eecs2001

Introduction to the Theory of Computation

Lecture 7

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Finite Automata

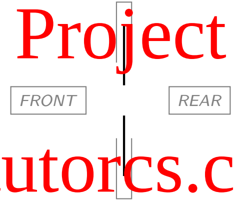
Reading: ITC Section 1.1

A finite automaton or finite state machine is a simple computational model.

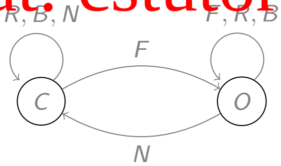
We will work with this model of computation for the next part of this course.

A simple automaton–sliding door example

Consider an automatic sliding door sliding door with two pads that receive signals if someone is standing on them:

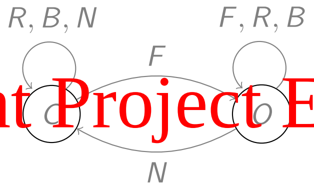


We can model the controller of the sliding door as a simple automaton:



Here we use: $C = CLOSED$, $O = OPEN$, $F = FRONT$, $R = REAR$, $B = BOTH$, $N = NEITHER$

A simple automaton–sliding door example



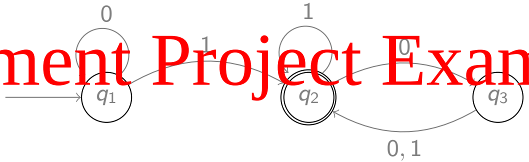
Here we use: $C = CLOSED$, $O = OPEN$, $F = FRONT$, $R = REAR$, $B = BOTH$, $N = NEITHER$

The behavior of the door can be described in terms of the following transition function:

	NEITHER	FRONT	REAR	BOTH
CLOSED	CLOSED	OPEN	CLOSED	CLOSED
OPEN	CLOSED	OPEN	OPEN	OPEN

State diagram of M_1

We can use a state diagram to describe a finite automaton M_1:



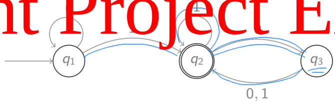
Interpretation of the state diagram: The arrow "coming out of nowhere" going into the leftmost state, signals, that this state is the start state. This automaton can read letters from the alphabet $\Sigma = \{0, 1\}$. Being in some state q, receiving letter σ, the computation finds the outgoing edge from q that has a label σ, and moves along that arrow to a new state.

Examples:

- If we feed the string 10010 to M_1, we move through the states $q_1, q_2, q_3, q_2, q_2, q_3$, and end up in state q_3, which is not an accept state.
- If we feed the string 1101 to M_1, we end up in state q_2, which is an accept state (accept states are the nodes with a double circle).
- If we feed the empty string ϵ to M_1, we end up in state q_1, which is not an accept state.

State diagram of M_1

We can use a state diagram to describe a finite automaton M_1:



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Examples:

- If we feed the string 10010 to M_1, results in q_3

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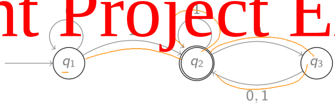

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Examples:

- If we feed the string 10010 to M_1, we move through the states $q_1, q_2, q_3, q_2, q_2, q_3$, and end up in state q_3, which is not an accept state.
- If we feed the string 110[illegible] to M_1 [illegible] is accepted.

Definition

A finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

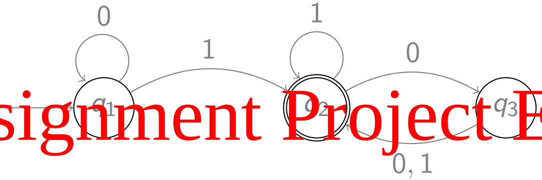
1. Q is a finite set called the set of states,
2. Σ is a finite set called the alphabet,
3. $\delta : Q \times \Sigma \to Q$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.

Formal definition of a finite

Definition

A deterministic finite automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q is a finite set called the set of states,
2. Σ is a finite set called the alphabet,
3. $\delta : Q \times \Sigma \to Q$ is the transition function,
4. $q_0 \in Q$ is the start state, and
5. $F \subseteq Q$ is the set of accept states.



The above state diagram corresponds to the following formal description: $M_1 = (Q, \Sigma, \delta, q_1, F)$, where

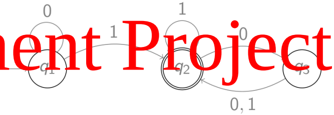
1. $Q = \{q_1, q_2, q_3\}$,
2. $\Sigma = \{0, 1\}$,
3. δ is defined by the following table:

δ	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

4. q_1 is the start state, and
5. $F \subseteq \{q_2\}$.

Given the description of an automaton, we can ask: which strings will lead to an accept state when fed into the automaton? As we have seen in the example computations with M_1 before, some strings do and others don't. The set of strings that do lead to an accept state form a language over Σ, the language of M_1.

Formal description of M_1



The above state diagram corresponds to the following formal description:
$M_1 = (Q, \Sigma, \delta, q_1, F)$, where

$Q = \{q_1, q_2, q_3\}$

$\Sigma = \{0, 1\}$

q_1 is starting state

$F = \{q_2\}$

δ	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

Language accepted by an automaton

Let Σ be the alphabet of some automaton M. Then we let

$$L(M) = \{w \in \Sigma^k \mid k \in \mathbb{N} \text{ and } w \text{ is accepted by } M\}$$

denote the language of machine M. The is $L(M)$ is the set of all words over Σ that are accepted by machine M.

For the language $A = L(M)$ we also say machine M recognizes (or accepts) A.

Definition

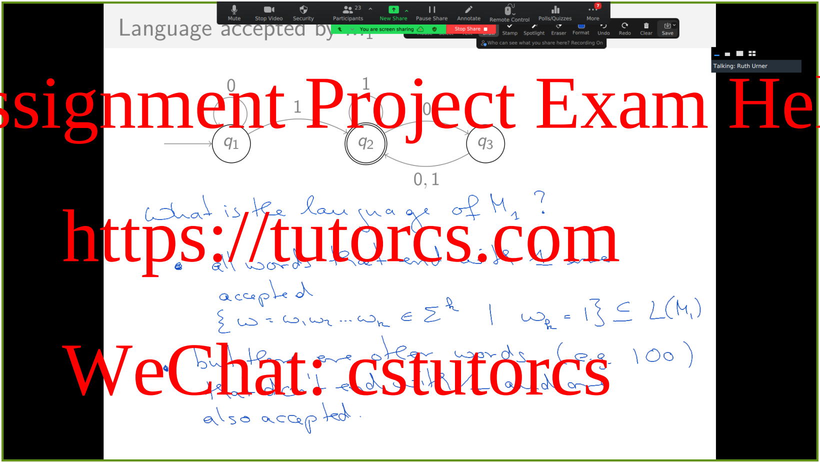
Let $M = (Q, \Sigma, \delta, q_0, F)$ a finite automaton and $\mathbf{w} = w_1 w_2 \dots w_n$ a string over Σ. We say that M **accepts** $\mathbf{w}$ if there exists a sequence $s_0 s_1 s_2 \dots s_n$ of states such that

1. $s_0 = q_0$,
2. $\delta(s_i, w_{i+1}) = s_{i+1}$ for $i = 0, 1, \dots, n_1$, and
3. $s_n \in F$.

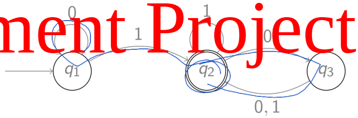
For the machine M_1 we get

$L(M_1) = \{\mathbf{w} \mid \mathbf{w}$ contains at least one 1 and the number of 0s after the last 1 is even$\}$

Task for you: convince yourself that this is exactly the set of words accepted by this automaton.


Language accepted by M_1
q_1
q_2
q_3
0
1
1
0
0, 1
What is the language of M_1?
all words that end with 1 are
accepted
$\{ w = w_1 w_2 \dots w_k \in \Sigma^k \mid w_k = 1 \} \subseteq L(M_1)$
(e.g. 100)
also accepted.
Talking: Ruth Urner

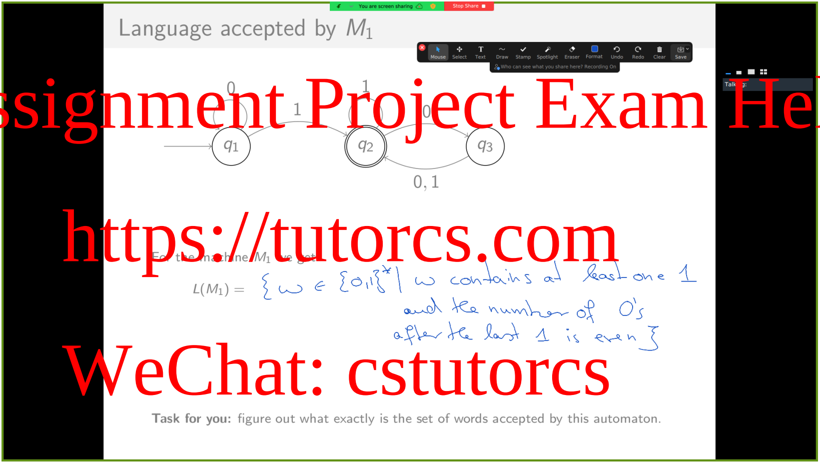
Language accepted by M_1

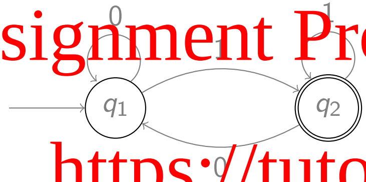


For the machine M_1 we get

$L(M_1) =$

Task for you: figure out what exactly is the set of words accepted by this automaton.

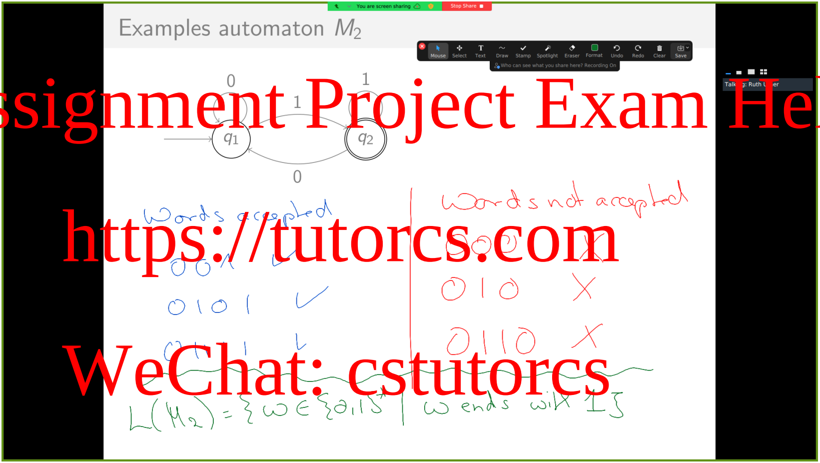

Language accepted by M_1
0
1
1
0
q_1
q_2
q_3
0, 1
For the machine M_1 we get
$L(M_1) = \{ w \in \{0,1\}^* \mid w$ contains at least one 1
and the number of 0's
after the last 1 is even$\}$
Task for you: figure out what exactly is the set of words accepted by this automaton.



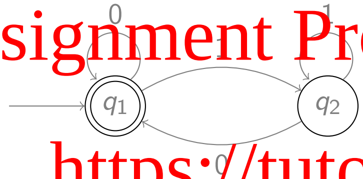
For the machine M_2 we get

$$L(M_2) = \{w \in \{0,1\}^* \mid w \text{ ends with letter } 1\}$$

Task for you: convince yourself that this is exactly the set of words accepted by this automaton.


You are screen sharing
Stop Share
Mouse Select Text Draw Stamp Spotlight Eraser Format Undo Redo Clear Save
Examples automaton M_2
0
1
1
q_1
q_2
0
Words accepted
0101
Words not accepted
010
0110
$L(M_2) = \{ w \in \{0,1\}^* \mid w \text{ ends with } 1 \}$

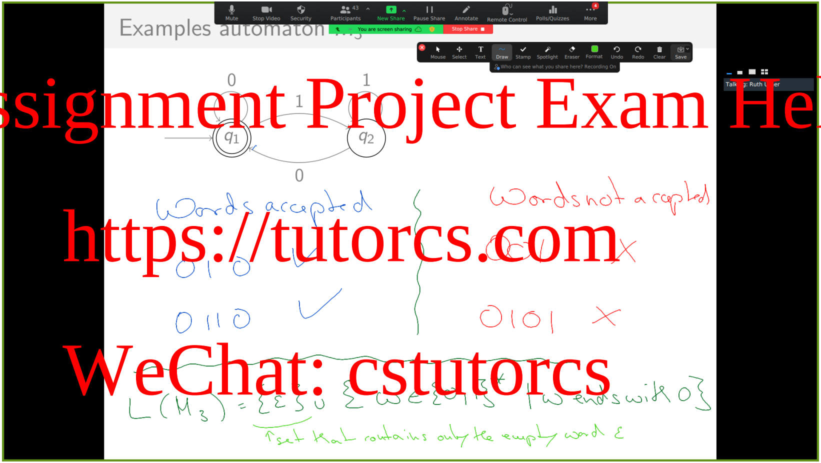
Examples automaton M_3



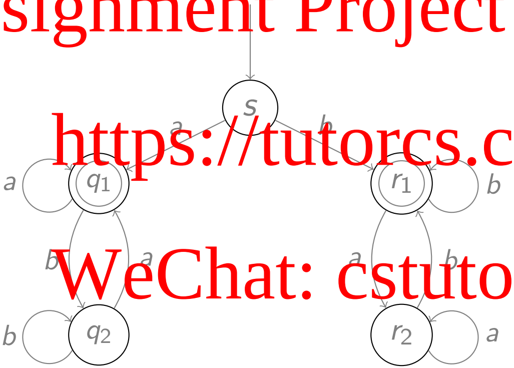
For the machine M_3 we get

$$L(M_3) = \{w \mid w \in \{0,1\}^* \text{ is the empty string or ends with letter } 0\}$$

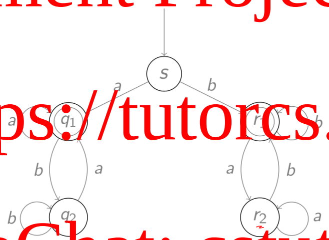
Task for you: convince yourself that this is exactly the set of words accepted by this automaton.


Examples automaton M_3
0
1
1
0
q_1
q_2
Words accepted
0110
Words not accepted
0101
$L(M_3) = \{\varepsilon\} \cup \{w \in \{0,1\}^* \mid w \text{ ends with } 0\}$
set that contains only the empty word ε

More examples: M_4



More examples: M_4



Words accepted

aabba ✓

baaabab ✓

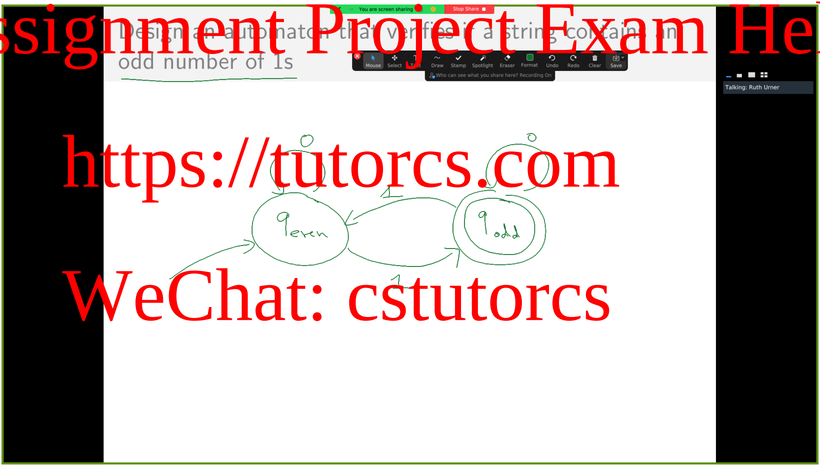
Words that are not accepted

abb X

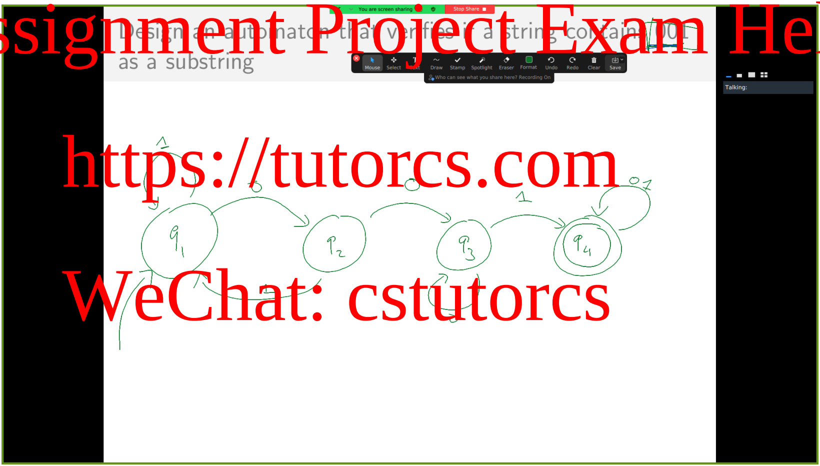
baaba X

$L(M_4) = \{ w \in \{a,b\}^* \mid w \neq \varepsilon$ and w starts and ends with the same letter$\}$

Design an automaton that verifies if a string contains an odd number of 1s



Design an automaton that verifies if a string contains 001 as a substring



Regular Languages

Reading: ITC Section 1.1

Definition

A language L over some alphabet Σ is called a regular language if there exists a finite automaton M such that $L = L(M)$, that is, if there exists a finite automaton that recognizes it.

Regular language

over some alphabet Σ

Definition

A language L is called a regular language if there exists a finite automaton M such that $L = L(M)$, that is, if there exists a finite automaton that recognizes it.

Examples of regular languages

- $\{ w \in \{0,1\}^* \mid w$ contains 001 as a substring $\}$
- $\{ w \in \{0,1\}^* \mid$ [illegible] odd $\}$

Definition:
Let A and B be languages. Then we define the following operations that each form a new language:

- **Union:** $A \cup B \;=\; \{\mathbf{w} \;|\; \mathbf{w} \in A \text{ or } \mathbf{w} \in B\}$.
- **Concatenation:** $A \circ B \;=\; \{\mathbf{wv} \;|\; \mathbf{w} \in A \text{ and } \mathbf{v} \in B\}$.
- **Star:**
  $A^* \;=\; \{\mathbf{w}_1\mathbf{w}_2\ldots\mathbf{w}_k \;|\; k \geq 0 \text{ and } \mathbf{w}_i \in A \;\forall\; i \in \{0, 1, \ldots, k\}\}$

Regular o…

In particular, for an alphabet Σ, Σ^* is the set of all finite words over Σ.

$A = \{01, 001, 0001\}$

Ex: $B = \{11, 110\}$

$A \times B \ni (01, 110)$

Definition:

Let A and B be languages. Then we define the following operations that each form a new language:

- **Union:** $A \cup B = \{\mathbf{w} \mid \mathbf{w} \in A \text{ or } \mathbf{w} \in B\}$.

Ex $A \cup B = \{01, 001, 0001, 11, 110\}$

- **Concatenation:** $A \circ B = \{\mathbf{wv} \mid \mathbf{w} \in A \text{ and } \mathbf{v} \in B\}$.

Ex $A \circ B = \{011, 0011, 00011, 0111, 00111, 000111, 01110, 001110, 0001110, 0110, 00110, 000110\}$

- **Star:**

$A^* = \{\mathbf{w}_1\mathbf{w}_2\dots\mathbf{w}_k \mid k \geq 0 \text{ and } \mathbf{w}_i \in A\ \forall\, i \in \{0, 1, \dots, k\}\}$

Ex $A^* = \{\varepsilon, 01, 001, 0001, 0101, 01001, 001001, 010001, 0001001, 001001, 0010001, 0001001, 010101, 0100101, \dots\}$

infinite set of words…

Regular operations

$\Sigma = \{0,1\}$

$A = \{01, 001, 0001\}$

Definition:

Let A and B be languages. Then we define the following operations that each form a new language:

- **Union:** $A \cup B = \{\mathbf{w} \mid \mathbf{w} \in A \text{ or } \mathbf{w} \in B\}$.
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- **Star:** $A^* = \{\mathbf{w}_1\mathbf{w}_2 \ldots \mathbf{w}_k \mid k \geq 0 \text{ and } \mathbf{w}_i \in A\ \forall i \in \{0,1,\ldots,k\}\}$

Ex $A^* = \{\varepsilon, 01, 001, 0001, 0101, 01001, 010001, 001001, 0010001, 0001001, 010101, 0100101, \ldots\}$

infinite set of words...

$A \times B \ni (01, 110)$

$B \subsetneq C^*$

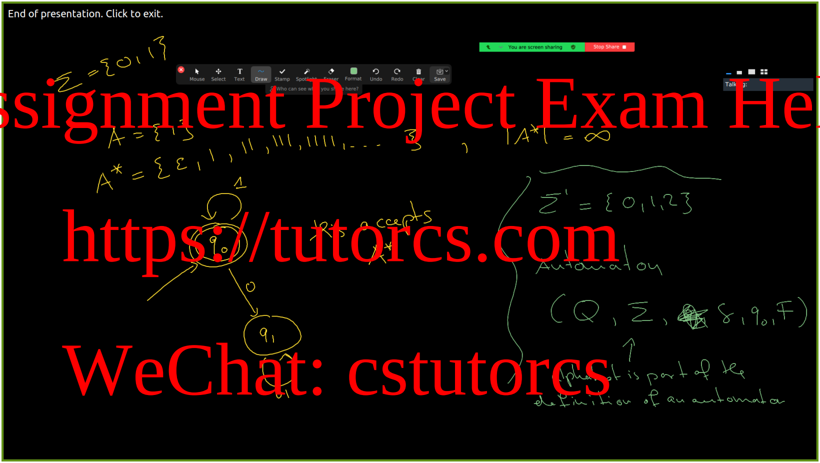
$A \subsetneq C^*$

$B \subsetneq D^*$

$A \subsetneq D^*$

$B \notin A^*$

In particular for an alphabet Σ, Σ^* is the set of all finite words over Σ.


End of presentation. Click to exit.
Σ = {0,1}
A = {1}
A* = {ε, 1, 11, 111, 1111, ... }
|A*| = ∞
accepts A*
Σ' = {0,1,2}
Automaton
(Q, Σ, δ, q0, F)
alphabet is part of the definition of an automaton