QCC 2022 HW3 Solutions

1. A circuit for the Quantum Discrete Fourier Transform (QDFT) with $N = 4$ ($n = 2$ qubits) is shown in Fig. 1. Let $|v_1\rangle = |10\rangle$ and $|v_2\rangle = |11\rangle$ be used at the input of this circuit and states $|w_1\rangle$ and $|w_2\rangle$ be the states at the output of the circuit.

Find $|w_1\rangle$ and $|w_2\rangle$.

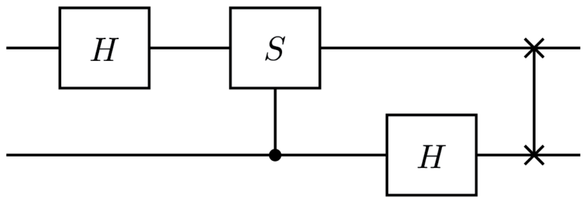

Figure 1: Quantum DFT, $n = 2$

The evolution (the state after each of the gates) of $|v_1\rangle$ is:

$$
\begin{aligned}
H &: \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \\
S &: \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \\
H &: \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle(|0\rangle + |1\rangle) - |1\rangle(|0\rangle + |1\rangle)) = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)
\end{aligned}
$$

Swapping: $|w_1\rangle = \frac{1}{2}(|00\rangle + |10\rangle - |01\rangle - |11\rangle) = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$

For $|v_2\rangle$ we have:

$$
\begin{aligned}
H &: \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)|1\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \\
S &: \frac{1}{\sqrt{2}}(|01\rangle - i|11\rangle) \\
H &: \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(|0\rangle(|0\rangle - |1\rangle) - i|1\rangle(|0\rangle - |1\rangle)) = \frac{1}{2}(|00\rangle - |01\rangle - i|10\rangle + i|11\rangle)
\end{aligned}
$$

Swapping: $|w_2\rangle = \frac{1}{2}(|00\rangle - |10\rangle - i|01\rangle + i|11\rangle) = \frac{1}{2}(|00\rangle - i|01\rangle - |10\rangle + i|11\rangle)$

2. A circuit for the inverse QDFT is shown in Fig. 2. Let the input states be $w_1\rangle$, $|w_2\rangle$ obtained in the previous problem. Let $|u_1\rangle$ and $|u_2\rangle$ be the output states.

Find $|u_1\rangle$ and $|u_2\rangle$.

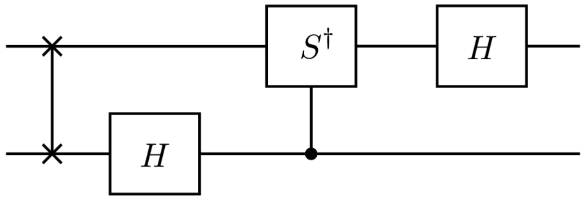

Figure 2: Inverse Quantum DFT, $n = 2$

Reminder: $S^\dagger$ denotes the Hermitian conjugation of S.

The evolution (the state after each of the gates) of $|w_1\rangle$ is:

Swapping: $\frac{1}{2}(|00\rangle - |10\rangle + |01\rangle - |11\rangle)$

$$
\begin{aligned}
H &: \frac{1}{2\sqrt{2}}[|0\rangle(|0\rangle + |1\rangle) - |1\rangle(|0\rangle + |1\rangle) + |0\rangle(|0\rangle - |1\rangle) - |1\rangle(|0\rangle - |1\rangle)) \\
&= \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle \\
S^\dagger &: \frac{1}{\sqrt{2}}(|00\rangle - |10\rangle) \\
H &: \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}[(|0\rangle + |1\rangle)|0\rangle - (|0\rangle - |1\rangle)|0\rangle] = |10\rangle = |v_1\rangle.
\end{aligned}
$$

The evolution (the state after each of the gates) of $|w_2\rangle$ is:

Swapping: $\frac{1}{2}(|00\rangle - i|10\rangle - |01\rangle + i|11\rangle)$

$$
\begin{aligned}
H &: \frac{1}{2\sqrt{2}}[|0\rangle(|0\rangle + |1\rangle) - i|1\rangle(|0\rangle + |1\rangle) - |0\rangle(|0\rangle - |1\rangle) + i|1\rangle(|0\rangle - |1\rangle)) \\
&= \frac{1}{\sqrt{2}}(|01\rangle - i|11\rangle \\
S^\dagger &: \frac{1}{\sqrt{2}}(|01\rangle - i(-i)|11\rangle) = \frac{1}{\sqrt{2}}(|01\rangle - |11\rangle) \\
H &: \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}[(|0\rangle + |1\rangle)|1\rangle - (|0\rangle - |1\rangle)|1\rangle] = |11\rangle = |v_2\rangle.
\end{aligned}
$$