STP 421 - Spring 2023
Problem Set 3
Due: February 27 2023

Instructions:

- **Show your work and attempt every problem!** Partial credit will be given if you can show some progress towards a solution. If you can't solve a problem in full generality, try to solve some special cases.
- You may work in groups and consult outside resources (textbooks, web sites, etc.). However, you should acknowledge any assistance received from other people or outside resources. If working in a group, please list the names of your group members.
- Submitted solutions should either be typed, preferably using LaTeX, or neatly handwritten.
- Solutions should be submitted as PDF documents through Canvas. If submitting scanned images, please assemble all pages into a single PDF document prior to uploading to Canvas.

1.) Suppose that X is a discrete random variable which takes values in the set $\{-2, -1, 1, 2\}$ with the following probabilities: $\mathbb{P}(X = -2) = \mathbb{P}(X = 2) = 2/5$ and $\mathbb{P}(X = -1) = \mathbb{P}(X = 1) = 1/10$. Calculate the following quantities:

(a) $\mathbb{E}[X]$;

(b) $\mathrm{Var}(X)$;

(c) $\mathbb{E}[\sin(\pi X/2)]$.

2.) Let Z be an integer-valued random variable with probability mass function

$$p_Z(n) = \begin{cases} Cn^{-2} & \text{if } n \neq 0 \\ 0 & \text{if } n = 0. \end{cases}$$

(a) Find the value of the normalizing constant C.

(b) Calculate the probability of the event "Z is even."

(c) Calculate the expected value of Z.

3.) Suppose that X and Y are independent Bernoulli random variables with parameter $p = 1/2$ and let $W = \frac{X}{X+Y+1}$.

(a) Find the probability mass function of W.

(b) Calculate $\mathbb{E}[W]$.

4.) Suppose that N is uniformly distributed on the integers $\{0, 1, \cdots, n\}$. Conditional on $N = k$, let X be uniformly distributed on the integers $\{0, 1, \cdots, k\}$.

(a) Find the probability mass function of X.

(b) Calculate $\mathbb{E}[N|X = k]$ for $k = 0, \cdots, n$.

5.) Suppose that X and Y are independent integer-valued random variables with probability mass functions p_X and p_Y and let $Z = X + Y$.

(a) Find an expression for the probability mass function p_Z in terms of p_X and p_Y.

(b) Use your result from (a) to show that if X and Y are independent binomial random variables with parameters (n_1, p) and (n_2, p), then their sum $Z = X + Y$ is Binomial with parameters $(n_1 + n_2, p)$.

6.) Suppose that five balls are sampled with replacement from an urn that contains 10 red balls, 10 blue balls, 5 green balls and 5 yellow balls and let R, B, G and Y, respectively, denote the numbers of red, blue, green or yellow balls in the sample.

(a) Find the joint probability mass function of (R, B, G, Y).

(b) Find the probability mass functions of B and of $R + G$.

(c) Calculate $\mathbb{E}[G + Y - 2R]$.

7.) Let $s \geq 1$ and suppose that X is a positive integer-valued random variable with distribution

$$\mathbb{P}(X = k) = \frac{k^{-s}}{\xi(s)}, \quad k = 1, 2, 3, \cdots$$

where

$$\xi(s) \equiv \sum_{n=1}^{\infty} n^{-s}.$$

(a) Let $p_1, p_2, \cdots$ be an enumeration of the prime numbers and define $E_{p_i} \equiv \{X \equiv 0 \mod p_i\}$ to be the event that X is divisible by p_i. Show that the events $E_{p_1}, E_{p_2}, \cdots$ are independent.
(b) Use the result from (a) to prove that

$$\frac{1}{\xi(s)} = \prod_{i=1}^{\infty} \left(1 - p_i^{-s}\right).$$

8.) Suppose that X, Y and Z are discrete real-valued random variables.

(a) Show that $\mathbb{E}[X + Y|Z] = \mathbb{E}[X|Z] + \mathbb{E}[Y|Z]$.

(b) Calculate $\mathbb{E}[g(X)|X]$, where $g : \mathcal{R} \to \mathcal{R}$ is a real-valued function.

(c) Calculate $\mathbb{E}[X|Y]$ assuming that X is independent of Y.

9.) Suppose that $Z_0, Z_1, \cdots$ is a sequence of independent, identically-distributed random variables with distribution $\mathbb{P}(Z_i = 1) = \mathbb{P}(Z_i = -1) = 1/2$ and define the sequence $X_0, X_1, \cdots$ recursively by setting $X_0 = 0$ and $X_{n+1} = X_n + Z_n$ for each $n \geq 1$. Calculate $\mathbb{E}[X_{n+m}|X_n]$ for $n, m \geq 0$.

10.) Although the sex ratio at birth is close to $1:1$ in human populations, biased sex ratios are found in many other species. This is true of many parasitoids, which lay their eggs on another host organism; when the eggs hatch, the young enter into the still living host and consume it from within, often taking care to avoid eating essential organs. In many of these species, mating occurs within the host organism between siblings.

Suppose that a parasitoid always lays three eggs on its host and that each egg survives to reproductive maturity, independently of the others, with probability s. In addition, suppose that the sex of each surviving offspring is female with probability ϕ and male with probability $1 - \phi$ and that sex is also independently determined for each offspring. Surviving females can only reproduce if there is at least one surviving male present within the host, which they exit after being inseminated.

(a) Find a formula in terms of s and ϕ for the expected number of inseminated females that emerge from each parasitized host.

(b) For each s, find the value of ϕ that maximizes this expected value. This is said to be the optimal sex ratio, ϕ_{ESS}.

(c) Explain the relationship between the survival probability s and the optimal sex ratio.