CS262 Logic and Verification

Lecture 8: Natural deduction

Introduction

Two proof systems we have seen so far: semantic tableau and resolution

Both are well-suited for automation and computer implementation (will implement a resolution prover as a coursework assignment)

This week we will look at another proof system: **natural deduction**

Natural deduction

Formalizes the kind of reasoning people do in informal arguments

Unlike tableau and resolution, natural deduction not well-suited for computer automation

Unlike the first two, natural deduction is not based on any normal form expansion (CNF or DNF)

Central notion: nested subordinate proofs (='lemmas'), in which we derive conclusions from certain assumptions, and then discharge the assumptions to produce assumption-free results

Typical rules

Implication rule: If one can derive Y from X as an assumption, then one can discharge the assumption and conclude that $X \to Y$ holds unconditionally

$$\boxed{\begin{array}{c} X \\ \vdots \\ Y \end{array}}$$

$$X \to Y$$

Subordinate proofs/lemmas are contained in boxes

The first formula X in a box is an **assumption**

We can assume anything, but the question is whether the assumptions help in making useful conclusions

Typical rules

Modus Ponens rule: From X and $X \to Y$ we can conclude Y

$$\frac{X \quad X \to Y}{Y}$$

A formula is called **active** at some stage if it does not occur in a closed box

We may only use active formulas at any stage

Rules are paired, one for **introducing** $\to$ and one for **eliminating** $\to$

Example

Natural deduction proof of $(p \to (q \to r)) \to (q \to (p \to r))$

Rules

Constant rules:

$$\frac{\bot}{X} \qquad \frac{}{\neg\bot}$$

Negation rules:

$$\frac{\begin{array}{c} X \\ \neg X \end{array}}{\bot} \qquad \begin{array}{c}\boxed{\begin{array}{c} X \\ \vdots \\ \bot \end{array}} \\ \neg X \end{array} \qquad \begin{array}{c}\boxed{\begin{array}{c} \neg X \\ \vdots \\ \bot \end{array}} \\ X \end{array}$$

Primary connective rules:

$\alpha\mathbf{E}$

$$\frac{\alpha}{\alpha_1} \qquad \frac{\alpha}{\alpha_2}$$

$\alpha\mathbf{I}$

$$\frac{\begin{array}{c} \alpha_1 \\ \alpha_2 \end{array}}{\alpha}$$

$\beta\mathbf{E}$

$$\frac{\begin{array}{c} \neg\beta_1 \\ \beta \end{array}}{\beta_2} \qquad \frac{\begin{array}{c} \neg\beta_2 \\ \beta \end{array}}{\beta_1}$$

$\beta\mathbf{I}$

$$\begin{array}{c}\boxed{\begin{array}{c} \neg\beta_1 \\ \vdots \\ \beta_2 \end{array}} \\ \beta \end{array} \qquad \begin{array}{c}\boxed{\begin{array}{c} \neg\beta_2 \\ \vdots \\ \beta_1 \end{array}} \\ \beta \end{array}$$

Remarks

Primary connective rules come in two flavors: **I**ntroduction + **E**limination

Last two negation rules embody the principle of proof by contradiction

Second constant rule has no premises

Order of premises does not matter, but all premises must be active

A rule is **derived** if it does not strengthen the proof system

Any occurence of the rule can be translated away using the 'official' rules

Double negation

$$\frac{\neg\neg X}{X} \qquad \frac{X}{\neg\neg X}$$

Copy rule

$$\frac{X}{X}$$

Implication

$$\frac{\boxed{\begin{array}{c} X \\ \vdots \\ Y \end{array}}}{X \to Y}$$

Modus ponens

$$\frac{X \qquad X \to Y}{Y}$$

Modus tollens

$$\frac{\neg Y \qquad X \to Y}{\neg X}$$

Excluded middle

$$\frac{}{X \vee \neg X}$$

Proof strategies

Think backwards: What rules could be applied in the last step, and based on that come up with assumptions that should be made to apply those rules.

In proving X, assume $\neg X$ and produce $\perp$, then use negation rule

This needs a lot of practice and experience

Example: Prove $p \to (q \to p)$

Proving consequences

Want to prove propositional consequences $S \models X$

S-introduction rule for natural deduction: At any stage, any member of S may be used as a line.

W.l.o.g., we may introduce them as the initial lines of the proof (recall copy rule). These formulas are sometimes called **premises** (no boxes!).

We write $S \vdash_d X$ if there is a natural deduction derivation of X from S

Example: Prove $\{p \to q, q \to r\} \vdash_d p \to r$

Soundness and completeness

Theorem (Soundness and completeness)

We have $S \models X$ if and only if $S \vdash_d X$.

Proof omitted here