Financial Econometrics

Slides-01: RETURN PROPERTIES Part II

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Shape Characteristics: Population

Let X_t be a random variable with pdf $f(x)$

$$\begin{aligned}
\mu &= E[X_t] : \text{ center} \\
\sigma^2 &= var(X_t) = E[(X_t-\mu)^2] : \text{ spread} \\
skewness(X_t) = S(X) &= E\left[\frac{(X_t-\mu)^3}{\sigma^3}\right] : \text{ symmetry} \\
kurtosis(X_t) = K(X) &= E\left[\frac{(X_t-\mu)^4}{\sigma^4}\right] : \text{ tail thickness} \\
K(X) - 3 &: \quad \text{Excess kurtosis}
\end{aligned}$$

Note: The k^{th} moment and central moment of X_t are:

$$\begin{aligned}
m'_k &= E[X_t^k] \\
m_k &= E[(X_t-\mu)^k]
\end{aligned}$$

Shape Characteristics of Random Variable

- Why are the mean and variance of returns important?

  They are concerned with long-term return and risk, respectively.

- Why is return symmetry of interest in financial study?

  Symmetry has important implications in holding short or long financial positions and in risk management.

- Why is kurtosis important?

  Related to volatility forecasting, efficiency in estimation and tests, etc.

  High kurtosis implies heavy (or long) tails in distribution.

Examle: Normal Random Variable

Normal Distribution

$X \sim N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad -\infty \le x \le \infty$$

$$\begin{aligned} E[X] &= \mu \\ \text{var}(X) &= \sigma^2 \\ \text{skew}(X) &= 0 \\ \text{kurt}(X) &= 3 \\ m_k &= 0 \text{ for } k \text{ odd} \end{aligned}$$

Shape Characteristics: Sample moments

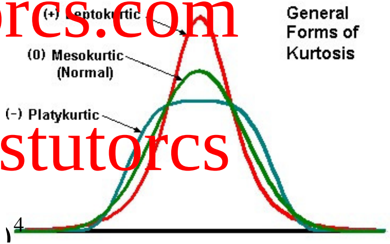
Sample moments

Let $\{r_t, \ldots, r_T\}$ denote a random sample of size T where r_t is a realization of the random variable $\tilde{r}$.

$$
\begin{aligned}
\hat{\mu} &= \frac{1}{T}\sum_{t=1}^{T} r_t,\ \hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t-\hat{\mu})^2 = \hat{m}_2 \\
\widehat{\text{skew}} &= \frac{\hat{m}_3}{\hat{\sigma}^3},\ \widehat{\text{kurt}} = \frac{\hat{m}_4}{\hat{\sigma}^3} \\
\hat{m}_k &= \frac{1}{T-1}\sum_{t=1}^{T}(r_t-\hat{\mu})^k,
\end{aligned}
$$

Note: we divide by $T-1$ to get unbiased estimates. Check software to see how moments are computed.

Shape Characteristics: Visually



Testing for normality

- **QQ-plot:** plot standardized empirical quantiles vs. theoretical quantiles from specified distribution. Note: Shapiro Wilks (SW) test for normality: correlation coefficient between values used in QQ-plot
- **Jarque-Bera (JB)** test for normality

$$\begin{aligned} JB &= \frac{T}{6}\left(\hat{skew}^2 + \frac{(\hat{kurt}-3)^2}{4}\right) \\ &\sim_A \chi^2(2) \end{aligned}$$

Note: if r_t is $N(\mu, \sigma^2)$ then:

$$\sqrt{T}\hat{skew} \sim N(0,6), \text{ and } \sqrt{T}(\hat{kurt}-3) \sim N(0,24)$$

Shape Characterirtics: Normality test

The null hypothesis:
H_0 : Data (the return) X_t are Normally distributed.

1. **Skewness test:** $Z_{sk} = \frac{sk\hat{e}w}{\sqrt{6/T}} \sim N(0,1)$

   Reject H_0 if $|z_{sk}|$ is too large (> 1.96, at 5%)

2. **Kurtosis test:** $Z_{kt} = \frac{kurt-3}{\sqrt{24/T}} \sim N(0,1)$

   Reject H_0 if $|z_{kt}|$ is too large (> 1.96, at 5%).

3. **Jaque-Bera test:** $JB = Z_{ks}^2 + Z_{kt}^2 \sim \chi_2^2$

   Reject JB is too large (> 5.99 at 5%)

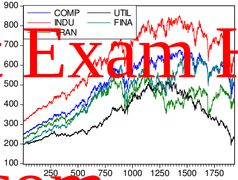
Example: Descriptive Statistics

- Descriptive statistics

eg. NYSE index prices: (19950103-20020830)

Composite, Industrial,

Trans, Utility, Finance.



Descriptive statistics of log returns.

	Composite	Industrial	Trans	Utility	Finance
Mean	0.035	0.034	0.031	0.007	0.052
Std. Dev.	1.006	1.009	1.320	1.087	1.310
Skewness	-0.316	-0.386	-1.044	-0.275	-0.042
Kurtosis	7.224	7.755	18.103	5.637	5.772

Correlations of log returns

	Composite	Industrial	Trans	Utility	Finance
Composite	1				
Industrial	0.983	1			
Trans	0.731	0.708	1		
Utility	0.769	0.711	0.505	1	
Finance	0.885	0.800	0.668	0.623	1

Portfolio variance and diversification:

$$Z = \frac{1}{2}(Y + X),$$

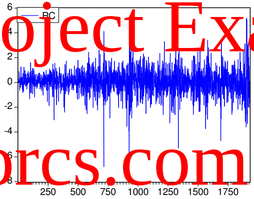
$$\mathrm{Var}(Z) = \frac{1}{4}[\mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X, Y)]$$

Example: Descriptive Statistics

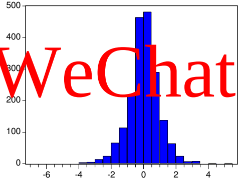
– Normality test

eg. Comp. index log return

time series plot



histogram



Series: RC	
Sample 1 1931	
Observations 1930	
Mean	0.035300
Median	0.0322[illegible]5
Maximum	[illegible]787[illegible]4
Minimum	-6.7[illegible]11[illegible]2
Std. Dev.	1.006207
Skewness	-0.315728
Kurtosis	7.224376
Jarque-Bera	1467.129
Probability	0.000000

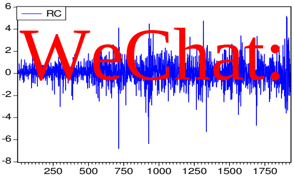
p value = $P(\chi^2_{(2)} > JB)$

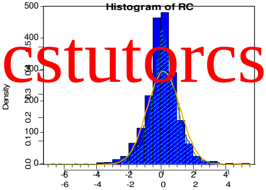
Stylized Fact: Large kurtosis

Some stylised facts about index return series

- concentration around zero with a few large “outliers”
- large standard deviations (volatile)
- negative skewness (longer tail at the negative side)
- large kurtosis (tail probabilities larger than normal)
- large variation followed by large ones (clustering)

leptokurtic





Series: RC
Sample 1 1931
Observations 1930

Mean	0.035300
Median	0.052285
Maximum	5.178704
Minimum	-6.791142
Std. Dev.	1.006207
Skewness	-0.315728
Kurtosis	7.224376
Jarque-Bera	1467.129
Probability	0.000000

Descriptive statistics: Autocorrelation

- **Predictability**
  - We say X_{t+1} is predictable if information at t, eg. $\{X_t, X_{t-1}, \cdots\}$, helps to improve our prediction of X_{t+1}.
  - In particular, X_{t+1} is predictable if X_{t+1} is correlated with X_{t-j} for some $j > 0$ (ie. $Cov(X_{t+1}, X_{t-j}) \neq 0$).
- **Autocorrelation Function (ACF)**
  - Autocovariance: $\gamma_j = Cov(X_t, X_{t-j}) = Cov(X_t, X_{t+j})$
  - Sample autocovariance: $\hat{\gamma}_j = \frac{1}{T}\sum_{t=j+1}^{T}(X_t - \overline{X})(X_{t-j} - \overline{X})$
  - Autocorrelation: $\rho_j = \frac{\gamma_j}{\gamma_0}$

    Sample Autocorrelation: $\hat{\rho}_j = \frac{\hat{\gamma}_j}{\hat{\gamma}_0}$
- **Partial autocorrelation (PAC)**
  - PAC p_j is a measure of the **direct** relation between X_t and X_{t-j} for $j = 1, 2, \cdots$
  - p_j is the correlation between X_t and X_{t-j} after controlling for the effects of X_t and $X_{t-1} \cdots X_{t-j+1}$
  - $\hat{p}_1 = \hat{\phi}_{11}$ in $X_t = \phi_{10} + \phi_{11}X_{t-1} + e_{1t}$
  - $\hat{p}_2 = \hat{\phi}_{11}$ in $X_t = \phi_{20} + \phi_{21}X_{t-1} + \phi_{22}X_{t-2} + e_{2t}, \cdots$

Test for autocorrelation

The null hypothesis: H_0: There is no autocorrelation (White noise process)

1. **Autocorrelation test:** $\sqrt{T}\hat{\rho}_j \sim N(0,1)$ under the null hypothesis
   Reject if $|\hat{\rho}_j|$ is too large ($> 1.96/\sqrt{T}$, at 5% significance level)

Joint Hypothesis Tests

- We can also test the joint hypothesis that all m of the ρ_k correlation coefficients are simultaneously equal to zero using the Q-statistic developed by Box and Pierce:

$$Q = T \sum_{k=1}^{m} \hat{\rho}_k^2$$

where T=sample size, m=maximum lag length

- The Q-statistic is asymptotically distributed as a χ_m^2.
- However, the Box Pierce test has poor small sample properties, so a variant has been developed, called the Ljung-Box statistic:

$$Q^* = T(T+2) \sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{T-k} \sim \chi_m^2$$

- This statistic is very useful as a portmanteau (general) test of linear dependence in time series.

An ACF Example

- Question:

Suppose that a researcher had estimated the first 5 autocorrelation coefficients using a series of length 100 observations, and found them to be (from 1 to 5): 0.207, -0.013, 0.086, 0.005, -0.022.

Test each of the individual coefficient for significance, and use both the Box-Pierce and Ljung-Box tests to establish whether they are jointly significant.

Solution

A coefficient would be significant if it lies outside $(-0.196, +0.196)$ at the 5% level, so only the first autocorrelation coefficient is significant.

$Q = 5.09$ and $Q* = 5.26$

Compared with a tabulated χ^2(5)=11.1 at the 5% level, so the 5 coefficients are jointly insignificant.

Example: ACF/PACF

- Descriptive statistics

e.g. NYSE composite return

Correlogram of RC

Sample: 1 1931
Included observations: 1930

J	AC	PAC	Q-Stat	Prob
1	0.068	0.068	9.0448	0.003
2	0.046	-0.051	13.106	0.001
3	-0.031	-0.024	14.940	0.002
4	-0.001	0.001	14.942	0.005
5	-0.052	-0.055	20.226	0.001
6	-0.014	-0.008	20.624	0.002
7	-0.033	-0.037	22.761	0.002
8	0.011	0.012	22.992	0.003
9	0.033	0.028	25.155	0.003
10	0.028	0.021	26.723	0.003
11	-0.043	-0.044	30.332	0.001
12	0.054	0.061	35.987	0.000
13	0.015	0.004	36.403	0.001
14	-0.013	-0.009	36.720	0.001
15	-0.002	0.008	36.726	0.001
16	-0.027	-0.031	38.181	0.001
17	0.014	0.025	38.588	0.002
18	-0.007	-0.015	38.677	0.003
19	0.022	0.026	39.639	0.004
20	0.006	0.004	39.710	0.005

AC test at 5% level:

$1.96/\sqrt{T} = 0.04462$,

H_0 is rejected at

$j = 1,2,5,12$

LB test at 5% level:

H_0 is rejected for all m, as all p-values are less than 0.05.

Example: ACF/PACF of squared Returns

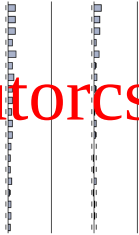
- Descriptive statistics
  - What about squared returns?

    Usually strongly correlated.
  - Why squared returns?

    $E(r_t^2) \approx Var(r_t)$

  eg. NYSE Composite

  return squared



Correlogram of RC2

Sample: 1 1931
Included observations: 1930

Lag	AC	PAC	Q-Stat	Prob
1	0.179	0.179	62.044	0.000
2	0.168	0.140	116.47	0.000
3	0.175	0.131	176.00	0.000
4	0.109	0.044	198.89	0.000
5	0.184	0.130	264.80	0.000
6	0.113	0.035	289.54	0.000
7	0.136	0.067	325.24	0.000
8	0.112	0.030	349.60	0.000
9	0.108	0.039	372.11	0.000
10	0.095	0.015	389.80	0.000
11	0.107	0.042	411.90	0.000
12	0.102	0.028	432.21	0.000
13	0.066	-0.006	440.72	0.000
14	0.053	-0.016	446.26	0.000
15	0.041	-0.016	449.58	0.000
16	0.097	0.052	467.86	0.000
17	0.040	-0.018	470.96	0.000
18	0.068	0.024	479.94	0.000
19	0.063	0.014	487.77	0.000
20	0.045	0.005	491.70	0.000

Summary of stylized Facts

KEY stylised facts about financial return series

1. the returns have small, often non-significant autocorrelations (no linear return predictability)

2. the squared returns have strong positive autocorrelations (predictability in volatility, volatility clustering)
3. large kurtosis (heavy tails, tail probabilities larger than normal)

Summary

- Characterizing Financial time series:
  - asset price and returns
  - stylised facts about index return series

- Normality tests: Z_{ks}, Z_{kt}, JB
- Predictability in returns
  - Autocovariance and autocorrelation
  - Tests for autocorrelation: AC test and Q_m
- Next week: Application of linear regression in Finance (asset pricing)