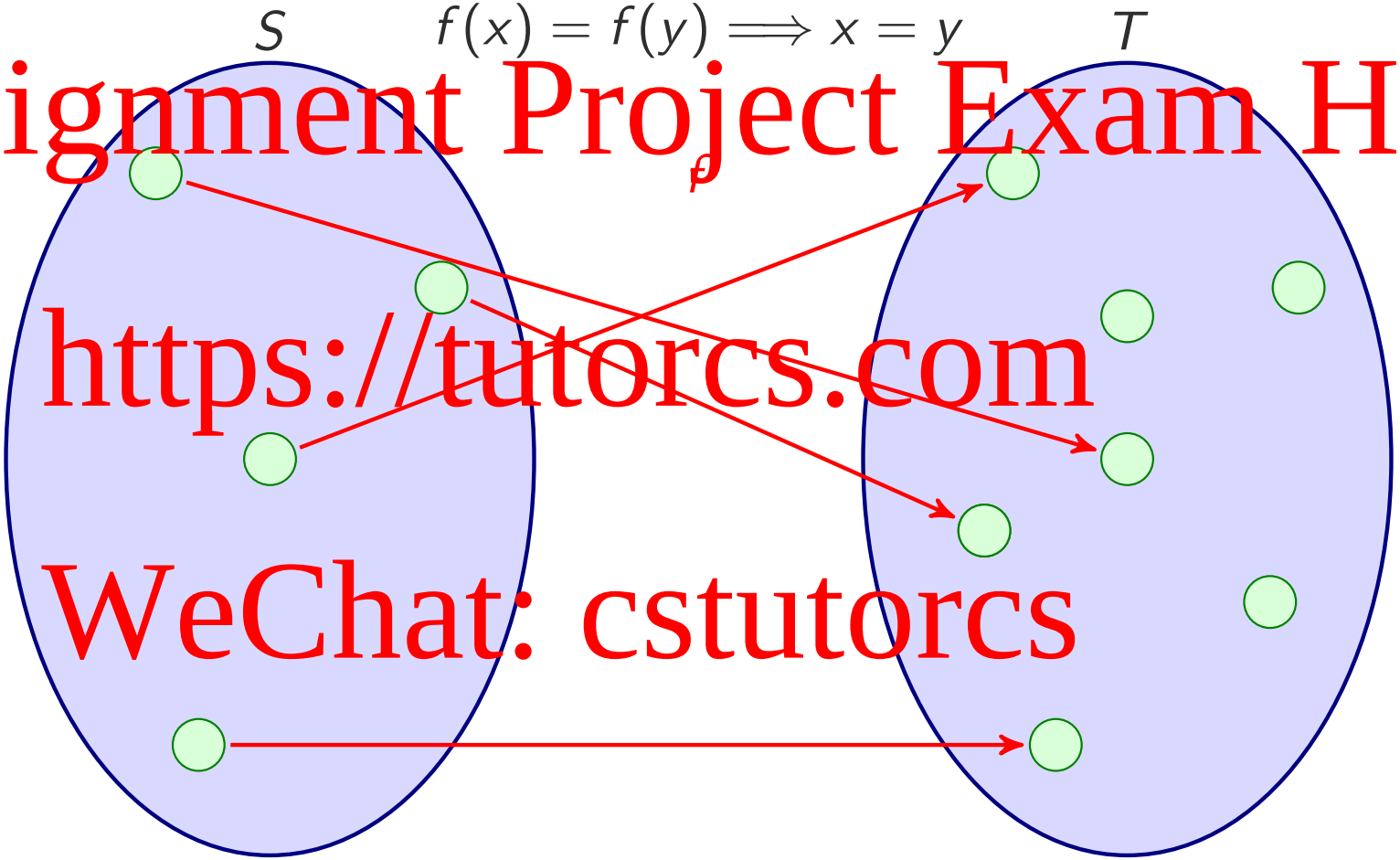
**Definition**

- **partial function** is a binary relation that is (Fun).
- A **function** is a binary relation that is (Fun) and (Tot).
- An **injection** is a function that is (Inj).
- A **surjection** is a function that is (Sur).
- A **bijection** is a function that is (Bij).

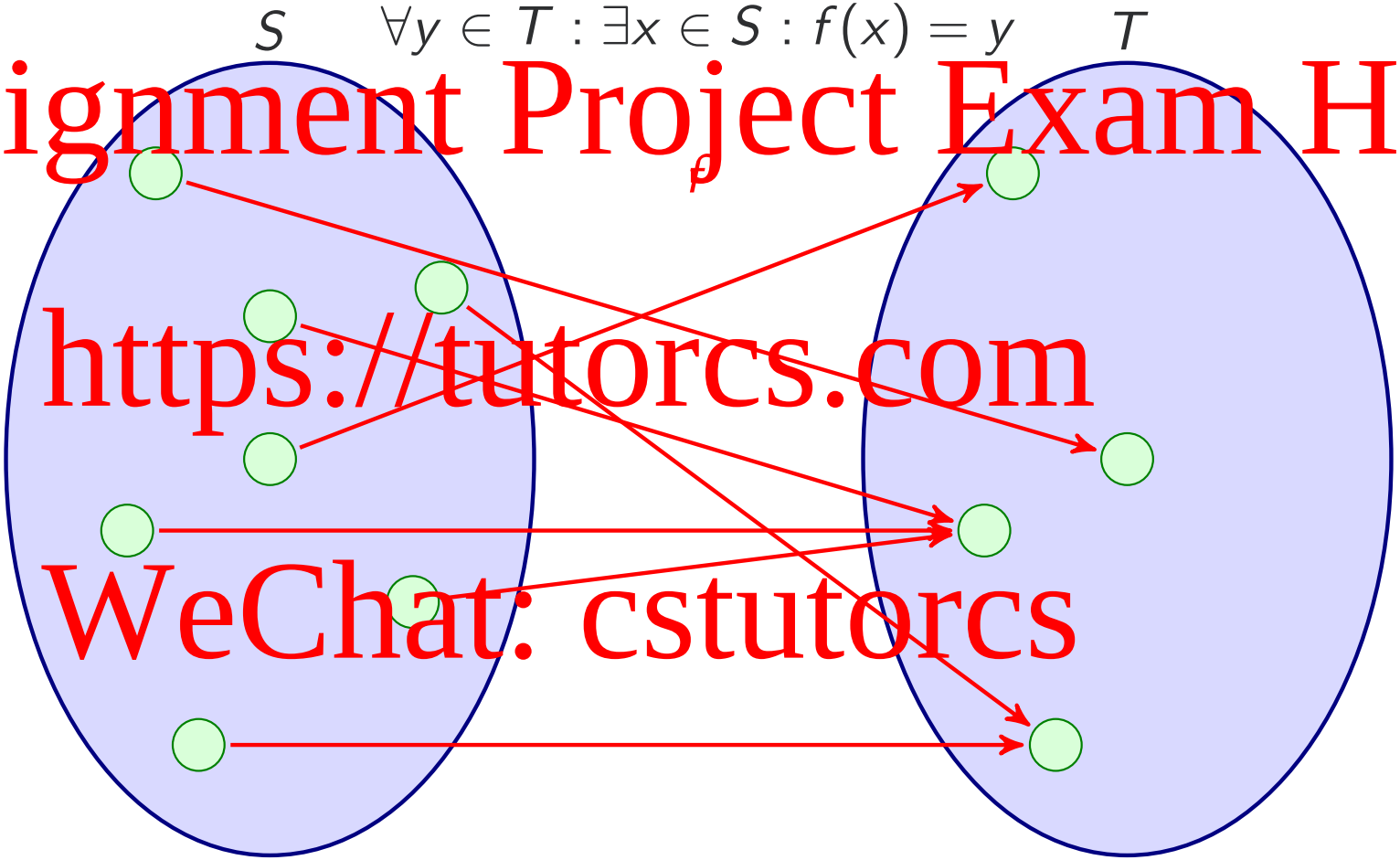
# Graphical representation: Function



# Graphical representation: Injection



# Graphical representation: Surjection



# Properties of Binary Relations $R \subseteq S \times S$

**Definition**

| | | |
|---|---|---|
| (R) | reflexive | For all $x \in S$: $(x, x) \in R$ |
| (AR) | antireflexive | For all $x \in S$: $(x, x) \notin R$ |
| (S) | symmetric | For all $x, y \in S$: If $(x, y) \in R$ then $(y, x) \in R$ |
| (AS) | antisymmetric | For all $x, y \in S$: If $(x, y)$ and $(y, x) \in R$ then $x = y$ |
| (T) | transitive | For all $x, y, z \in S$: If $(x, y)$ and $(y, z) \in R$ then $(x, z) \in R$ |



**NB**

- *Properties have to hold for all elements*
- *(S), (AS), (T) are conditional statements – they will hold if there is nothing which satisfies the 'if' part*

# Relation properties: Examples

## Examples

**(R)** Reflexivity: $(x, x) \in R$ for all $x$

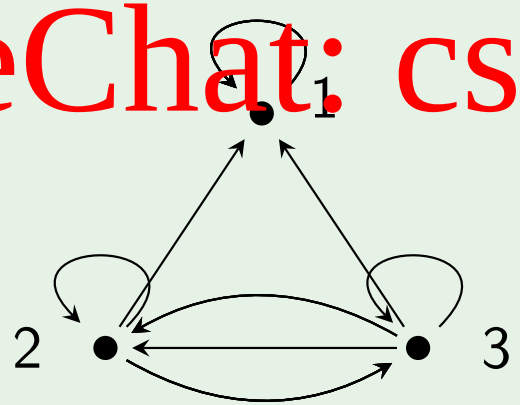
**(AR)** Antireflexivity: $(x, x) \notin R$ for all $x$

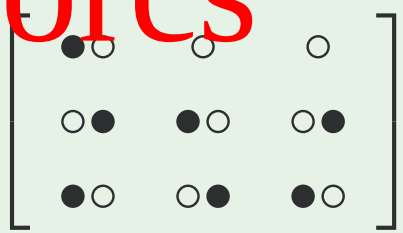
**(S)** Symmetry: If $(x, y) \in R$ then $(y, x) \in R$ for all $x$, $y$

**(AS)** Antisymmetry: $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all $x$, $y$

**(T)** Transitivity: $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x$, $y$, $z$.





# Interaction of Properties



A relation *can* be both symmetric and antisymmetric. Namely, when $R$ consists only of some pairs $(x, x), x \in S$.

A relation *cannot* be simultaneously reflexive and antireflexive (unless $S = \emptyset$).



**NB**

$$\left.\begin{array}{r} \textit{nonreflexive} \\ \textit{nonsymmetric} \end{array}\right\} \textit{ is not the same as } \left\{\begin{array}{l} \textit{antireflexive/irreflexive} \\ \textit{antisymmetric} \end{array}\right.$$

# Exercises



## Exercises

RW: 3.1.1 The following relations are on $S = \{1, 2, 3\}$. Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?



(a) $(m, n) \in R$ if $m + n = 3$?

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$?

# Exercises



## Exercises

RW: 3.1.1 The following relations are on $S = \{1, 2, 3\}$. Which of the properties (R), (AR), (S), (AS), (T) does each satisfy?



(a) $(m, n) \in R$ if $m + n = 3$? (AR) and (S)

(e) $(m, n) \in R$ if $\max\{m, n\} = 3$? (S)

# Exercises

## Exercises



PW 3.2.10 Give examples of relations with specified properties

(a) (AS), (T), not (R)



(b) (S), not (R), not (T)

# Exercises

## Exercises



IPW 3.2.10 Give examples of relations with specified properties

(a) (AS), (T), not (R)

- Strict order of numbers $x < y$
- $\leq$ but with some pairs $(x, x)$ removed
- Being a prime divisor: $(p, n) \in R$ iff $p$ is prime and $p|n$
  - Not reflexive: $(1, 1) \notin R$
  - Transitivity is meaningful only for the pairs $(p, p), (p, n)$ $p|n$ for $p$ prime





(b) (S), not (R), not (T)

Simplest example - inequality

# Exercises



**Exercises**

RW: 3.6.10 (supp)

$R$ is a relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^2 \times \mathbb{N}^2$

$(m, n)\, R\, (p, q)$ if $m \equiv_{(3)} p$ or $n \equiv_{(5)} q$.



(a) Is $R$ reflexive?

(b) Is $R$ symmetric?



(c) Is $R$ transitive?

# Exercises



**Exercises**

RW: 3.6.10 (supp)

$R$ is a relation on $\mathbb{N} \times \mathbb{N}$, i.e. it is a subset of $\mathbb{N}^2 \times \mathbb{N}^2$

$(m, n)\, R\, (p, q)$ if $m =_{(3)} p$ or $n =_{(5)} q$.



(a) Is $R$ reflexive? Yes: $m =_{(3)} m$ so $(m, n)R(m, n)$.

(b) Is $R$ symmetric? Yes: by symmetry of $\cdot =_{(n)} \cdot$.



(c) Is $R$ transitive? No: Consider $(1, 1)$, $(1, 4)$ and $(2, 4)$.

# Exercises

## Exercises

Complete the following table of common relations (over $\mathbb{Z}$) and their properties:

| | $(R)$ | $(AR)$ | $(S)$ | $(AS)$ | $(T)$ |
|---|---|---|---|---|---|
| $=$ | | | | | |
| $\leq$ | | | | | |
| $<$ | | | | | |
| $\emptyset$ | | | | | |
| $\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$ | | | | | |
| $\mid$ | | | | | |
| $=_{(3)}$ | | | | | |

# Exercises

## Exercises

Complete the following table of common relations (over $\mathbb{Z}$) and their properties:

$\mathcal{U} = \mathbb{Z} \times \mathbb{Z}$

| | $(R)$ | $(AR)$ | $(S)$ | $(AS)$ | $(T)$ | |
|---|---|---|---|---|---|---|
| $=$ | ✓ | | ✓ | ✓ | ✓ | |
| $\leq$ | ✓ | | | ✓ | ✓ | |
| $<$ | | ✓ | | ✓ | ✓ | |
| $\emptyset$ | | ✓ | ✓ | ✓ | ✓ | |
| $\mathcal{U}$ | ✓ | | ✓ | | ✓ | |
| $\mid$ | ✓ | | | * | ✓ | * True for $\mid \subseteq \mathbb{N} \times \mathbb{N}$ |
| $=_{(3)}$ | ✓ | | ✓ | | ✓ | |

# Outline

# Functions



**Definition**

A **function**, $f : S \to T$, is a binary relation $f \subseteq S \times T$ that satisfies (Fun) and (Tot). That is, for all $s \in S$ there is *exactly one* $t \in T$ such that $(s, t) \in f$.

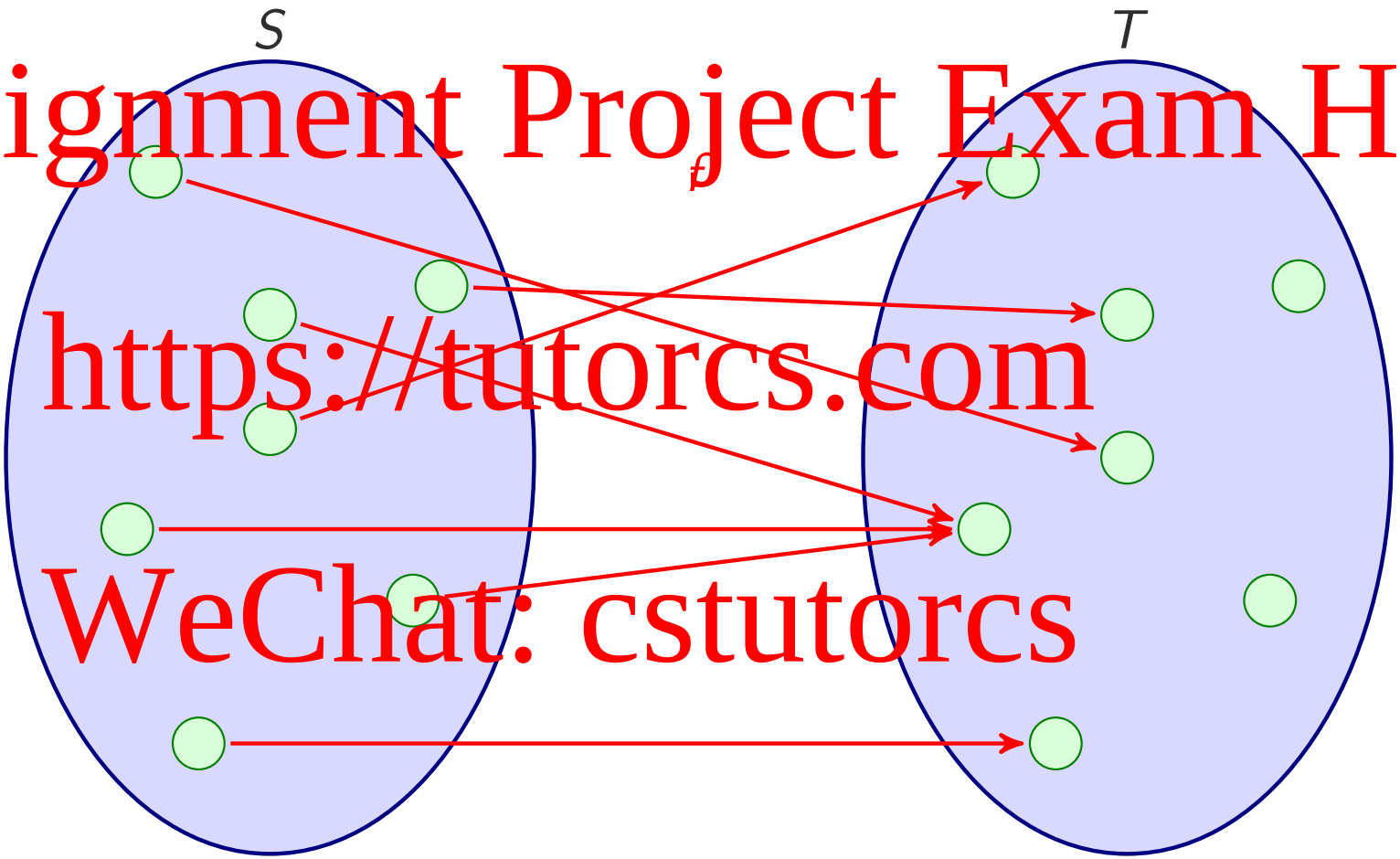


We write $f(s)$ for the unique element related to $s$.



We write $T^S$ for the set of all functions from $S$ to $T$.

# Graphical representation



# Functions

$f : S \longrightarrow T$ describes pairing of the sets: it means that $f$ assigns to every element $s \in S$ a unique element $t \in T$. To emphasise where a specific element is sent, we can write $f : x \mapsto y$, which means the same as $f(x) = y$



| | | Symbol | |
|---|---|---|---|
| $S$ | **domain** of $f$ | $\mathrm{Dom}(f)$ | (inputs) |
| $T$ | **co-domain** of $f$ | $\mathrm{Cod}(f)$ | (*possible* outputs) |
| $f(S)$ | **image** of $f$ | $\mathrm{Im}(f)$ | (*actual* outputs) |

$= \{ f(x) : x \in \mathrm{Dom}(f) \}$

**Important!**

The domain and co-domain are critical aspects of a function's definition.

$$f : \mathbb{N} \to \mathbb{Z} \quad \text{given by} \quad f(x) \mapsto x^2$$



and

$$g : \mathbb{N} \to \mathbb{N} \quad \text{given by} \quad g(x) \mapsto x^2$$

are different functions even though they have the same behaviour!

**Question**

$f^{\leftarrow}$ *is a relation, when is it a function?*