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Introduction to the Theory of Computation

Lecture 12

Ruth Urner

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Reminder: Midterm on March 1!

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Q: show regular languages are closed under OPD(·,·,·)

You can use:

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Regular languages are closed under

- unions
- concatenation
- star-operation
- intersection
- complements

Regular Expressions

Reading: ITC Section 1.3

A regular expression is a compact way of defining a set of words. It is a sequence of symbols that represent a language over some alphabet Σ.

Inductive definition of sets – general pattern

An inductive definition of a set consists of

1. **A universe set** U
2. **A core set** $C \subseteq U$
3. **A finite set** $O = \{o_1, o_2, \ldots o_n\}$ **of operations from** $o_i : U^{r_i} \to U$ **for some arities** $r_i \in \mathbb{N}$

We define $\mathcal{I}(U, C, O)$ as the set of elements that we obtain by starting with the core set and putting all those elements of U into $\mathcal{I}(U, C, O)$ that one can reach by successively applying the operations in O.

Let Σ be some alphabet. We define the set $\mathcal{R}$ of regular expressions over Σ inductively by setting $\mathcal{R}_\Sigma = \mathcal{I}(U, C, \mathcal{O})$, where

1. The universe U is the set of all strings over $\Sigma \cup \{(,)\} \cup \circ, {}^*, \epsilon, \emptyset\}$

2. The core set C is the set of al symbols in Σ and $\epsilon, \emptyset$ and two additional symbols: $C = \Sigma \cup \{\epsilon, \emptyset\}$.

3. Three operations:
   - $o_\cup(R_1, R_2) = (R_1 \cup R_2)$,
   - $o_\circ(R_1, R_2) = (R_1 \circ R_2)$,
   - $o_*(R) = (R^*)$.

Regular expression–inductive definition

Let Σ be some alphabet. We define the set $\mathcal{R}$ of regular expressions over Σ inductively by setting $\mathcal{R}_\Sigma = \mathcal{I}(U, C, O)$, where

1. The universe U is the set of all strings over $\Sigma \cup \{(,), \cup, \circ, ^*, \epsilon, \emptyset\}$.
2. The core set C is the set of al symbols in Σ and $\epsilon, \emptyset$ and two additional symbols: $C = \Sigma \cup \{\epsilon, \emptyset\}$.
3. Three operations:
   - $o_\cup(R_1, R_2) = (R_1 \cup R_2)$,
   - $o_\circ(R_1, R_2) = (R_1 \circ R_2)$,
   - $o_*(R) = (R^*)$.

Examples: $\Sigma = \{0, 1\}$

0

$(1 \cup 0)$ ✓

$(1\ 0$ ✗

ε ✓

$(\varepsilon \circ 1)$ ✓

$\in \mathcal{R}_\Sigma$

member of universe

$)(\circ\emptyset$

these two are not regular expressions

Exercise: Prove (by induction) the number of "(" is always equal to the number of ")" in a regular expression

The language of a regular expression

Each regular expression $R \in \mathcal{R}_\Sigma$ over some alphabet Σ represents a language over Σ. We define the interpretation $L(R)$ of a regular expression R according to the inductive definition:

Members of the core-set:

- The expression a for $a \in \Sigma$ represents the language $\{a\}$, that is $L(a) = \{a\}$.
- The expression ϵ represents the language $\{\epsilon\}$, that is $L(\epsilon) = \{\epsilon\}$.
- The expression $\emptyset$ represents the language $\emptyset$,that is $L(\emptyset) = \emptyset$.

Result of operation: For regular expressions R_1, R_2 and R, we define:

- $L((R_1 \cup R_2)) \;=\; L(R_1) \cup L(R_2)$
- $L((R_1 \circ R_2)) \;=\; L(R_1) \circ L(R_2)$
- $L((R^*)) \;=\; L(R)^*$

We call $L(R)$ the language of R.

$\Sigma = \{0, 1\}$

$0 \in \mathcal{R}_\Sigma$

$L(0) = \{0\}$

$0, \varepsilon \in \mathcal{R}_\Sigma$

$(0 \cup \varepsilon) \in \mathcal{R}_\Sigma$

$L((0 \cup \varepsilon)) = \{0, \varepsilon\}$

$(0 \cup \emptyset) \in \mathcal{R}_\Sigma$

$L((0 \cup \emptyset)) = \{0\}$

The language of a regular expression

Each regular expression $R \in \mathcal{R}_\Sigma$ over some alphabet Σ represents a language over Σ. We define the interpretation $L(R)$ of a regular expression R according to the inductive definition:

Members of the core-set:

- The expression a for $a \in \Sigma$ represents the language $\{a\}$, that is $L(a) = \{a\}$.
- The expression ϵ represents the language $\{\epsilon\}$, that is $L(\epsilon) = \{\epsilon\}$.
- The expression $\emptyset$ represents the language $\emptyset$, that is $L(\emptyset) = \emptyset$.

Result of operation: For regular expressions R_1, R_2 and R, we define:

- $L(\boxed{(R_1 \cup R_2)}) = L(R_1) \cup L(R_2)$

Regular expression–additional notation

For convenience and readability, we use the following notational conventions:

1. For an alphabet Σ, we use Σ as a regular expression representing all words of length 1 over Σ.And then Σ^* is a regular expression for the set of all words over Σ.

2. We often omit brackets. The order of precedence then is: $*$, $\circ$, $\cup$.
3. The $\circ$-symbol is typically omitted: we use R_1R_2 as shorthand for $R_1 \circ R_2$.
4. We let R^+ be shorthand for RR^*.

5. We let R^k be the k times repeated concatenation of R with itself: $R^k = R \circ R \circ R \circ \ldots \circ R$.

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- 0^*10^*
- $\Sigma^*1\Sigma^*$
- $\Sigma^*001\Sigma^*$
- $1^*(01^+)^*$
- $(\Sigma\Sigma)^*$
- $(\Sigma\Sigma\Sigma)^*$

Regular expressions–examples

precedence: $*$ $\circ$ $\cup$

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- $0^*10^* = ((0^*)\circ 1 \circ (0^*))$

$= \{w \in \Sigma^* \mid w \text{ contains exactly one } 1\}$

$1 \in L(0^*10^*)$

$0010 \in L(0^*1\,0^*)$

$10000 \in L(0^*10^*)$

$L((0^*))$ = {set of all words that have only letter 0}

ε is included

$L((0^*)) = \{\varepsilon, 0, 00, 000, \dots\}$

$L(1) = \{1\}$

Regular expressions–examples

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- 0^*10^*
- $\Sigma^*1\Sigma^* = (\Sigma^*)1(\Sigma^*) = (\Sigma^*) \circ 1 \circ (\Sigma^*)$

$L(\Sigma^*1\Sigma^*) = \{w \in \Sigma^* \mid w$ contains at least one $1\}$

$1 \in L(\Sigma^* 1 \Sigma^*)$

$\underbrace{010110}_{\in\Sigma^*}1\underbrace{010}_{\in\Sigma^*} \in L(\Sigma^* 1 \Sigma^*)$

Regular expressions–examples

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- 0^*10^*
- $\Sigma^*1\Sigma^*$
- $\Sigma^*001\Sigma^*$

$L(\Sigma^*001\Sigma^*) = \{w \in \Sigma^* \mid w$ contains 001 as a substring$\}$

Regular expressions–examples

$* \circ \cup$

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- 0^*10^*
- $\Sigma^*1\Sigma^*$
- $\Sigma^*001\Sigma^*$
- $1^*(01^+)^*$

$L(1^*(01^+)^*) = \{w \in \Sigma^* \mid$ every 0 is followed by a $1\}$

$1\,1\,1\,0\,1\,0\,1\,1\,1\,0\,1\,1$

$\epsilon\ 1^*\quad \epsilon 1^+\quad 1^+\quad 1^+$

Regular expr

precedence $* \circ \cup$

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ.

- 0^*10^*
- $\Sigma^*1\Sigma^*$
- $\Sigma^*001\Sigma^*$
- $1^*(01^+)^*$

$L(1^*(01\,1^*)^*) = \{w \in \Sigma^* \mid \text{every } 0 \text{ is followed by a } 1\}$

Regular expr...

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ.

- 0^*10^*
- $\Sigma^*1\Sigma^*$
- $\Sigma^*001\Sigma^*$
- $1^*(01^+)^*$
- $(\Sigma\Sigma)^*$

$L(\Sigma) = \{w \in \Sigma^* \mid w \text{ has only one letter}\}$

$= \{0, 1\}$

$L((\Sigma \circ \Sigma)) = \{00, 01, 10, 11\}$

$L((\Sigma\Sigma)^*) = \{w \in \Sigma^* \mid w \text{ has an even length}\}$

$A^+ = \{w \in \Sigma^* \mid w$ is the concatenation of one or more words from $A\}$

A^* always includes ε

Regular exp

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- 0^*10^*
- $\Sigma^*1\Sigma^*$
- $\Sigma^*001\Sigma^*$
- $1^*(01^+)^*$
- $(\Sigma\Sigma)^*$
- $(\Sigma\Sigma\Sigma)^*$

$L((\Sigma\Sigma\Sigma)^*) = \{w \in \Sigma^* \mid$ the length of w is divisible by $3\}$

Regular expressions–examples

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- $01 \cup 10$
- $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$
- $(0 \cup \epsilon)1^*$
- $(0 \cup \epsilon)(1 \cup \epsilon)$
- $1^*\emptyset$
- $\emptyset^*$

Regular expr

$* \quad \circ \quad \cup$

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- $01 \cup 10 = ((0 \circ 1) \cup (1 \circ 0))$

Construction sequence

1. (core set)
2. 0 (core - set)
3. $(0 \circ 1)$ (applying $O_\circ$ to 2+1)
4. $(1 \circ 0)$ (applying $O_\circ$ to 1+2)
5. $((0 \circ 1) \cup (1 \circ 0))$ (applying $O_\cup$ to 3,4)

$L(1) = \{1\}$

$L(0) = \{0\}$

$L((0 \circ 1)) = \{01\}$

$L((1 \circ 0)) = \{10\}$

$L(((0 \circ 1) \cup (1 \circ 0))) = \{01, 10\}$

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Talking:

Regular expr[...]

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- $01 \cup 10$
- $(0\Sigma^*0) \cup (1\Sigma^*1) \cup 0 \cup 1$

$L(0) = \{0\}$

$L(1) = \{1\}$

$L(0\Sigma^*0) = \{w \in \Sigma^* \mid w$ starts and ends with 0 and has length at least 2$\}$

$L(1\Sigma^*1) = \{$ ... 1 ... $\}$

$L(0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1) = \{w \in \Sigma^* \mid w \neq \varepsilon$ and w starts and ends on the same letter$\}$

Regular expr

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- $01 \cup 10$
- $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$
- $(0 \cup \epsilon)1^*$

$L(0 \cup \varepsilon) = \{0, \varepsilon\}$

$L((0 \cup \varepsilon)1^*) = \{w \in \Sigma^* \mid w$ has only 1 or contains a single 0 at the start$\}$

Regular expr

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- $01 \cup 10$
- $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$
- $(0 \cup \epsilon)1^*$
- $(0 \cup \epsilon)(1 \cup \epsilon)$
- $1^*\emptyset$
- $\emptyset^*$

$L(1^*\emptyset) = \emptyset$

(concatenating any language with $\emptyset$ results in $\emptyset$)

Regular expr

We let $\Sigma = \{0, 1\}$.

Now we can interpret the following regular expressions over Σ:

- $01 \cup 10$
- $0\Sigma^*0 \cup 1\Sigma^*1 \cup 0 \cup 1$
- $(0 \cup \epsilon)1^*$
- $(0 \cup \epsilon)(1 \cup \epsilon)$
- $1^*\emptyset$ $L(1^*\emptyset) = \emptyset$
- $\emptyset^*$

$L(\emptyset^*) = \{\varepsilon\}$

Regular expressions–more examples

Let R be some regular expression. Then we have:

- $L(R \cup \emptyset) = L(R)$
- $L(R \circ \epsilon) = L(R)$
- $L(R \cup \epsilon)$ is not necessarily equal to $L(R)$
- $L(R \circ \emptyset)$ is not necessarily equal to $L(R)$

Regular expr

Let R be some regular expression. Then we have:

- $L(R \cup \emptyset) = L(R)$
- $L(R \circ \epsilon) = L(R)$
- $L(R \cup \epsilon)$ is not necessarily equal to $L(R)$

Example: $L(R) = \{1, 10\}$, $L(R \cup \varepsilon) = \{\varepsilon, 1, 10\}$

- $L(R \circ \emptyset)$ is not necessarily equal to $L(R)$

Example: $L(R) = \{1, 10\}$, $L(R \circ \emptyset) = \emptyset$

Theorem
A language is regular if and only if some regular expression describes it.

Lemma 1
If a language is described by a regular expression, then it is regular.

Lemma 2
If a language is regular, then there is regular expression that describes it.

Expressive p

Theorem
A language is regular if and only if some regular expression describes it.

$\Longleftrightarrow$

$\Longleftarrow$

Lemma 1
If a language is described by a regular expression, then it is regular.

$\Longrightarrow$

Lemma 2
If a language is regular, then there is regular expression that describes it.

you can find the proof in the textbook

Expressive power of regular expressions–proof of Lemma 1

Let Σ be some alphabet. We prove by induction (according to the inductive definition of regular expressions) that for every regular expression R there exists and NFA that recognizes the language $L(R)$ (and this implies that $L(R)$ is a regular language).

Base case

We prove that the claim is true for members of the core-set.

1. If $R = a$ for some member of the alphabet Σ then we have $L(R) = \{a\}$. We can construct an NFA recognizing $L(R)$ as follows:

2. If $R = \epsilon$, then we have $L(R) = \{\epsilon\}$. We can construct an NFA recognizing $L(R)$ as follows:

3. If $R = \emptyset$, then we have $L(R) = \emptyset$. We can construct an NFA recognizing $L(R)$ as follows:

Expressive p... proof of Lemma 1

Let Σ be some alphabet. We prove by induction (according to the inductive definition of regular expressions) that for every regular expression R there exists an *NFA* that recognizes the language $L(R)$ (and this implies that $L(R)$ is a regular language).

Base case

We prove that the claim is true for members of the core-set.

1. If $R = a$ for some member of the alphabet Σ then we have $L(R) = \{a\}$. We can construct an *NFA* recognizing $L(R)$ as follows:

2. If $R = \epsilon$, then we have $L(R) = \{\epsilon\}$. We can construct an *NFA* recognizing $L(R)$ as follows:

3. If $R = \emptyset$, then we have $L(R) = \emptyset$. We can construct an *NFA* recognizing $L(R)$ as follows:

Expressive power of regular expressions–proof of Lemma 1

Induction Hypothesis
We assume that for two regular expressions R_1 and R_2 there exists NFAs N_1 and N_2 such that

$$L(R_1) = L(N_1) \quad \text{and} \quad L(R_2) = L(N_2)$$

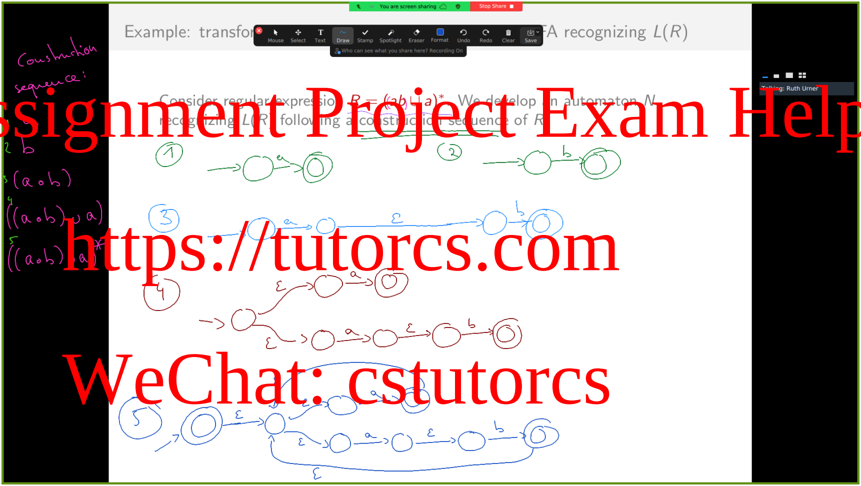
Induction Step
We need to show that, given the induction hypothesis, there exist NFAs that recognize the languages obtained by applying the three operations to the expressions R_1 and R_2.

1. We have $o_\cup(R_1, R_2) = (R_1 \cup R_2)$ and $L((R_1 \cup R_2)) = L(R_1) \cup L(R_2)$. By the induction hypothesis, there exist NFAs N_1 and N_2 recognizing $L(R_1)$ and $L(R_2)$. We have seen in Lecture 10, how to construct an NFA N recognizing the language $L(R_1) \cup L(R_2)$ (regular languages are closed under unions).

2. We have $o_\circ(R_1, R_2) = (R_1 \circ R_2)$ and $L((R_1 \circ R_2)) = L(R_1) \circ L(R_2)$. By the induction hypothesis, there exist NFAs N_1 and N_2 recognizing $L(R_1)$ and $L(R_2)$. We have seen in Lecture 10, how to construct an NFA N recognizing the language $L(R_1) \circ L(R_2)$ (regular languages are closed under concatenation).

3. We have $o_*(R_1) = (R_1^*)$ and $L((R_1^*)) = L(R_1)^*$. By the induction hypothesis, there exists an NFA N_1 recognizing $L(R_1)$. We have seen in Lecture 10, how to construct an NFA N recognizing the language $L(R_1)^*$ (regular languages are closed under the star-operation).

Consider regular expression $R = (ab \cup a)^*$. We develop an automaton N recognizing $L(R)$ following a construction sequence of R:


Example: transfor
FA recognizing L(R)
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Consider regular expression R = (ab ∪ a)*. We develop an automaton N
recognizing L(R) following a construction sequence of R
Construction sequence:
2 b
3 (a∘b)
4 ((a∘b)∪a)
5 ((a∘b)∪a)*
1
2
3
4
5
a
b
ε

Non-regular Languages

Reading: ITC Section 1.4

Proving a language is not regular

We have seen multiple techniques to prove that a language L over some alphabet Σ is regular by now:

- It suffices to prove that there exists a DFA that recognizes (accepts) L.

- It suffices to prove that there exists an NFA that recognizes (accepts) L.

- It suffices to prove that there exists a regular expression that is interpreted as L.

To prove that some language L is not regular, we need to prove that none of these is possible.

Here is a general technique for showing that some object t is not a member of some set S:

- Prove that all members of S have some property P.
- Prove that t doesn't have property P.
- Conclude that $t \notin S$.

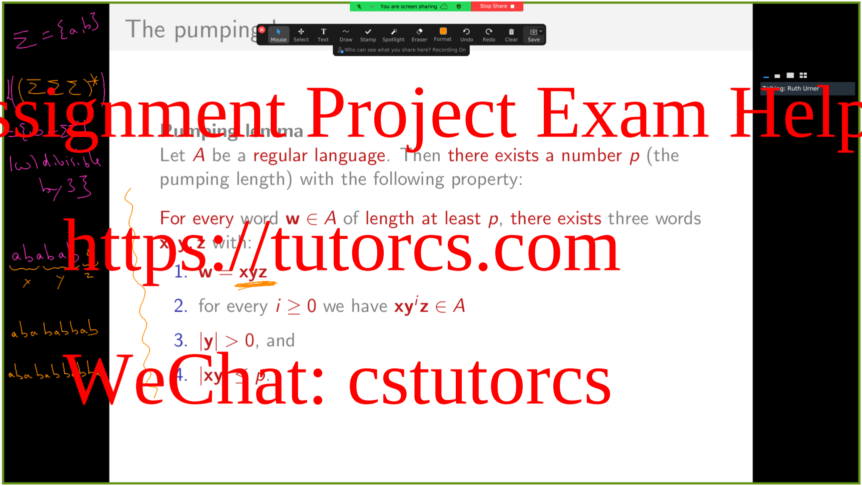
We will next state (and prove) a property that all regular languages have.

Pumping lemma

Let A be a regular language. Then there exists a number p (the pumping length) with the following property:

For every word $\mathbf{w} \in A$ of length at least p, there exists three words $\mathbf{x}, \mathbf{y}, \mathbf{z}$ with:

1. $\mathbf{w} = \mathbf{xyz}$
2. for every $i \geq 0$ we have $\mathbf{xy}^i\mathbf{z} \in A$
3. $|\mathbf{y}| > 0$, and
4. $|\mathbf{xy}| \leq p$.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The pumping l
Pumping lemma
Let A be a regular language. Then there exists a number p (the pumping length) with the following property:
For every word w ∈ A of length at least p, there exists three words x, y, z with:
1. w = xyz
2. for every i ≥ 0 we have xy^i z ∈ A
3. |y| > 0, and
4. |xy| ≤ p.
Σ = {a, b}
|w| divisible by 3}
ababab
x y z
ababbabbab