1. (10 marks) Let I be an interval, let $f : I \to \mathbb{R}$ be differentiable and let f' be bounded on I. Prove that f is Lipschitz continuous on I i.e. prove that there exists a $K > 0$ such that for all $x, u \in I$ it holds that $|f(x) - f(u)| \leq K|x - u|$.

Solution:

Since f' is bounded on I there exists a $K > 0$ such that $|f'(x)| \leq K$ for all $x \in I$. We will prove that for this K it holds that $|f(x) - f(u)| \leq K|x - u|$ for all $x, u \in I$. If $x = u$ this is obvious. Now assume that $x \neq u$; due to symmetry we may assume w.l.o.g. that $x < u$. We will now apply the mean value theorem to the function f on the interval $[x, u]$; the conditions of this theorem are satisfied since f is differentiable (and thus also especially continuous) on all of $I \supseteq [x, u]$. Thus there exists a $c \in]x, u[$ (and thus in I) such that

$$f'(c) = \frac{f(x) - f(u)}{x - u}$$

Then $f(x) - f(u) = f'(c)(x - u)$ and thus $|f(x) - f(u)| = |f'(c)| \cdot |x - u| \leq K|x - u|$.

In any case we thus have $|f(x) - f(u)| \leq K|x - u|$ which is what we had to prove.

2. (10 marks) Let $f : [-1,1] \to \mathbb{R}$, $f(x) := \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Let $P := \{x_0 = -1, x_1, \dots, x_{n-1}, x_n = 1\}$ be an arbitrary partition of $[-1,1]$, and let $1 \leq k \leq n$ be such that $0 \in]x_{k-1}, x_k]$. Prove that $U(f,P) = 2$, and that

$$L(f,P) = \begin{cases} 2 + x_{k-1} - x_k & \text{if } 0 \in]x_{k-1}, x_k[\\ 2 + x_{k-1} - x_{k+1} & \text{if } 0 = x_k \end{cases}$$

(b) Prove that if the mesh of P is less than δ, then $2 - 2\delta < L(f,P)$.

(c) Prove directly from the definition of Riemann integrability and parts (a) and (b) that f is Riemann integrable on $[-1,1]$ and determine $\int_{-1}^{1} f$. No other method will be accepted!

Solution:

(a) We have that $0 \leq f(x) \leq 1$ for all $x \in [-1,1]$ and that f is different from 1 at at most one point on any of the partition intervals. Thus for all $1 \leq i \leq n$ we have

$$M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\} = 1$$

and

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1}) \overset{\text{Telesc. sum}}{=} x_n - x_0 = 1 - (-1) = 2$$

Now we'll determine $L(f,P)$.

<u>1. Case</u>: $x_k \neq 0$ i.e. $x_k \in]x_{k-1}, x_k[$

In this case, the only partition interval containing 0 is $[x_{k-1}, x_k]$. Hence

$$m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\} = \begin{cases} 1 & \text{if } i \neq k \\ 0 & \text{if } i = k \end{cases}$$

Thus

$$\begin{aligned} L(f,P) &= \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{k-1} (x_i - x_{i-1}) + \sum_{i=k+1}^{n} (x_i - x_{i-1}) \\ &\overset{\text{Telesc. sum}}{=} (x_{k-1} - x_0) + (x_n - x_k) = (x_{k-1} + 1) + (1 - x_k) = 2 + x_{k-1} - x_k \end{aligned}$$

<u>2. Case</u>: $x_k = 0$

In this case, 0 is contained in exactly two partition intervals: $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$. Hence

$$m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\} = \begin{cases} 1 & \text{if } i \neq k, k+1 \\ 0 & \text{if } i = k \text{ or } i = k+1 \end{cases}$$

Thus

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{k-1}(x_i - x_{i-1}) + \sum_{i=k+2}^{n}(x_i - x_{i-1})$$
$$\overset{\text{Telesc. sum}}{=} (x_{k-1} - x_0) + (x_n - x_{k+1}) = (x_{k-1} + 1) + (1 - x_{k+1}) = 2 + x_{k-1} - x_{k+1}$$

This proves the formula for $L(f,P)$ in all cases.

(b) Let the mesh of P be less than δ. Then

$$L(f,P) = \begin{cases} 2 + x_{k-1} - x_k = 2 - (x_k - x_{k-1}) > 2 - \delta > 2 - 2\delta \\ 2 + x_{k-1} - x_{k+1} = 2 - (x_{k+1} - x_k) - (x_k - x_{k-1}) > 2 - \delta - \delta = 2 - 2\delta \end{cases}$$

We thus have $2 - \delta < L(f,P)$ in all cases.

(c) It follows from parts (a) and (b) that for any partition P of $[-1,1]$ of mesh less than δ it holds that

$$2 - 2\delta < L(f,P) \leq L(f) \leq U(f) \leq U(f,P) = 2$$

Consequently, $2 - 2\delta < L(f) \leq U(f) \leq 2$ for all $\delta > 0$. Hence $2 = L(f) = U(f)$, which means that f is Riemann integrable on $[-1,1]$, and that $\int_{-1}^{1} f = 2$.

3. (10 marks) Let $S := \{\frac{1}{n} : n \in \mathbb{N}\}$. Consider the function $f : [0,1] \to \mathbb{R}$,

$$f(x) := \begin{cases} \sin(1/x) & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

Use the squeeze theorem to prove that f is Riemann integrable on $[0,1]$ and compute $\int_0^1 f$.

Solution:

1. Solution

Let $\varepsilon > 0$. Note that $-1 \leq f(x) \leq 1$ for all $x \in [0,1]$; this motivates the following definitions for $\alpha, \omega : [0,1] \to \mathbb{R}$:

$$\alpha(x) := \begin{cases} -1 & \text{if } x \in [0, \frac{\varepsilon}{4}] \\ f(x) & \text{if } x \in]\frac{\varepsilon}{4}, 1] \end{cases}$$

and

$$\omega(x) := \begin{cases} 1 & \text{if } x \in [0, \frac{\varepsilon}{4}] \\ f(x) & \text{if } x \in]\frac{\varepsilon}{4}, 1] \end{cases}$$

Then α and ω are constant on $[0, \frac{\varepsilon}{4}]$, and are thus Riemann integrable on this interval.

Now note that $]\frac{\varepsilon}{4}, 1] \cap S$ is finite since $\frac{1}{n} > \frac{\varepsilon}{4} \Leftrightarrow n < \frac{4}{\varepsilon}$ which is only satisfied by finitely many $n \in \mathbb{N}$. This means that α and ω are constantly zero on $[\frac{\varepsilon}{4}, 1]$ except at finitely many points. It now follows from assignment 4 that both α and ω are Riemann integrable on $[\frac{\varepsilon}{4}, 1]$. Finally, it follows from additivity that α and ω are Riemann integrable on $[0,1]$ and that

$$\int_0^1 \alpha = \int_0^{\frac{\varepsilon}{4}} \alpha = (-1)\frac{\varepsilon}{4} = -\frac{\varepsilon}{4}$$
$$\int_0^1 \omega = \int_0^{\frac{\varepsilon}{4}} \omega = 1 \cdot \frac{\varepsilon}{4} = \frac{\varepsilon}{4}$$

Hence

$$\int_0^1 (\omega - \alpha) = \frac{\varepsilon}{2} < \varepsilon$$

α and ω thus satisfy all conditions of the squeeze theorem and we conclude that f is Riemann integrable on $[0,1]$. Finally, it follows from $\alpha(x) \leq f(x) \leq \omega(x)$ for all $x \in [0,1]$ that

$$\int_0^1 \alpha = -\frac{\varepsilon}{4} \leq \int_0^1 f \leq \int_0^1 \omega = \frac{\varepsilon}{4}$$

i.e. that $-\frac{\varepsilon}{4} \leq \int_0^1 f \leq \frac{\varepsilon}{4}$ for all $\varepsilon > 0$. But this means that $\int_0^1 f = 0$.

2. Solution

Let $\varepsilon > 0$. Consider the following functions $\alpha, \omega : [0,1] \to \mathbb{R}$:

$$\alpha(x) := \begin{cases} -1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

and

$$\omega(x) := \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

Then $\alpha(x) \leq f(x) \leq \omega(x)$ for all $x \in [0,1]$ since the sine function only attains values between -1 and 1. Note that ω is Riemann integrable on $[0,1]$ with $\int_0^1 \omega = 0$ as shown on assignment 5. Note furthermore that $\alpha = -\omega$ which implies that α is Riemann integrable on $[0,1]$ as well. Lastly,

$$\int_0^1 (\omega - \alpha) = \int_0^1 2\omega = 2\int_0^1 \omega = 0 < \varepsilon$$

All conditions of the squeeze theorem are thus satisfied, and we conclude that f is Riemann integrable on $[0,1]$. Finally, it follows from $\alpha(x) \leq f(x) \leq \omega(x)$ on $[0,1]$ that

$$0 = \int_0^1 \alpha \leq \int_0^1 f \leq \int_0^1 \omega = 0$$

This implies that $\int_0^1 f = 0$.

4. (10 marks) Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) := \dfrac{1}{1+nx^2}$ for all $n \in \mathbb{N}$.

   (a) Prove that (f_n) converges pointwise on $\mathbb{R}$ and find the limit function f.

   (b) Prove that (f_n) converges uniformly on $[a, \infty[$ for any $a > 0$.

   (c) Prove that (f_n) does *not* converge uniformly on $[0, \infty[$.

   (d) Does (f_n) converge uniformly on $]0, \infty[$? Justify!

Solution:

(a) Let $x \neq 0$. Then

$$\lim\left(f_n(x)\right) = \lim\left(\frac{1}{1+nx^2}\right) = \lim\left(\frac{\frac{1}{n}}{\frac{1}{n}+x^2}\right) = \frac{\lim\left(\frac{1}{n}\right)}{\lim\left(\frac{1}{n}+x^2\right)} = \frac{0}{x^2} = 0$$

for all $x \neq 0$. For $x = 0$ we get: $f_n(0) = 1$ for all $n \in \mathbb{N}$. Thus $\lim\left(f_n(0)\right) = 1$. Summarizing: $f_n \to f$ on $\mathbb{R}$ where $f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$

(b)

$$|f_n(x) - f(x)| = \left|\frac{1}{1+nx^2} - 0\right| = \frac{1}{1+nx^2} < \frac{1}{nx^2} \leq \frac{1}{na^2}$$

for all $x \in [a, \infty[$ and for all $n \in \mathbb{N}$.

Now let $\varepsilon > 0$ and let $N > \frac{1}{a^2\varepsilon}$. Then $|f_n(x) - f(x)| \leq \dfrac{1}{na^2} \leq \dfrac{1}{Na^2} < \varepsilon$ for all $x \in [a, \infty[$ and all $n \in \mathbb{N}$. Hence $f_n \rightrightarrows f$ on $[a, \infty[$.

(c) <u>1. Solution</u>: f_n is continuous on $\mathbb{R}$ and thus on $[0, \infty[$ for all $n \in \mathbb{N}$. However, f is discontinuous at $x = 0$. Since uniform convergence preserves continuity it follows immediately that $f_n \not\rightrightarrows f$ on $[0, \infty[$.

<u>2. Solution</u>: We use the sequential criterion for non-uniform convergence. Let $x_n := \frac{1}{\sqrt{n}}$ and let $\varepsilon := \frac{1}{2}$. Then

$$|f_n(x_n) - f(x_n)| = \left|\frac{1}{1+n\cdot\frac{1}{n}} - 0\right| = \frac{1}{2} \geq \varepsilon$$

for all $n \in \mathbb{N}$. Thus $f_n \not\rightrightarrows f$ on $[0, \infty[$.

(d) Since $x_n \in]0, \infty[$ for all $n \in \mathbb{N}$, it follows from the 2. solution to part (c) above that $f_n \not\rightrightarrows f$ on $]0, \infty[$.