MULT20015 Elements of Quantum Computing

Assignment 1

Due: 5pm, 20th August 2021

Instructions: Work on your own, attempt all questions. Submit your completed written work electronically as a pdf (no other formats accepted) to LMS, with name and student number on the front, on or before the due date. Please show all working. Instructions on LMS submission to follow.

Total marks = 20. Number of questions = 3.

1. [1 + [0.5+0.5+0.5+2.0] = 4.5 marks]

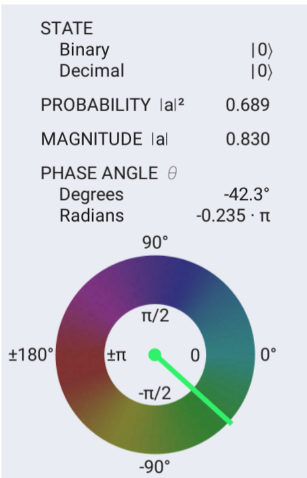
(a) Consider the following single-qubit operators in matrix form, corresponding to rotations by angle θ_{R} about X, Y or Z axes:

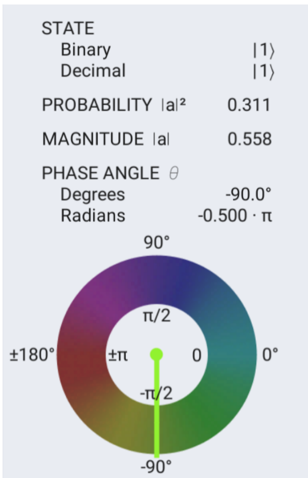
$$R_X(\theta_{\mathrm{R}}) = e^{i\pi/2}\begin{bmatrix} \cos\frac{\theta_{\mathrm{R}}}{2} & -i\sin\frac{\theta_{\mathrm{R}}}{2} \\ -i\sin\frac{\theta_{\mathrm{R}}}{2} & \cos\frac{\theta_{\mathrm{R}}}{2} \end{bmatrix}, \quad R_Y(\theta_{\mathrm{R}}) = e^{i\pi/2}\begin{bmatrix} \cos\frac{\theta_{\mathrm{R}}}{2} & -\sin\frac{\theta_{\mathrm{R}}}{2} \\ \sin\frac{\theta_{\mathrm{R}}}{2} & \cos\frac{\theta_{\mathrm{R}}}{2} \end{bmatrix},$$

$$R_Z(\theta_{\mathrm{R}}) = e^{i\pi/2}\begin{bmatrix} \cos\frac{\theta_{\mathrm{R}}}{2} - i\sin\frac{\theta_{\mathrm{R}}}{2} & 0 \\ 0 & \cos\frac{\theta_{\mathrm{R}}}{2} + i\sin\frac{\theta_{\mathrm{R}}}{2} \end{bmatrix}.$$

Explain how these operators are related to the familiar Pauli matrices, X, Y and Z given in lectures. (NB. For future reference, these expressions explicitly include a global phase of $\pi/2$)

(b) A particular single qubit state $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$, is obtained by rotating about the x+z axis from an initial state $|0\rangle$ (zero global phase). The amplitudes are given by QUI as:





(i) Plot the amplitudes a_0 and a_1 in the complex plane.

(ii) Convert the state to QUI polar notation form: $|\psi\rangle = |a_0|e^{i\theta_0}|0\rangle + |a_1|e^{i\theta_1}|1\rangle$ with the basis state phase angles θ_0 and θ_1 in radians (expressed as multiples of π), and degrees.

(iii) Now convert to "Bloch sphere" form $|\psi\rangle = e^{i\theta_{\text{global}}}\left(\cos\frac{\theta_{\text{B}}}{2}|0\rangle + \sin\frac{\theta_{\text{B}}}{2}e^{i\phi_B}|1\rangle\right)$, specifying the global phase θ_{global} and Bloch angles θ_{B} and ϕ_B. Hence, plot on the Bloch sphere.

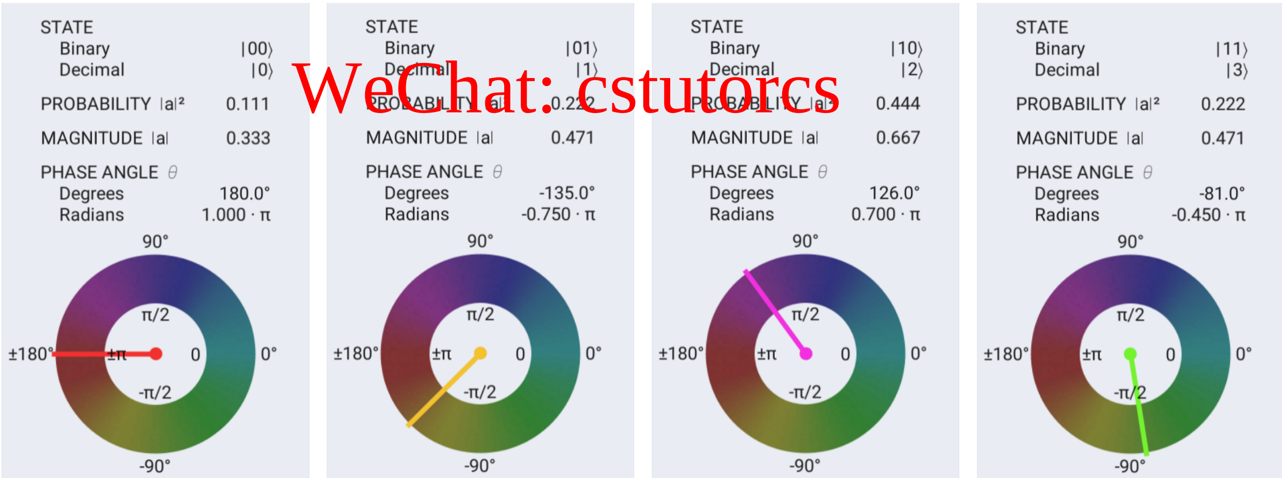
(iv) In matrix form rotation by θ_{R} about the x+z axis, i.e. $R_{x+z}(\theta_{\text{R}})$, is given by the 2x2 matrix (NB. zero global phase):

$$R_{x+z}(\theta_{\text{R}}) = \begin{bmatrix} \cos\left({}^{\theta_{\text{R}}}/_{2}\right) - \frac{i}{\sqrt{2}}\sin\left({}^{\theta_{\text{R}}}/_{2}\right) & -\frac{i}{\sqrt{2}}\sin\left({}^{\theta_{\text{R}}}/_{2}\right) \\ -\frac{i}{\sqrt{2}}\sin\left({}^{\theta_{\text{R}}}/_{2}\right) & \cos\left({}^{\theta_{\text{R}}}/_{2}\right) + \frac{i}{\sqrt{2}}\sin\left({}^{\theta_{\text{R}}}/_{2}\right) \end{bmatrix}$$

Find the angle θ_{R} that produced the state $|\psi\rangle$ from the initial state $|0\rangle$, in both radians (multiples of π) and degrees. Verify using QUI (submit appropriate screenshots with your answers). Explain quantitatively (using diagrams) how the operation moves the state across the Bloch Sphere.

2. [0.5 + 1.5 = 2 marks]

(a) For the QUI state information cards shown below, write out the two-qubit state in both ket and matrix forms.



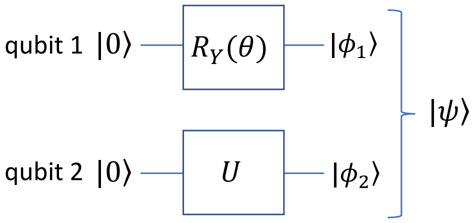
(b) Consider the following two-qubit state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}e^{-i\pi/2}|00\rangle + \frac{1}{2}e^{i\pi/4}|01\rangle + \frac{1}{\sqrt{12}}e^{-i\pi/2}|10\rangle + Qe^{i\pi/4}|11\rangle$$

where Q is a real positive parameter. Determine Q for which the state is correctly normalised. Consider a measurement carried out on the first qubit. Calculate the probability of obtaining a "1" result. Write the resulting state after a measurement on the first qubit produces "1".

3. [0.5 + 1 + 0.5 + 0.5 + 3.0 + 3.0 + 3.0 + 2.0 = 13.5 marks]

Consider the following circuit over two qubits producing the state $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$:



where the rotation around the Y-axis (with zero global phase this time) is given as (dropping the rotation "R" subscript for brevity):

$$R_Y(\theta) = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix},$$

and U is a single qubit operation to be specified. In matrix notation, we are given U parametrised by $(\alpha, \beta \in \mathbb{C})$:

$$U = \begin{bmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{bmatrix}.$$

(a) Find a condition on (α, β) which ensures U is unitary.

(b) For general angle θ and parameters (α, β) which specify the single qubit operations, find the individual qubit states $|\phi_1\rangle$ and $|\phi_2\rangle$ in matrix form, and write them out in ket form. Investigate the normalisation of $|\phi_1\rangle$ and $|\phi_2\rangle$. What can you say about the normalisation of $|\phi_2\rangle$ and the condition of unitarity of U you obtained in (a)?

(c) Now write out the combined state $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$ in ket form in expanded shorthand notation with basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Is the state $|\psi\rangle$ entangled? Explain your answer.

(d) A CNOT gate is introduced which takes $|\psi\rangle \rightarrow |\psi'\rangle$, with qubit 1 as the control. Write out the new state $|\psi'\rangle$ in ket form, keeping the angle θ and parameters (α, β) general.

(e) Find the angle θ and parameters (α, β) which produce the following state:

$$|\psi'\rangle = \frac{\sqrt{3}}{4}|00\rangle + \frac{1}{4}e^{-i\pi/2}|01\rangle + \frac{\sqrt{3}}{4}e^{-i\pi/2}|10\rangle + \frac{3}{4}|11\rangle$$

(f) For the parameters determined in part (e) write out the two-qubit operator $R_Y(\theta) \otimes U$ as a 4x4 matrix. Using the 4x4 matrix representation of the CNOT gate given in lectures, verify by explicit calculation that you obtain the state given in (e).

(g) Examine the $R_Y(\theta)$ and U operations so derived and compare with the cartesian operations in 1(a), bearing in mind those expressions included a factor of $e^{i\pi/2}$ corresponding to a global phase of $\pi/2$ (not important to know exactly why, only that you need to take into account when programming the QUI). Program the circuit in the QUI and verify that you have obtained the required state. Supply appropriate screen shots.

(h) Here you will calculate the amount of entanglement in the state $|\psi'\rangle$, which is reported in the QUI by hovering over the time slider between qubit lines. The amount of entanglement between two sets of qubits is quantified by the *entanglement entropy*, a value measured in number of bits. To calculate the entanglement entropy between two qubits in an arbitrary normalized state $a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle$, construct the matrix

$$A = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix},$$

and calculate the eigenvalues λ_1 and λ_2 of the matrix product $AA^\dagger$. The entanglement entropy between the qubits (in number of bits) is given by:

$$H = -\lambda_1 \log_2(\lambda_1) - \lambda_2 \log_2(\lambda_2).$$

For the state in question, $|\psi'\rangle = \frac{\sqrt{3}}{4}|00\rangle + \frac{1}{4}e^{-i\pi/2}|01\rangle + \frac{\sqrt{3}}{4}e^{-i\pi/2}|10\rangle + \frac{3}{4}|11\rangle$, find the eigenvalues λ_1 and λ_2 and calculate the entanglement entropy between the two qubits in number of bits. Confirm that this value matches that reported by the QUI. Supply appropriate screen shots.