ECON3206/5206 Financial Econometrics

Slides-11: GARCH, VaR and Extensions

Dr. Rachida Ouysse
School of Economics[1]

Lecture Plan

- Forecasting Volatility with GARCH
- Volatility and Risk: VaR
- Typical estimates of GARCH parameters

  A measure of volatility persistence
- Integrated GARCH and EWMA

Forecasting volatility with GARCH(1,1)

Forecasting volatility

At first sight, forecasting the volatility in the error terms may not seem very useful.

However, keep in mind that

$$var\left(y_t \,|y_{t-1}, y_{t-2}, \dots\right) = var\left(\mu_t \,|\mu_{t-1}, \mu_{t-2}, \dots\right)$$

Therefore, these models are very useful as they can add a model for the volatility of a time series to traditional ARMA models.

- forecasting the volatility of stock returns is useful e.g. in option pricing as this requires the expected volatility of the underlying asset over de lifetime of the option as an input

Forecasting volatility with GARCH(1,1)

Consider the following GARCH(1,1) model

$$y_t = \mu + \mu_t \qquad \mu_t \sim N\left(0, \sigma_t^2\right)$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 \mu_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

Generate one-, two- and three-step-ahead forecasts for the conditional variance of y_t at time T.

- First update the equations for the conditional variance:

$$\sigma_{T+1}^2 = \alpha_0 + \alpha_1 \mu_T^2 + \beta_1 \sigma_T^2$$
$$\sigma_{T+2}^2 = \alpha_0 + \alpha_1 \mu_{T+1}^2 + \beta_1 \sigma_{T+1}^2$$
$$\sigma_{T+3}^2 = \alpha_0 + \alpha_1 \mu_{T+2}^2 + \beta_1 \sigma_{T+2}^2$$

Forecasting volatility with GARCH(1,1)

- Then let $\sigma^{2f}_{1,T}$ be the one-step-ahead forecast for σ^2 at time T

$$
\begin{aligned}
\sigma^{2f}_{1,T} &= E_T\left(\sigma^2_{T+1}\right) = \alpha_0 + \alpha_1 \mu_T^2 + \beta_1 \sigma_T^2 \\
\sigma^{2f}_{2,T} &= E_T\left(\sigma^2_{T+2}\right) = \alpha_0 + \alpha_1 E_T\left(\mu^2_{T+1}\right) + \beta_1 \sigma^{2f}_{1,T} \\
&= \alpha_0 + \alpha_1 E_T\left(\sigma^2_{T+1}\right) + \beta_1 \sigma^{2f}_{1,T} \\
&= \alpha_0 + \left(\alpha_1 + \beta_1\right) \sigma^{2f}_{1,T} \\
&= \alpha_0 + \left(\alpha_1 + \beta_1\right)\left(\alpha_0 + \alpha_1 \mu_T^2 + \beta_1 \sigma_T^2\right) \\
\sigma^{2f}_{3,T} &= E_T\left(\sigma^2_{T+3}\right) = \alpha_0 + \left(\alpha_1 + \beta_1\right) \sigma^{2f}_{2,T} \\
&= \alpha_0 + \alpha_0\left(\alpha_1 + \beta_1\right) + \left(\alpha_1 + \beta_1\right)^2 \sigma^{2f}_{1,T} \\
\sigma^{2f}_{s,T} &= E_T\left(\sigma^2_{T+s}\right) = \alpha_0 \sum_{i=1}^{s-1} \left(\alpha_1 + \beta_1\right)^{i-1} + \left(\alpha_1 + \beta_1\right)^{s-1} \sigma^{2f}_{1,T}
\end{aligned}
$$

- For $s \to \infty \quad \sigma^{2f}_{s,T} = \alpha_0 / \left(1 - \left(\alpha_1 + \beta_1\right)\right)$ if $\left|\alpha_1 + \beta_1\right| < 1$

Example: Forecasting volatility with GARCH(1,1)

Example: forecasting mean and variance from an AR(1)-GARCH(1,1) for returns on the S&P500 index in a hold-out sample of 100 observations.

EViews: in the *Equation Window* select *Forecast*

Note that volatility is highly persistent!

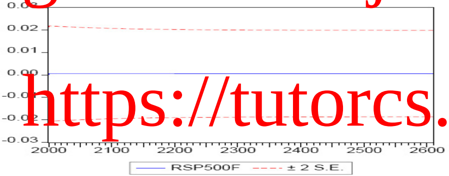
- forecasted volatility converges only slowly to the unconditional mean, which is equal to

$$\sigma^2 = \frac{0.000000792}{1 - 0.068012 - 0.923437} = 0.000093$$

- there is a great deal of predictability in volatility!

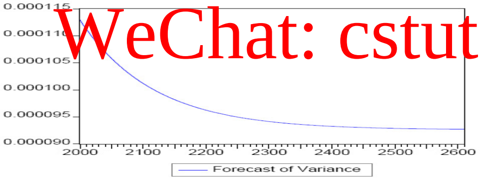
Forecasting volatility with GARCH(1,1)

Figure 20: Forecasting mean and volatility



Forecast: RSP500F
Actual: RSP500
Forecast sample: 2000 2610
Included observations: 611

Root Mean Squared Error	0.012137
Mean Absolute Error	0.008811
Mean Abs. Percent Error	129.1004
Theil Inequality Coefficient	0.952475
Bias Proportion	0.000094
Variance Proportion	0.995943
Covariance Proportion	0.003963



Volatility and Risk: Risks of Large Losses

- Amaranth h/f ($6.5 billion in one week in September 2006)
- Credit Lyonnais ($5.0 billion in 1990)
- Long-Term Capital Management h/f ($4.6 billion in 1998)
- **Lehman Brothers ($3.9 billion in September 2008)**
- Orange County ($2 billion in 1994)
- Barings ($1.4 billion in 1995)
- Daiwa Bank ($1.1 billion in 1995)
- Allied Irish Bank ($0.7 billion in 2002)
- China Aviation Oil ($0.6 billion in 2004)

Value at Risk VaR

Risk managers/regulators are often interested in the following statement:
"I am 99% certain that my portfolio of assets will not lose more than $V over the next **period** and have sufficient reserves to cover losses lower than this level. "
period is often one day, but can be a month, quarter, year

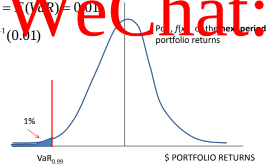
$$(1-\alpha)100\%\ \text{VaR} : \text{VaR}_{1-\alpha} \quad = \quad F^{-1}(\alpha) \times \text{Value of Investment}$$

VaR is the maximum portfolio loss in a given period (eg, 1 day) with a given probability (eg, 0.99).

99% Value at Risk

$$\int_{-\infty}^{VaR} f(y) = F(VaR) = 0.01$$

$$VaR = F^{-1}(0.01)$$



Conditional Value at Risk

- Consider $AR(1) - GARCH(1,1)$ for the portfolio return y_t
  $y_t = c + \beta_1 y_{t-1} + \mu_t$, where $\mu_t|\Omega_{t-1} \sim N(0, \sigma_t^2)$
  $\sigma_t^2 = \alpha_0 + \alpha_1 \mu_{t-1}^2 + \beta_1 \sigma_{t-1}^2$
  - $\nu_t = \frac{\mu_t}{\sigma_t} = \frac{y_t - y_{t|t-1}}{\sigma_t} \sim N(0,1)$, where $y_{t|t-1} = E(y_t|\Omega_{t-1})$
  - $P(\nu_t < -2.326) = 0.01 = 1 - 0.99$ implies:

$$P(y_t < y_{t|t-1} - 2.326\sigma_t) = 0.01$$

  - $\text{VaR}_{0.99} = \frac{1}{100}(y_{t|t-1} - 2.326\sigma_t) \times$ Portfolio Value

Example 1

eg. NYSE composite return (continued)

Portfolio valued at \$1m at T = 2002-08-29.

AR(1)-GARCH(1,1): σ_{T+1} =1.64196, $y_{T+1|T}$ =0.05132.

VaR = $\frac{1}{100}\left(y_{T+1|T} - 2.326\sigma_{T+1}\right) \times$ \$1m = –\$37,678

If using the sample mean, sample variance and normality, we find

VaR = $\frac{1}{100}$ [.0353 – 2.326(1.0062)] ×\$1m = – \$23,051.

VaR using Empirical Quantile

BUT normality is often rejected?

- GARCH is able to account for clustering, such that the standardised shock (ν_t) can be viewed as iid.
- To compute VaR, we only need the lower quantile of ν_t, which can be estimated by the **empirical quantile** of the **standardised residuals**.

- Instead of using the $N(0,1)$ to find $F^{-1}(\alpha)$, we need to use the distrubution of the the estimated standardised residuals ν_t
  - $\nu_t = \frac{\mu_t}{\sigma_t} = \frac{y_t - y_{t|t-1}}{\sigma_t} \sim$ iid(0,1)
  - $P(\nu_t < Q_{0.01}) = 0.01 = 1 - 0.99$ implies $P(y_t < y_{t|t-1} - Q_{0.01}\sigma_t) = 0.01 = 1 - 0.99$

$$\text{VaR}_{0.99} = \frac{1}{100}(y_{t|t-1} - Q_{0.01}\sigma_t) \times \text{Portfolio Value}$$

Example 2

eg. NYSE composite return (continued)

Portfolio valued at \$1m at T = 2002-08-29.

AR(1)-GARCH(1,1): $\sigma_{T+1} = 1.64196$, $y_{T+1|T} = 0.05132$.

The 1% quantile of v_t: $q_{0.01} = -2.873$

VaR = $\frac{1}{100}\left(y_{T+1|T} - 2.873\sigma_{T+1}\right)$ ×\$1m = −\$46,660

For ARCH(5): $\sigma_{T+1} = 1.25322$, $y_{T+1|T} = 0.05037$,

$q_{0.01} = -2.774$, VaR = −\$34,260

A measure of persistence: half-life time

- Let $\omega_t = \mu_t^2 - \sigma_t^2$, then μ_t^2 has an ARMA(1,1) representation:$\mu_t^2 = \alpha_0 + (\alpha_1 + \beta_1)\mu_{t-1}^2 + \omega_t - \beta_1\omega_{t-1}$
- When the shocks are zero, ie, $\omega = 0$ for all t, by substitution:

$$\mu_t^2 = \alpha_0\left[1 + \cdots + (\alpha_1 + \beta_1)^{t-1}\right] + (\alpha_1 + \beta_1)^t \mu_0^2$$

The impact of μ_0^2 on μ_t^2 is $(\alpha_1 + \beta_1)^t$, ceteris paribus.

- Half-life time, t_H, is defined as the number of periods required for the impact to be halved:

$$(\alpha_1 + \beta_1)^{t_H}\mu_0^2 = \frac{1}{2}\mu_0^2, \text{ or } t_H = \frac{\ln(1/2)}{\ln(\alpha_1 + \beta_1)}$$

eg. Composite return: $\alpha_1 + \beta_1 = 0.996$, $t_H = 172.9$ (days).

Integrated GARCH: iGARCH

- What happens if $\alpha_1 + \beta_1 = 1$? (known as iGARCH)
- When $\alpha_0 > 0$, the unconditional variance is NOT finite and grows with t: $E(\sigma_t^2) = \alpha_0 t + E(\sigma_0^2)$.

  True because $E(\sigma_t^2) = \alpha_0 + (\alpha_1 + \beta_1)E(\sigma_{t-1}^2) = \alpha_0 + E(\sigma_{t-1}^2)$

  We may write $\alpha_0 = (1 - \alpha_1 - \beta_1)\omega$, where ω is the unconditional variance of μ_t for $\alpha_1 + \beta_1 = 1$.
- When $\alpha_1 + \beta_1 = 1$ and $\alpha_0 = 0$, the conditional variance is an EWMA of μ_t^2:

$$\sigma_t^2 = (1 - \beta_1)\mu_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

  which, as an EWMA, is not mean-reverting.

eg. NYSE composite return: The above explains why GARCH is very slow to revert to the average level

Summary

- Forecasting with the GARCH follows similar recursive structure as an autoregressive model.
- The long run forcast of volatility converges to the unconditional variance of the process.

- Application to VaR: measures the risk exposure and the maximum amount of loss in dollar value forecast for the next period:
  - The VaR involves the mean, and variance of the distribution of returns/payoffs of investment,
  - GARCH/ARCH models allow us to compute Conditional VaR,
  - The unconditional mean and variance underestimates the VaR: conditional VaR bigger in absolute value than Unconditional VaR (based on the mean and sample variance)
  - The normal distribution quantile leads to underestimating the VaR compared to using the empirical quantile.
- The Half-life time measures the amount of persistence in the GARCH.