Financial Econometrics

Slides-03: Linear Regression with Time Series Diagnostics Tests, Robust Inference& Model Stability

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Testing the CAPM: Mobil Exxon

The CAPM implies that the market rewards investors for the market risk

$$E(R_i) - R_f = \beta_i \left[E(R_m - R_f)\right]$$

where R_i is the return on an asset i, and R_m is the return on the market index.

- To estimate the CAPM: Run an OLS regression of excess returns on asset i, $X_{i,t}$, on the market excess return $X_{m,t}$

$$X_{i,t} = \alpha_i + \beta_i X_{m,t} + \mu_t$$

- If the CAPM holds, the null hypothesis $H_0 : \alpha_i = 0$ $H_a : \alpha_i \neq 0$ (two-tailed test)

Dependent Variable: E_MOBIL
Method: Least Squares
Sample: 1978M01 1987M12
Included observations: 120

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.004241	0.00588[illegible]	0.721037	0.4723
E_MARKET	0.[illegible]605	0.085615	8.347761	0.0000

R-squared	0.371287	Mean dependent var	0.009353
Adjusted R-squared	0.365959	S.D. dependent var	0.080468
S.E. of regression	0.064074	Akaike info criterion	-2.641019
Sum squared resid	0.484452	Schwarz criterion	-2.594561
Log likelihood	160.4612	F-statistic	69.68511
Durbin-Watson stat	2.087124	Prob(F-statistic)	0.000000

observed value $\hat{t} = 0.721$, p-value $= 0.472$: decision: Do not reject the null.

- Joint hypothesis: $H_0 : \alpha_i = 0, \beta_i = 1$: observed $\hat{F} = 5.623$, $df = (2, 118)$ p-value$= P\left(F_{2,118} > \hat{F}\right) = 0.0046 \Longrightarrow$ decision: Reject the null. PythonCode

Arbitrage Pricing Theory (APT)

What determines the expected return of an asset?

Excess returns: $X_{i,t} = R_{i,t} - R_{f,t}$ and $X_{m,t} = R_{m,t} - R_{f,t}$

1. CAPM:

$$
\begin{aligned}
E(X_{i,t}) &= \alpha_i + \beta_i E(X_{m,t}) \\
RP_i &= \alpha_i + \beta_i RP_m
\end{aligned}
$$

where RP_i: risk premium for asset i, RP_m: market risk premium

2. APT (Arbitrage Pricing Theory)

$$
E(X_{i,t}) = RP_i = \alpha_i + \beta_i RP_m + \beta_{other} RP_{other factors}
$$

Arbitrage Pricing Theory (APT)

What determines the expected return of an asset?

- APT (Arbitrage Pricing Theory): if there are r risk factors priced in the fiunancial market, then:

$$RP_i = \alpha_i + \beta_i RP_m + \beta_{i,1} RP_1 + \cdots \beta_{i,r} RP_r$$

  - RP_j is the **risk premium** for exposure to factor j risk;, $j = 1, \cdots, r$.
  - $\beta_{i,j}$ is the sensitivity of the asset to factor j; it also measures asset i's exposure to the factor risk j

So what are the other risk factors in the APT?

A well established APT Model in the finance literature is the Fama&French three factor model: ▸ Fama French

$$RP_i = \alpha_i + \beta_{i.m} RP_m + \beta_{i.s} RP_s + \beta_{i.h} RP_h + \beta_{i.u} RP_u \tag{1}$$

- RP_m is the market risk premium
- RP_s is the size factor risk premium (*small market capitalisation*)
- RP_h is the value factor risk premium (*high book-to-market stocks*)
- RP_u is the momentum risk factor premium (prior gains)
- $\beta_{i.m}$, $\beta_{i.s}$, $\beta_{i.h}$ and $\beta_{i.u}$ are the betas for the market risk, size factor, value factor and momentum respectively

Example 1: Expected Return

Based on the following data and a risk-free rate of return of 2%, compute expected return under APT model.

	Beta of each factor	Factor	Risk Premium
$\beta_{i.m}$	1.2	RP_m	5.1
$\beta_{i.s}$	0.8	RP_s	0.5
$\beta_{i.h}$	0.2	RP_h	0.95
$\beta_{i.u}$	-0.1	RP_u	2.5

Solution:

$$E(R_i) = R_f + \alpha_i + \beta_{i.m}RP_m + \beta_{i.s}RP_s + \beta_{i.h}RP_h + \beta_{i.u}RP_u$$
$$= 2\% + 1.2{*}5.1\% + 0.8{*}0.5\% + (0.2){*}0.95\% + (-0.1){*}2.5\%$$

$E(R_i) = 8.46\%$

Question 1: NIKE

Given the risk-free rate of return of 1.0%, average return of Nike (i.e: S&P 500 company with small market cap) of 15.88% p.a and the data provided in table below:

Q1(a) Compute the expected return of the NIKE under APT model;
Q1(b) Determine the alpha return of the NIKE;
Q1(c) Construct a portfolio comprising S&P500 index fund (market portfolio), Wilshire 5000 index fund, Russell 1000 value index fund and US T-Bills to replicate the expected return of Nike.

Table 1: Factor beta, returns and risk premium

Factor	Beta	R	Risk Premium
RP_m	0.7877	14.5%	13.49%
RP_s	0.6701	14.65%	0.15%
RP_v	-0.0288	10.38%	-4.12%

(i) RP_m is the market risk premium, i.e: excess of S&P500 return over the risk-free rate of return;
(ii) RP_s is the size factor risk premium , i.e: excess of Wilshire 5000 index returns over the S&P500 returns (iii) RP_v is the value factor risk premium , i.e: excess of Russell 1000 index returns over the S&P500 returns.

Solution to Question 1

$E(R_{NKE}) = R_f + \beta_{NKE,m}(R_m - R_f) + \beta_{NKE,s}(R_s - R_m) + \beta_{NKE,v}(R_v - R_m)$

(i) $E(R_{NKE}) =$
$1.0\% + (0.7877)(13.49\%) + (0.6701)(0.15\%) + (-0.0288)(-4.12\%) = 11.85\%$

(ii) α_{NKE} = Actual return - Expected return $= 15.88\% - 11.85\% = 4.03\%$

(iii) **Replicating portfolios' weights:**

$$\begin{aligned}
E(R_i) &= R_f + \beta_{i,m} E\left(R_m - R_f\right) + \beta_{i,s} E\left(R_s - R_m\right) + \beta_{i,v} E\left(R_v - R_m\right) \\
&= R_f\left(1 - \beta_{i,m}\right) + \left(\beta_{\mathbf{i,m}} - \beta_{\mathbf{i,s}} - \beta_{\mathbf{i,v}}\right) E(R_m) + \beta_{i,s} E(R_s) + \beta_{i,v} E(R_v) \\
&= w_{i,Rf} R_f + w_{i,m} E(R_m) + w_{i,s} E(R_s) + w_{i,v} E(R_v)
\end{aligned}$$

$w_{i,Rf} = 1 - \beta_m = 1 - 0.7877 = 0.2123$
$w_{i,m} = \beta_{i,m} - \beta_{i,s} - \beta_{i,v} = 0.7877 - (0.6701) - (-0.0288) = 0.1464$
$w_s = \beta_{i,s} = 0.6701$
$w_v = \beta_{i,v} = -0.0288$

Solution to Question 1 continued

Replicating portfolio: (a) long position :21.23% in US T-Bills,
(b) long position: 14.64% in market portfolio (or S&P500 market index fund).
(c) long position: 67.01% in Wilshire 5000 index fund, and
(d) short 2.88% in Russell 1000 value index fund.

Computing the expected return of replicating portfolio:
$R_f = 1.0\%$ $W_{Rf} = 0.2123$
$E(R_m) = 14.5\%$ $W_m = 0.1464$
$E(R_s) = 14.65\%$ $W_s = 0.6701$
$E(R_v) = 10.38\%$ $W_v = -0.0288$
$E(R_{AAPL}) =$
$0.2123^*1.0\% + 0.1464^*14.5\% + 0.6701^*14.65\% - 0.0288^*10.38\%$
$= 11.85\%$

Estimating & Testing the APT: Exxon Example

What determines the expected return of Exxon Mobil?

The APT extends the CAPM to allow for additional risk factors $X_{u,t}$ (eg unexpected macro events , unexpected changes in firm profits, etc)

$X_{u,t} = (INF, OIL)$ $H_0 : \gamma_{INF} = \gamma_{OIL} = 0,$

Do we reject?

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.004	0.006	0.721	0.472
E_MKT	0.713	0.086	8.271	0.000
INF	0.440	0.641	0.687	0.494
OIL	0.341	0.637	0.536	0.593
Test Statistic	Value	df	Probability	
F-statistic	0.6965	(2, 116)	0.5004	

1. PythonCodeAPT

Test for Autocorrelation

. H_0 : **No autocorrelation in the error term μ_t**

1. **Durbin-Watson test (DW):**
   Reject H_0 if DW is too different from 2.
2. LM test for autocorrelation (Breush-Godfrey):
   - Run OLS on the original regression
     $$Y_t = \beta_0 + \beta_1 X_{1t} + \cdots + \beta_K X_{Kt} + \mu_t \quad (2)$$
     and save residuals e_t
   - Run OLS on the auxiliary regression
     $$e_t = \gamma_0 + \gamma_1 X_{1t} + \cdots + \gamma_K X_{Kt} \quad (3)$$
     $$+\delta_1 e_{t-1} + \cdots + \delta_q e_{t-q} + error_t, \quad (4)$$
     and save $R-$squared R_a^2;
   - Reject H_0 if $(T-q)R_a^2 > \chi_q^2-$critical value.

Test for Heteroskedasticity

. H_0 : **Homoskedasticity of the error term** μ_t

1. LM test (White)
   - Suppose the original regression has only two regressors.
   - Run OLS on the original regression

$$Y_t = \beta_0 + \beta_1 X_{1t} + \beta_2 X_{2t} + \mu_t \qquad (5)$$

   and save residuals e_t

   - Run OLS on the auxiliary regression

$$\begin{aligned} e_t^2 &= \gamma_0 + \gamma_1 X_{1t} + \gamma_2 X_{2t} & (6)\\ &+\delta_1 X_{1t}^2 + \delta_2 X_{2t}^2 + \delta_3 X_{1t} X_{2t} + error_t, & (7) \end{aligned}$$

   and save $R-$squared R_a^2;

   - Reject H_0 if $TR_a^2 > \chi_m^2$ $-$critical value, where m is the number of regressors in the auxiliary regression, here $(m = 5)$

Notice the problem of m increasing with K $(m = 2K + \frac{K(K-1)}{2})$

Test for Heteroskedasticity

2. Alternative method for White LM test
   - Run OLS on the original regression

$$Y_t = \beta_0 + \beta_1 X_{1t} + \cdots + \beta_K X_{Kt} + \mu_t \tag{8}$$

and save residuals e_t, and predicted values $\hat{Y}_t$

   - Run OLS on the auxiliary regression

$$e_t^2 = \gamma_0 + \gamma_1 \hat{Y}_t + \gamma_2 \hat{Y}_t^2 + error_t, \tag{9}$$

and save $R-$squared R_a^2;

   - Reject H_0 if $TR_a^2 > \chi_2^2-$critical value.

Example:Mobil

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Breusch-Godfrey Serial Correlation LM Test:			
F-statistic	0.229380	Probability	0.795386
Obs*R^2	0.472709	Probability	0.789501

White Heteroskedasticity Test:			
F-statistic	3.587532	Probability	0.030751
Obs*R^2	6.933821	Probability	0.031213

Dependent Variable: E_MOBIL				
Method: Least Squares				
Sample: 1978M01 1987M12				
Included observations: 120				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.004241	0.005881	0.721087	0.4723
E_MARKET	0.714695	0.085615	8.347761	0.0000
R-squared	0.371287	Mean dependent var		0.009353
Adjusted R-squared	0.365959	S.D. dependent var		0.080468
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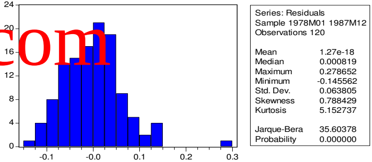

- No evidence for AC in the error term (large p-value).
- Strong evidence for heteroskedasticity (small p-value).
- Strong evidence for non-normality (small p-value).

Robust Standard Errors

- **The key assumption is** $E(\mu_t|X_t) = 0$
  (which my be weakened to $Cov(X_t, \mu_t) = 0$).
  Can we test for this 'key assumption'? How would the test look like?

- Even when there is heteroskedasticity or autocorrelation in μ_t, the OLS estimators are still consistent. However, the standard errors of the estimators are incorrect and MUST be corrected.
- In practice, we should always use **robust standard errors** that correct the effect of heteroskedasticity and/or autocorrelation:

  1. White standard errors (correct heteroskedasticity)
  2. Newey-West (HAC) standard errors (correct jointly heteroskedasticity and autocorrelation.)

Example:Mobil

OLS s.e.				
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E_MKT	0.714695	0.085615	8.347761	0.0000
White s.e.				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.004241	0.005620	0.754602	0.4520
E_MKT	0.714695	0.086243	8.287035	0.0000
Newey-West s.e.				
Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	0.004241	0.005130	0.826596	0.4101
E_MKT	0.714695	0.090799	7.871135	0.0000

Miscellaneous issues

- **Dynamics:** the lags of Y_t may be included in the RHS of the regression
  eg. Mobil" $X_{i,t} = \alpha + \beta X_{m,t} + \gamma X_{i,t-1} + \mu_{i,t}$
- **Dummy variable**
  - Stock market event:

$$Y_t = \beta_0 + \beta_1 X_t + \beta_2 D_t X_t + \mu_t, \quad (10)$$

  $D_t = 0$ pre crisis and $D_t = 1$ post crisis.
  The effect of X_t on Y_t is β_1 before the crisis but becomes $(\beta_1 + \beta_2)$ after the crisis.

  - Day-of-the-week effects: $\text{Fri}_t = \left\{ \begin{array}{ll} 0, & t \text{ is not on a Friday} \\ 1, & t \text{ is on a Friday} \end{array} \right\}$

Model Stability

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Model stability: Does Its structure changes over time?

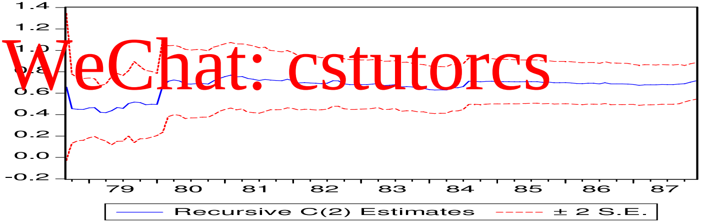
1. Recursive parameter estimates

Monitor changes in parameter estimates over time.

- Start from an initial sample of size τ, estimate the model, get $\hat{\beta}(\tau)$,
- add one observation to the sample, estimate the model, get the $\hat{\beta}(\tau+1)$,
- Continue recursively until last estimate with full sample $\hat{\beta}(T)$

- eg. Mobil: Stability of the CAPM Model $X_{i,t} = \alpha + \beta X_{m,t} + \mu_{i,t}$

Recursive estimates of the market beta β:



Model Stability

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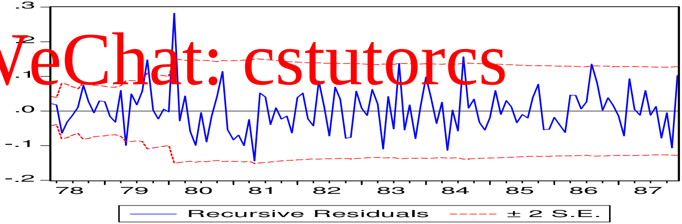
❷ Recursive residuals

- Estimate recursively the model parameters:
  $\hat{\beta}(\tau), \hat{\beta}(\tau+1), \cdots, \hat{\beta}(T)$,
- estimate recursive residuals: $e_{\tau+1|\tau} = Y_{\tau+1} - \mathbf{X}_{\tau+1}\hat{\beta}(\tau)$
  $e_{\tau+1|\tau}, e_{\tau+2|\tau+1}, \cdots, e_{T|T-1}$
  If the model is correct (stable),

$$W_{\tau+1|\tau} = \frac{e_{\tau+1|\tau}}{se(e_{\tau+1|\tau})} \sim N(0,1) \qquad (11)$$

Mobil Recursive Residuals (CAPM)



Model Stability

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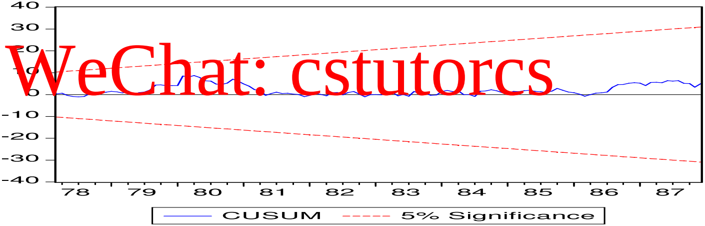
❸ **CUSUM Test** (cumulative sum of standardised recursive residuals)

$$CUSUM_t = \sum_{\tau=K+1}^{t} W_{\tau+1|\tau}, \quad t = K+1, K+2, \cdots, T-1 \tag{12}$$

Reject Stability if it goes outside the 95% bands

eg. Mobil CUSUM test:



Summary

1. Linear regression
   - What are the basic assumptions about linear regression
   - What are OLS estimators and their properties
   - What are the diagnostic statistics we have covered
   - Why we should use robust standard errors
   - What are recursive estimates of β
   - What is the CUSUM test
2. Applications in finance
   - CAPM is about the relationship of $\cdots$
   - APT is an extension of $\cdots$
   - These can be evaluated with a $\cdots$ model.