Financial Econometrics

Slides 14: Multivariate Volatility Models

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Multivariate GARCH Models

- Multivariate GARCH models are used to estimate and to forecast covariances and correlations.
- The basic formulation is similar to that of the GARCH model, but where the covariances as well as the variances are permitted to be time-varying.
- There are 3 main classes of multivariate GARCH formulation that are widely used: VECH, diagonal VECH and BEKK.

<u>VECH and Diagonal VECH</u>

- e.g. suppose that there are two variables used in the model. The conditional covariance matrix is denoted $H\ t$, and would be 2×2. H_t and VECH(H_t) are

$$H_t = \begin{bmatrix} h_{11t} & h_{12t} \\ h_{21t} & h_{22t} \end{bmatrix}, \qquad VEC(H_t) = \begin{bmatrix} h_{11t} \\ h_{22t} \\ h_{12t} \end{bmatrix}$$

VECH and Diagonal VECH

- In the case of the VECH, the conditional variances and covariances would each depend upon lagged values of all of the variances and covariances and on lags of the squares of both error terms and their cross products.

- In matrix form, it would be written

$$VECH(H_t) = C + AVECH(\Xi_{t-1}\Xi'_{t-1}) + BVECH(H_{t-1})$$
$$\Xi_t|\psi_{t-1} \sim N(0, H_t)$$

VECH and Diagonal VECH (Cont'd)

- Writing out all of the elements gives the 3 equations as

$$
\begin{aligned}
h_{11t} &= c_{11} + a_{11} u_{1t-1}^2 + a_{12} u_{2t-1}^2 + a_{13} u_{1t-1} u_{2t-1} + b_{11} h_{11t-1} \\
&\quad + b_{12} h_{22t-1} + b_{13} h_{12t-1} \\
h_{22t} &= c_{21} + a_{21} u_{1t-1}^2 + a_{22} u_{2t-1}^2 + a_{23} u_{1t-1} u_{2t-1} + b_{21} h_{11t-1} \\
&\quad + b_{22} h_{22t-1} + b_{23} h_{12t-1} \\
h_{12t} &= c_{31} + a_{31} u_{1t-1}^2 + a_{32} u_{2t-1}^2 + a_{33} u_{1t-1} u_{2t-1} + b_{31} h_{11t-1} \\
&\quad + b_{32} h_{22t-1} + b_{33} h_{12t-1}
\end{aligned}
$$

VECH and Diagonal VECH (Cont'd)

- Such a model would be hard to estimate. The diagonal VECH is much simpler and is specified, in the 2 variable case, as follows:

$$
\begin{aligned}
h_{11t} &= \alpha_0 + \alpha_1 u_{1t-1}^2 + \alpha_2 h_{11t-1} \\
h_{22t} &= \beta_0 + \beta_1 u_{2t-1}^2 + \beta_2 h_{22t-1} \\
h_{12t} &= \gamma_0 + \gamma_1 u_{1t-1} u_{2t-1} + \gamma_2 h_{12t-1}
\end{aligned}
$$

BEKK and Model Estimation for M-GARCH

- Neither the VECH nor the diagonal VECH ensure a positive definite variance-covariance matrix.

- An alternative approach is the BEKK model (Engle & Kroner, 1995).
- The BEKK Model uses a Quadratic form for the parameter matrices to ensure a positive definite variance / covariance matrix $H\ t$.

- In matrix form, the BEKK model is

$$H_t = W'W + A'H_{t-1}A + B'\Xi_{t-1}\Xi'_{t-1}B$$

BEKK and Model Estimation for M-GARCH (Cont'd)

- Model estimation for all classes of multivariate GARCH model is again performed using maximum likelihood with the following *LLF*:

$$\ell(\theta) = -\frac{TN}{2}\log 2\pi - \frac{1}{2}\sum_{t=1}^{T}\left(\log|H_t| + \Xi_t' H_t^{-1}\Xi_t\right)$$

where N is the number of variables in the system (assumed 2 above), θ is a vector containing all of the parameters, and T is the number of obs.

Correlation Models and the CCC

- The correlations between a pair of series at each point in time can be constructed by dividing the conditional covariances by the product of the conditional standard deviations from a VECH or BEKK model
- A subtly different approach would be to model the dynamics for the correlations directly
- In the constant conditional correlation (CCC) model, the correlations between the disturbances to be fixed through time
- Thus, although the conditional covariances are not fixed, they are tied to the variances
- The conditional variances in the fixed correlation model are identical to those of a set of univariate GARCH specifications (although they are estimated jointly):

$$h_{ii,t} = c_i + a_i \epsilon_{i,t-i}^2 + b_i h_{ii,t-1}, \qquad i = 1, \ldots, N$$

More on the CCC

- The off-diagonal elements of H_t, $h_{ij,t} (i \neq j)$, are defined indirectly via the correlations, denoted ρ_{ij}:

$$h_{ij,t} = \rho_{ij} h_{ii,t}^{1/2} h_{jj,t}^{1/2}, \qquad i, j = 1, \ldots, N, i < j$$

- Is it empirically plausible to assume that the correlations are constant through time?

- Several tests of this assumption have been developed, including a test based on the information matrix due and a Lagrange Multiplier test

- There is evidence against constant correlations, particularly in the context of stock returns.

The Dynamic Conditional Correlation Model

- Several different formulations of the dynamic conditional correlation (DCC) model are available, but a popular specification is due to Engle (2002)
- The model is related to the CCC formulation but where the correlations are allowed to vary over time.
- Define the variance-covariance matrix, H_t, as $H_t = D_t R_t D_t$
- D_t is a diagonal matrix containing the conditional standard deviations (i.e. the square roots of the conditional variances from univariate GARCH model estimations on each of the N individual series) on the leading diagonal
- R_t is the conditional correlation matrix
- Numerous parameterisations of R_t are possible, including an exponential smoothing approach

The DCC Model – A Possible Specification

- A possible specification is of the MGARCH form:

$$H_t = S \circ (\iota\iota' - A - B) + A \circ u_{t-1}u'_{t-1} + B \circ H_{t-1}$$

where:

- S is the unconditional correlation matrix of the vector of standardised residuals (from the first stage estimation), $u_t = D_t^{-1}\Xi_t$.

- ι is a vector of ones
- H_t is an $N \times N$ symmetric positive definite variance-covariance matrix.

- $\circ$ denotes the *Hadamard* or element-by-element matrix multiplication procedure.
- This specification for the intercept term simplifies estimation and reduces the number of parameters.

The DCC Model – A Possible Specification

- Engle (2002) proposes a GARCH-esque formulation for dynamically modelling H_t with the conditional correlation matrix, R_t then constructed as

$$R_t = diag\{Q_t^*\}^{-1} H_t diag\{Q_t^*\}^{-1}$$

where $diag(\cdot)$ denotes a matrix comprising the main diagonal elements of $(\cdot)$ and Q^* is a matrix that takes the square roots of each element in H.

- This operation is effectively taking the covariances in H_t and dividing them by the product of the appropriate standard deviations in Q_t^* to create a matrix of correlations.

DCC Model Estimation

- The model may be estimated in a single stage using ML although this will be difficult. So Engle advocates a two-stage procedure where each variable in the system is first modelled separately as a univariate GARCH

- A joint log-likelihood function for this stage could be constructed, which would simply be the sum (over N) of all of the log-likelihoods for the individual GARCH models

- In the second stage, the conditional likelihood is maximised with respect to any unknown parameters in the correlation matrix

DCC Model Estimation (Cont'd)

- The log-likelihood function for the second stage estimation will be of the form

$$\ell(\theta_2|\theta_1) = \sum_{t=1}^{T} \left(\log|R_t| + u_t' R_t^{-1} u_t \right)$$

- where θ_1 and θ_2 denote the parameters to be estimated in the 1^{st} and 2^{nd} stages respectively.

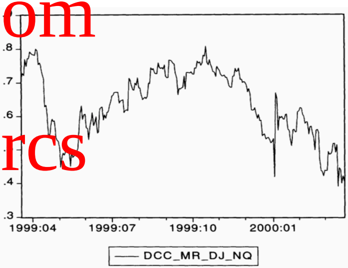
DCC Example

eg. Engle (2002): Dow Jones (tradition stocks) and NASDAQ (technology stocks) returns

$$\hat{a} = 0.039\,, \quad \hat{b} = 0.942.$$
$$[0.006] \qquad\qquad [0.010]$$

The conditional correlation varies a great deal.

Useful implications for portfolio management.



Asymmetric Multivariate GARCH

- Asymmetric models have become very popular in empirical applications, where the conditional variances and / or covariances are permitted to react differently to positive and negative innovations of the same magnitude

- In the multivariate context, this is usually achieved in the Glosten et al. (1993) framework

- Kroner and Ng (1998), for example, suggest the following extension to the BEKK formulation (with obvious related modifications for the VECH or diagonal VECH models)

$$H_t = W'W + A'H_{t-1}A + B'\Xi_{t-1}\Xi'_{t-1}B + D'z_{t-1}z'_{t-1}D$$

where z_{t-1} is an N-dimensional column vector with elements taking the value $-\epsilon_{t-1}$ if the corresponding element of ϵ_{t-1} is negative and zero otherwise.

An Example: Estimating a Time-Varying Hedge Ratio for FTSE Stock Index Returns (Brooks, Henry and Persand, 2002).

- Data comprises 3580 daily observations on the FTSE 100 stock index and stock index futures contract spanning the period 1 January 1985–9 April 1999.
- Several competing models for determining the optimal hedge ratio (OHR) are constructed. Define the hedge ratio as β.
  - No hedge (β=0)
  - Naïve hedge (β=1)
  - Multivariate GARCH hedges:
    - Symmetric BEKK
    - Asymmetric BEKK

      In both cases, estimating the OHR involves forming a 1-step ahead forecast and computing

$$OHR_{t+1} = \frac{h_{FS,t+1}}{h_{F,t+1}} | \Omega_t$$

OHR Results

In-sample				
	Unhedged	Naive hedge	Symmetric time-varying hedge	Asymmetric time-varying hedge
	$\beta = 0$	$\beta = -1$	$\beta_t = \frac{h_{FS,t}}{h_{F,t}}$	$\beta_t = \frac{h_{FS,t}}{h_{F,t}}$
(1)	(2)	(3)	(4)	(5)
Return	0.0389	−0.0003	0.0061	0.0060
	{2.3713}	{−0.0351}	{0.9562}	{0.9580}
Variance	0.8286	0.1718	0.1240	0.1211
Out-of-sample				
	Unhedged	Naive hedge	Symmetric time-varying hedge	Asymmetric time-varying hedge
	$\beta = 0$	$\beta = -1$	$\beta_t = \frac{h_{FS,t}}{h_{F,t}}$	$\beta_t = \frac{h_{FS,t}}{h_{F,t}}$
Return	0.0819	−0.0004	0.0120	0.0140
	{1.4958}	{0.0216}	{0.7761}	{0.9083}
Variance	1.4972	0.1696	0.1186	0.1188

Plot of the OHR from Multivariate GARCH

- OHR is time-varying and less than 1
- M-GARCH OHR provides a better hedge, both in-sample and out-of-sample.
- No role in calculating OHR for asymmetries