UNIVERSITY OF CANTERBURY
Te Whare Wānanga o Waitaha
CHRISTCHURCH NEW ZEALAND

**No electronic/communication devices are permitted.**

**Students may take exam question paper away after the exam.**

## Mathematics and Statistics

## EXAMINATION

End-of-year Examinations, 2019

## STAT317 / ECON 323 - 19S2 (C) Time Series Methods

## STAT456 / ECON614 - 19S2 (C) Time Series and Stochastic Processes



**Examination Duration:** 120 minutes

**Exam Conditions:**

Restricted Book exam: Approved materials only.

Calculators with a 'UC' sticker approved.

**Materials Permitted in the Exam Venue:**

Restricted Book exam materials.

One A4, double sided, handwritten page of notes.

**Materials to be Supplied to Students:**

1 x Standard 16-page UC answer book

**Instructions to Students:**

Use black or blue ink only (not pencil).

Students in STAT456 and ECON614 have to work on ALL SIX questions.

Students in STAT317 and ECON323 have to CHOOSE FIVE out of SIX questions.

Show ALL working.

If you use additional paper this must be tied within the exam booklet
and remember to write your name and student number on it.

**Questions Start on Page 3**

**Note**

Students in **STAT456** and **ECON614** have to work on ALL 6 questions.

Students in **STAT317** and **ECON323** have to CHOOSE 5 out of 6 questions. Only 5 questions will be marked.

Each question is worth 16 marks.

## Q.1 Basic Concepts [16 marks]

(a) Give the definition of weak white noise. [3 marks]

(b) Assume we apply the Ljung-Box test to a time series $X_t$, $t = 1, 2, \ldots, n$ and get the $p$-value of $p = 0.043$. How would you interpret this result? [3 marks]

(c) In a time series decomposition a time series is usually separated into three components. Name these three components. [3 marks]

(d) What is a trend in a time series? [2 marks]

(e) Give the definition of the autocovariance function of an arbitrary time series $X_t$, when $X_t$ is not necessarily stationary. [1 mark]

(f) Define an estimator of the autocorrelation function given a sample $x_1, x_2, \ldots, x_n$ when $X_t$ is stationary. If your definition involves other estimators then also define these. It must be clear in the end how the estimator of the autocorrelation function is computed from the sample. [4 marks]

## Q.2 Random Walk and Stationarity [16 marks]

(a) Random Walk

i. Give the definition for a random walk $X_t$, $t = 1, 2, 3, \ldots, n$ with drift $\delta$, volatility $\sigma$, and initial value $X_0 = 0$. [1 mark]

ii. Give an example for an application of the random walk model. [1 mark]

iii. Show that the estimator for the drift

$$\widehat{\delta} = \frac{1}{n-1}(X_n - X_1)$$

is unbiased. [4 marks]

iv. Give the formula for an unbiased estimator of the squared volatility $\sigma^2$. [1 mark]

(b) Stationarity

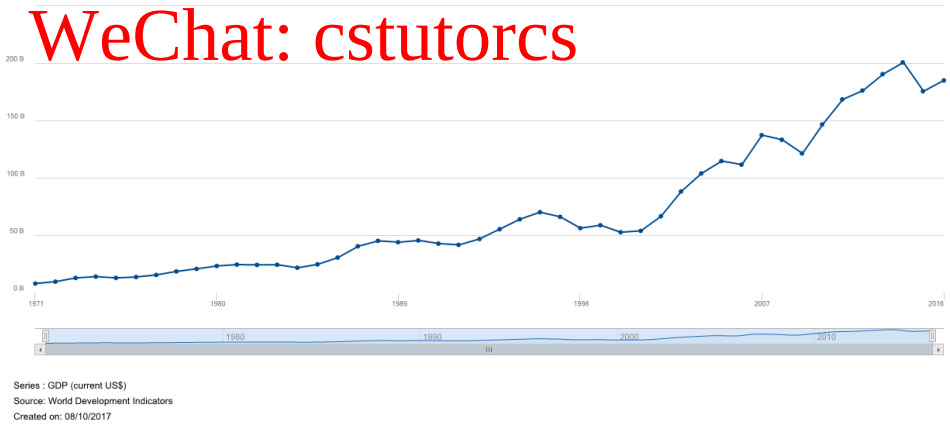
i. What are the conditions for a time series $X_t$, $t = 1, 2, 3, \ldots, n$ to be weakly stationary? [3 marks]

ii. Is a random walk stationary? Give a reason for your answer. [2 marks]

iii. Is the AR(1) process $X_t = 2 + 0.5X_{t-1}$ stationary? Give a reason for your answer. [2 marks]

iv. Would you assume that the GDP of New Zealand shown in the plot below is a stationary time series? Give a reason for your answer. [2 marks]





**Q.3 Time Series Regression [16 marks]**

We consider the multiple linear regression model

$$X_t = \beta_0 + \beta_1 Z_{t,1} + \beta_2 Z_{t,2}, \ldots, \beta_q Z_{t,q} + \varepsilon_t.$$

(a) Give the four Gauss-Markov assumptions for multiple linear regression.
Remember: These are the conditions that ensure that OLS is the best linear unbiased estimators. [4 marks]

(b) Which of the Gauss-Markov assumptions are typically violated for time series data? [2 marks]

(c) What are the two major drawbacks for the OLS estimator in time series regression with violated Gauss-Markov assumptions? [2 marks]

(d) Assume we observe a time series $X_t$, $t = 1, 2, 3, \ldots, n$ that appears to have an exponential trend. Explain how we can use linear regression to estimate this nonlinear trend? Write down the corresponding regression model. [4 marks]

(e) Write down a regression model that accounts for a trend and for seasonality that repeats after four observations. Avoid multicollinearity in the model. Show how all four seasonal components can be estimated with your model. [4 marks]

**Q.4 Periodogram [16 marks]**

(a) A monthly time series for $t = 1, \ldots, n$ is to be simulated from the following model:

$$X_t = 10 \sin\left(\frac{2\pi t}{12} + \frac{\pi}{4}\right) + W_t$$

where $W_t \sim WN(0, \sigma_W^2)$. What are the following quantities:

i. amplitude; [1 mark]

ii. frequency; [1 mark]

iii. period of the cycle; [1 mark]

iv. phase in radians; and [1 mark]

v. phase in time units. [2 marks]

(b) A general regression model for fitting a sine wave at a fixed frequency $f$ is given by:

$$X_t = A\cos(2\pi f\, t + \phi) + W_t$$

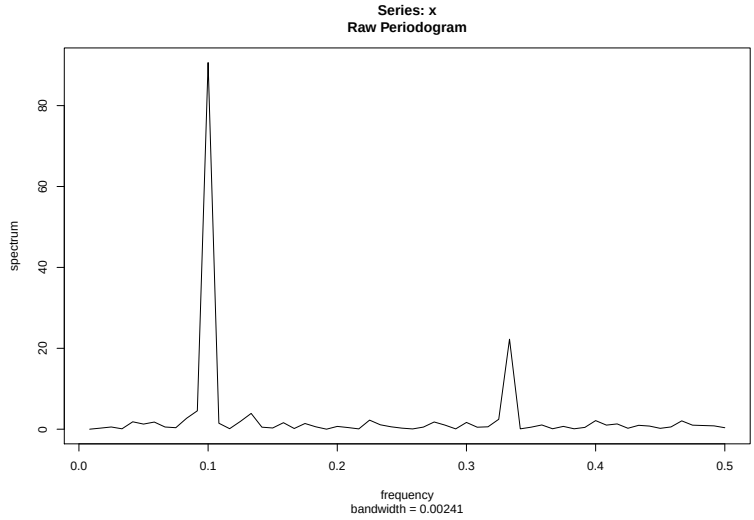
which has a non-linear parameter $\phi$. Use the relation:

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

to reformulate the above regression model as linear in the unknown coefficients. [2 marks]

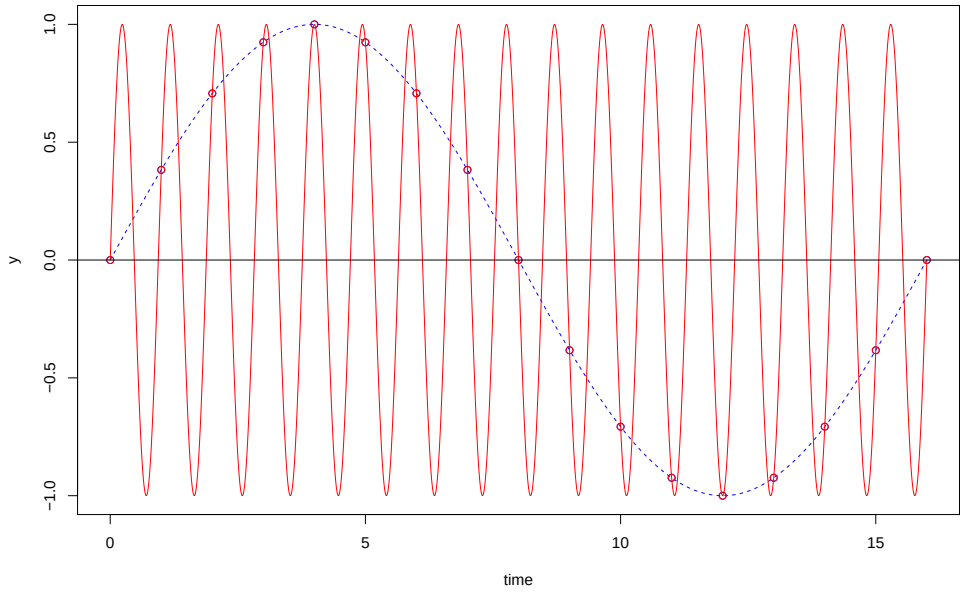


(c) Consider the following periodogram, identify the period of possible cycles in the underlying time series. [3 marks]




(d) The Nyquist frequency is the highest frequency considered in the periodogram. How does a sine wave behave at the Nyquist frequency? [2 marks]

(e) In the following diagram the time series is to be observed at times $t = 1, \ldots, 16$.




Use the diagram to explain the concepts of aliasing. [3 marks]

**Q.5 AR(1) and MA(1) Models [16 marks]**

(a) An AR(1) model can be written as

$$X_t = \phi_0 + \phi_1 X_{t-1} + W_t.$$

i. How is this model related to the random walk? [1 mark]

ii. Under the assumption of stationarity, derive the mean of $X_t$. [2 marks]

iii. Under the assumption of stationarity, derive the variance of $X_t$. [2 marks]

iv. Under the assumption of stationarity, show that the autocovariance function of $X_t$ at lags $|h| = 0, 1, \ldots$ is given by:

$$\gamma_X(h) = Cov(X_t, X_{t-h}) = \frac{\sigma_W^2}{1-\phi_1^2}\phi_1^{|h|}.$$

[3 marks]

v. State a condition on the values of $\phi_1$ that will make the AR(1) a causal stationary model. [2 marks]



(b) An MA(1) model can be written as

$$X_t = \mu + W_t + \theta W_{t-1}.$$



i. Derive the mean and variance of $X_t$. [2 marks]

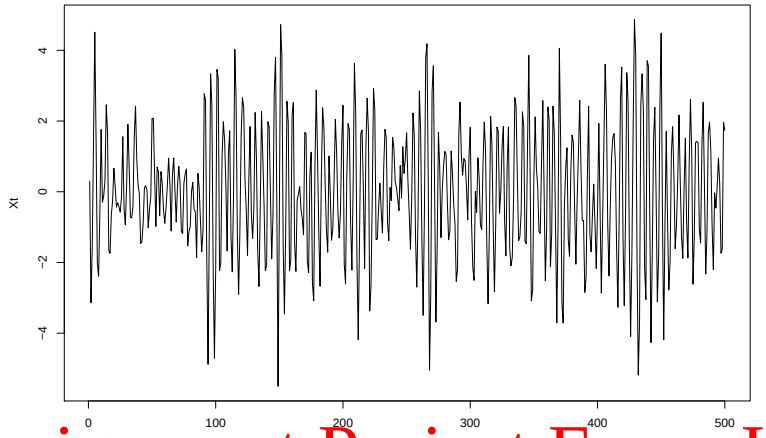
ii. Derive the autocovariance of $X_t$. [2 marks]



iii. State a condition that ensures the MA(1) model is invertible and explain why this property is useful. [2 marks]

**Q.6 Model Selection [16 marks]**

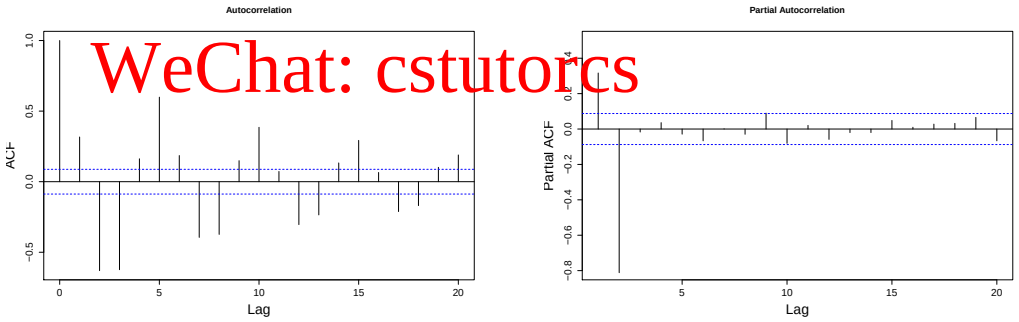
(a) Explain the key features of the dependence observed in the following plot of a times series. [2 marks]






(b) Explain how the features mentioned in part (a) are shown in the sample autocorrelation and partial autocorrelation functions below. [2 marks]








(c) Use these plots to identify whether a suitable model could be an AR($p$), MA($q$) or mixed ARMA($p, q$). Explain your choice and suggest the order of the model. [2 marks]

(d) Write down the backshift (or characteristic) polynomials for the ARMA($p, q$) model:

$$X_t - \phi_1 X_{t-1} - \cdots - \phi_p X_{t-p} = W_t + \theta_1 W_{t-1} + \cdots + \theta_q W_{t-q}$$

[2 marks]

(e) What are the conditions for invertibility and causal stationarity for an ARMA process? [2 marks]

(f) What condition is needed to avoid parameter redundancy for an ARMA process.

[1 mark]

(g) Identify the order of the following ARMA$(p, q)$ model and check it satisfies these three conditions.

$$X_t = 0.7X_{t-1} - 0.1X_{t-2} + W_t + 0.2W_{t-1}$$

If the model has parameter redundancy then rewrite the model without the redundancy.

You may find it useful to know that the roots of the quadratic $az^2 + bz + c$ are given by $z = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. [5 marks]

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