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Introduction to the Theory of Computation

Lecture 2

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Basic mathematical notation and terminology

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Reading: ITC Section 0.2

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Set-theory is the foundation of mathematics.

We will start with the set of natural numbers as a basic given set to start with:

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$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

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Further, we will assume the existence of the empty set: $\emptyset$

Sets

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Set-theory is the foundation of mathematics.

We will start with the set of natural numbers as a basic given set to start with:

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Further, we will assume the existence of the empty set: $\emptyset$

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$\emptyset$ does not contain any elements

$\emptyset$ is a subset of every other set

Elements of sets

A set consists of elements, denoted by the $\in$-relation:

$$a \in A$$

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means a is an element of set A.

To state that a is not an element of set A, we use the notation

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$$a \notin A.$$

Important properties:

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- Sets are not ordered and contain each element only once!
- The empty set has no elements.

Elements of sets

A set consists of elements, denoted by the $\in$-relation:

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Important properties:

- Sets are not ordered and contain each element only once!

$\{2,4,6,4\} = \{2,4,6\} = \{4,6,2\} = \{6,4,2,2\}$

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Subsets

Using the element relation, we can define what subsets are:

- $A \subseteq B$
- $A \subsetneq B$
- $A = B$

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Subsets

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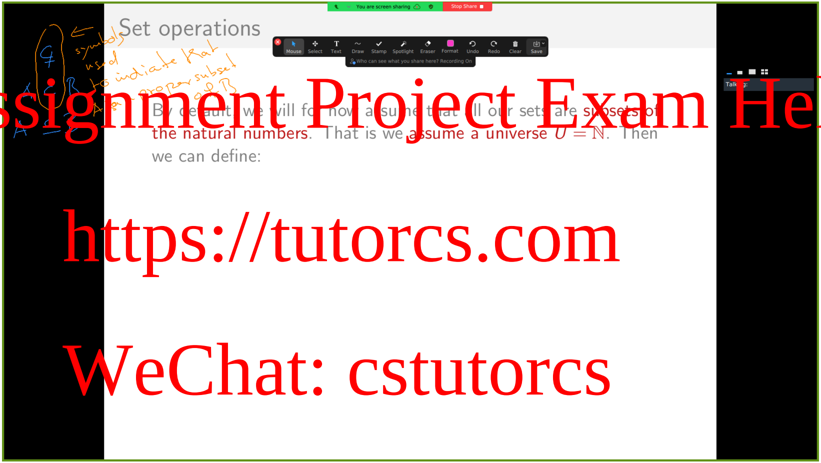
Using the element relation, we can define what subsets are:

- $A \subseteq B$ "A is a subset of B"
  if $c \in A$ then $c \in B$.
  $\{1,2,4\} \subseteq \mathbb{N}$, $\{1,2,4\} \subseteq \{1,2,4\}$
- $A \subsetneq B$ "A is a proper subset of B"
  $A \subseteq B$ and there is a $b \in B$ with $b \notin A$.
  $\{1,2,4\} \subsetneq \mathbb{N}$, $\{n \in \mathbb{N} \mid n \text{ even}\} \subsetneq \mathbb{N}$
- $A = B$ "A and B are equal as sets"
  $A \subseteq B$ and $B \subseteq A$
  $\{1,2,4,2\} = \{4,2,1\}$



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How to define sets

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Here are some tools for defining sets:

1. Make a list of the elements
   - Set of all students in this class
   - Set of odd natural numbers smaller than 10: $\{1, 3, 5, 7, 9\}$

   Problem: this technique fails for large or infinite sets

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2. Identify by a common characteristic
   - Odd natural numbers $\{n \in \mathbb{N} \mid n \text{ is not divisible by } 2\}$

   Problem: sometimes we don't know a precise defining characteristic

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3. Inductive definition
   - We'll see how to do this later.

How to define sets – a warning

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Warning:

When we define sets, we always need to specify from which universe (that is a possibly much larger ground set, for example the natural numbers) the elements of our set should be taken from!

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Example:

- Odd natural numbers $\{n \in \mathbb{N} \mid n \text{ is not divisible by } 2\}$
- Interval on the real line $\{x \in \mathbb{R} \mid 2 \leq x \leq 4\}$

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Otherwise we can fall into Russell's paradox...!

Russel's paradox

Consider the following definition of a set:

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$$R = \{r \mid r \notin r\}$$

That is, the set R contains all those sets that do not contain themselves as an element.

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Question: Is R an element of R?

Now, $R \in R$ implies that $R \notin R$, and vice versa ($R \notin R$ implies that $R \in R$) – a contradiction.

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Russel's paradox

Consider the following definition of a set

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Set operations

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By default, we will for now assume that all our sets are subsets of the natural numbers. That is we assume a universe $U = \mathbb{N}$. Then we can define:

- The set-difference $A \setminus B$ of two sets A and B

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- The intersection $A \cap B$ of two sets A and B

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- The union $A \cup B$ of two sets A and B

Set operations

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$A - B$

- The set-difference $\boxed{A \setminus B}$ of two sets A and B

always: $A \setminus B \subseteq A$

$A \setminus B = \{a \in A \mid a \notin B\}$

$A = \{1, 5, 17\}$ $B = \{1, 2, 3\}$

$A \setminus B = \{5, 17\}$

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Set operations

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- The set-difference $A \setminus B$ of two sets A and B

In general: $A \setminus B \neq B \setminus A$

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$A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3\}$

$A \setminus B = \{5, 7\}$, $B \setminus A = \{2\}$

Q: Is it possible that $A \setminus B = B \setminus A$ for some sets A and B?

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A: Yes. Eg. if $A = B = \emptyset$, then $A \setminus B = B \setminus A = \emptyset$

Or if $A = B = \{1, 2\}$, then $A \setminus B = B \setminus A = \emptyset$.

Set operations

By default we will for now assume that all our sets are subsets of the natural numbers. That is we assume a universe $U = \mathbb{N}$. Then we can define:

- ~~The set-difference $A \setminus B$ of two sets A and B~~

$A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3\}$

$A \cap B = \{1, 3\}$

- The intersection $A \cap B$ of two sets A and B

$A \cap B$ is the set of all elements that are members of both A and B.

$A \cap B = \{n \in \mathbb{N} \mid n \in A \text{ and } n \in B\}$

or universe

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Set operations

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- The set-difference $A \setminus B$ of two sets A and B

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- ~~The intersection $A \cap B$ of two sets A and B~~

$A = \{1, 3, 5, 7\}$, $B = \{1, 2, 3\}$

$A \cup B = \{1, 2, 3, 5, 7\}$

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- The union $A \cup B$ of two sets A and B

$A \cup B$ is the set of all elements in A and all elements in B. $A \cup B = \{n \in \mathbb{N} \mid n \in A \text{ or } n \in B\}$

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By default, we will for now assume that all our sets are subsets of the natural numbers. Then we can define:

- The complement $\bar{A}$ of a set A

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- The (cartesian) product) of $A \times B$ two sets A and B

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- The set A^k of k-tuples A^k of elements of A

More set operations

By default, we will for now assume that all our sets are subsets of the natural numbers. Then we can define:

- The complement $\bar{A}$ of a set A is the set of all elements in the universe that are not in A.

$A = \{n \in \mathbb{N} \mid n$ divisible by $2\} = \{$even numbers$\}$

$\bar{A} = \{$odd numbers$\} = \{n \in \mathbb{N} \mid n$ not divisible by $2\}$

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More set operations

By default we will for now assume that all our sets are subsets of the natural numbers. Then we can define:

- The complement $\bar{A}$ of a set A
- The (cartesian) product of $A \times B$ two sets A and B

is the set of all pairs of elements from A and B.

$A = \{1,3,5,7\}$, $B = \{1,2,3\}$

$A \times B = \{(1,1),(1,2),(1,3),(3,1),(3,2),(3,3),$
$(5,1),(5,2),(5,3),(7,1),(7,2),(7,3)\}$

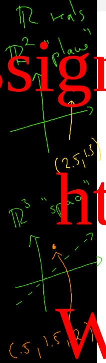
More set operations

By default we will for now assume that all our sets are subsets of the natural numbers. Then we can define:

- The complement $\bar{A}$ of a set A
- The (cartesian) product of $A \times B$ two sets A and B
- The set A^k of k-tuples A^k of elements of A

$A = \{1, 2, 4\}$,

$A^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2)$
$(1,1,4), (1,4,1), \ldots \quad (4,4,4)\}$



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More set operations

- The power set $\mathcal{P}(A)$ of a set A

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- The symmetric difference $A \Delta B$ between two sets A and B

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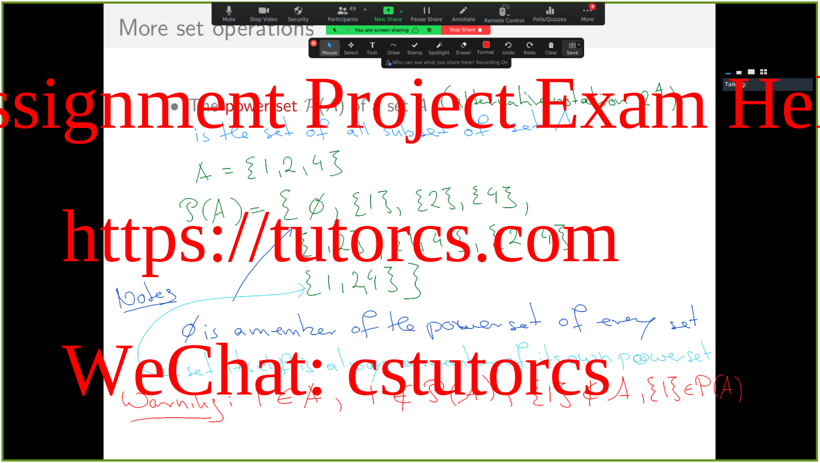

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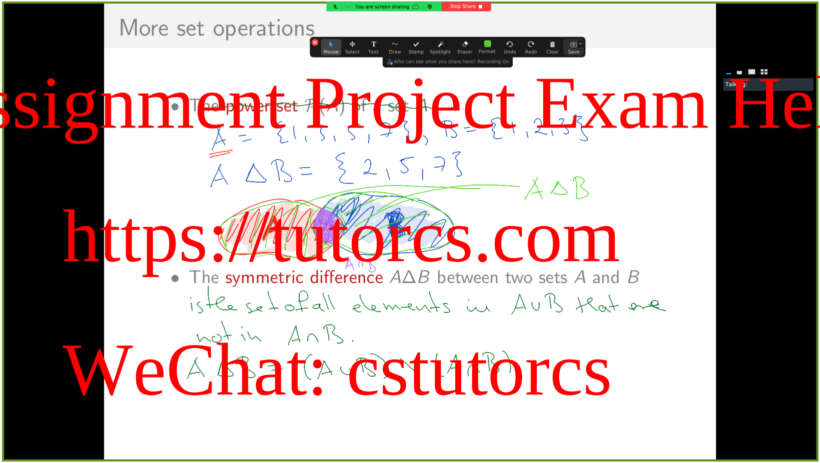

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Sets, multi-sets, sequences

- Recall: Sets contain each element only once and the order does not matter.

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- Multi-sets can contain elements multiple times, but do not contain an order of their elements.

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- Sequences or tuples are collections of elements in a fixed order.

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Sets, multi-sets, sequences

- Recall: Sets contain each element only once and the order does not matter.
- Multi-sets can contain elements multiple times, but do not contain an order of their elements.

by default the '{ }' brackets indicate sets not multisets

{1,1,1,5,5} is not equal to {1,5,5} as a mutiset, but equal as sets

{1,1,3,5} is equal to {5,1,3,1} as a multiset

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Sets, multi-sets, sequences

- Recall: Sets contain each element only once and the order does not matter.

- Multi-sets can contain elements multiple times, but do not contain an order of their elements.

- Sequences or tuples are collections of elements in a fixed order.

(3,4,5) is a 3-tuple

$(2i)_{i \in \mathbb{N}}$ is the sequence of even numbers

$= (0, 2, 4, 6, 8, 10, \ldots)$

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Definition:
A relation R between set A and set B is a subset of $A \times B$.

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We sometimes use notation aRb to state that $(a, b) \in R$.

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Relations

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Definition

A relation R between set A and set B is a subset of $A \times B$.

"$\leq$" is a relation over the natural numbers:

$\{(0,1),(0,0),(1,1),(0,2),(1,2),\ldots\}$

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$A = \{1,3,5,7\}$, $B = \{1,2,3\}$ then

$R = \{(1,1),(3,3)\}$ is a relation between A and B.

$R' = \{(3,2),(3,1),(5,3),(7,3)\}$

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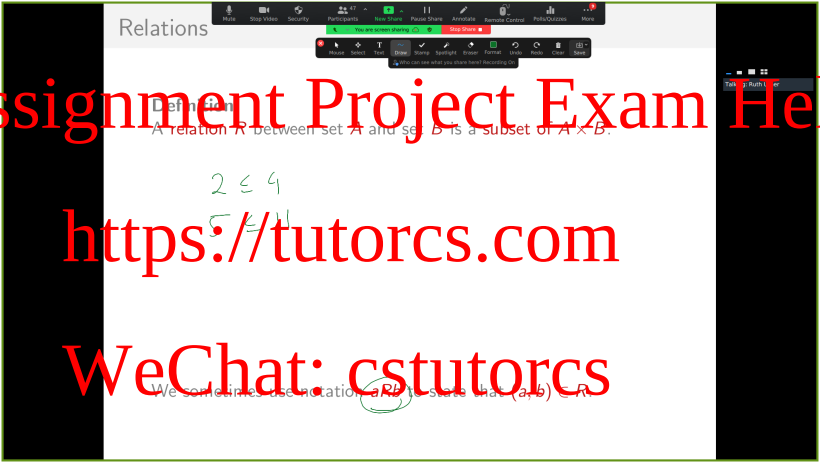

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- If R is a relation between A and itself, that is $R \subseteq A \times A$, we also call R a binary relation, or a relation of arity 2 on A.

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- A subset $R \subseteq A^k$ of k-tuples of elements of A, is also called a k-ary relation, or a relation of arity k on A.

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Definition:

A binary relation R on some set A is called an equivalence relation if it has the following properties:

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- reflexive
- symmetric
- transitive

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Equivalence relations

$\leq$ as a relation over $\mathbb{N}$ is reflexive and transitive but not symmetric

$=$ is an example of an equivalence relation

Definition:
A binary relation R on some set A is called an equivalence relation if it has the following properties:

- reflexive — for every $a \in A$, $(a,a) \in R$
- symmetric — if $(a,b) \in R$ for some $a,b \in A$, then $(b,a) \in R$
- transitive — if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$

$\equiv_{mod\,5}$ (congruence modulo 5) is an equivalence relation over the natural numbers

$6 \equiv_{mod\,5} 11$, $6 \equiv_{mod\,5} 21$, $11 \equiv_{mod\,5} 21$

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Definition:

A function f from set A to set B is a relation between A and B that satisfies the following properties:

- For every $a \in A$, there exists a $b \in B$ such at $(a, b) \in f$
- If $(a, b) \in f$ and $(a, b') \in f$ then $b \in b'$

Instead of $(a, b) \in f$, we usually use notation $f(a) = b$.
Alternatively, we also use the notation $f : a \mapsto b$.

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Functions

$f(x)=|x|$

$g(x) = \sin(x)$

not a function

Definition

A function f from set A to set B is a relation between A and B that satisfies the following properties:

- For every $a \in A$, there exists a $b \in B$ such at $(a, b) \in f$
- If $(a, b) \in f$ and $(a, b') \in f$, then $b = b'$.

$A = \{1, 3, 5, 7\}$ $B = \{1, 2, 3\}$

$f = R = \{(1,1), (5,3), (3,1), (7,2)\}$ is a fct

$R = \{(1,1), (5,1), (1,2), (7,2)\}$ is not a function; it violates both properties (input 1 gets mapped to two outputs and input 3 doesn't get mapped to any output)

Instead of $(a, b) \in f$, we usually use notation $f(a) = b$.

Alternatively, we also use the notation $f : a \mapsto b$.

x	$f(x)$
1	1
3	1
5	3
7	

for functions, we can use such tables to define them

Functions – important properties

We use the notation

$$f : A \to B$$

to state that f is a function from A to B.

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We call A the domain of the function f and B the range of the function f.

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Functions can be:

- one-to-one:
  We call a function $f : A \to B$ one-to-one if $f(a) = b$ and $f(a') = b$ for some $b \in B$ implies that $a = a'$.
- onto:
  We call a function $f : A \to B$ onto if for all $b \in B$ there exists an $a \in A$ with $f(a) = b$.

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