Reduction of Order Structures

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b:  y = y + 3;
c:  z = 2 * x;
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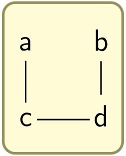

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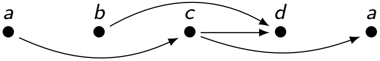

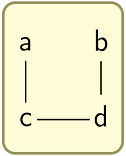

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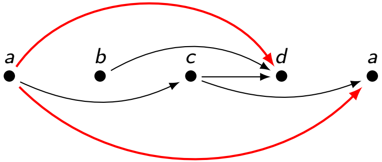

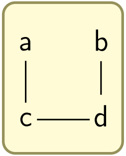

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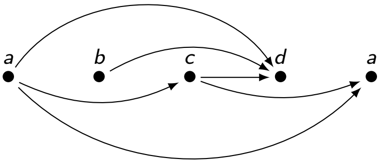

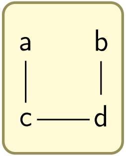

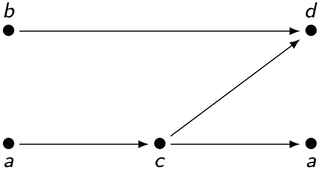

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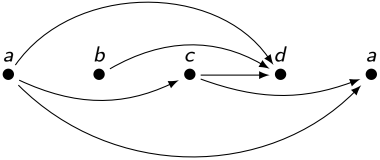

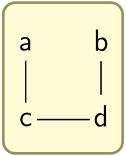

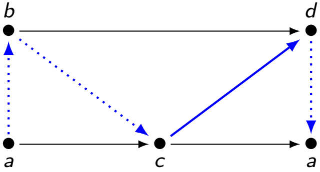

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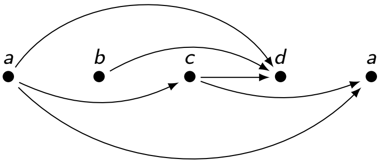

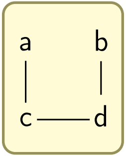


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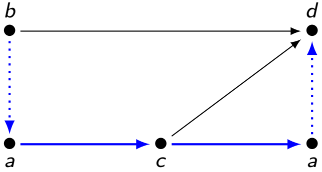
```



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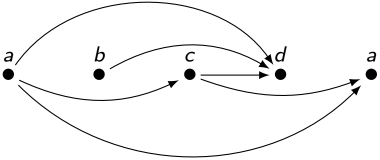

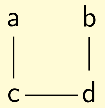


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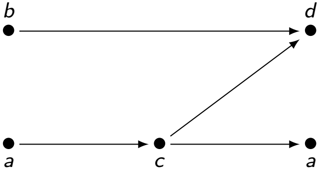

abcad
abcda
acabd
acbad
acbda
bacad
bacda

```
e: x = y;
f: y = x;
a: x = x + 1;
b: y = y + 3;
c: z = 2 * x;
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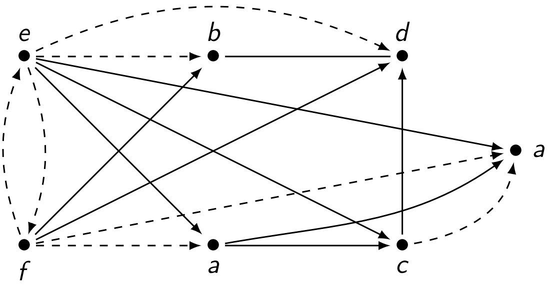
a: x = x + 1;
```

1. Mutual exclusion: ———

2. Weak causality: - - - →

3. (1) + (2): ——→



```
e: x = y;     } (e+f)
f: y = x;     }
a: x = x + 1;
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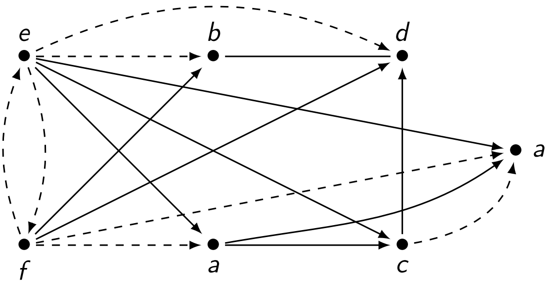

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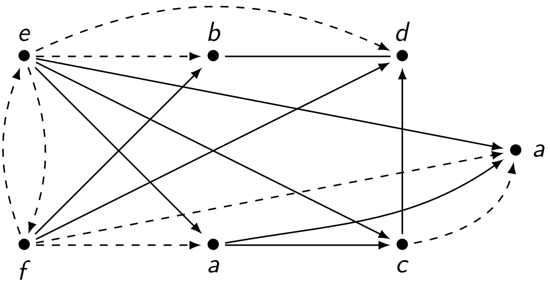
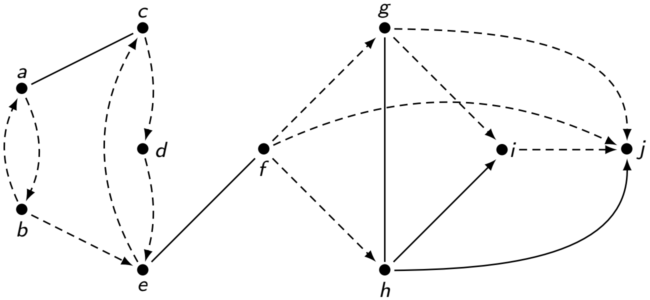


Reflexive order structure (*ros*)

A relational structure $S = (X, \rightleftharpoons, \sqsubset)$ where:

- $\rightleftharpoons \ \subset X \times X$ (*mutual exclusion*) – irreflexive and symmetric,
- $\sqsubset \subset X \times X$ (*weak causality*) – reflexive,
- $\rightleftharpoons$ and $\sqsubset$ are *separable*, i.e. $\left(\sqsubset^* \cap (\sqsubset^{-1})^* \cap \rightleftharpoons \right) = \varnothing$.

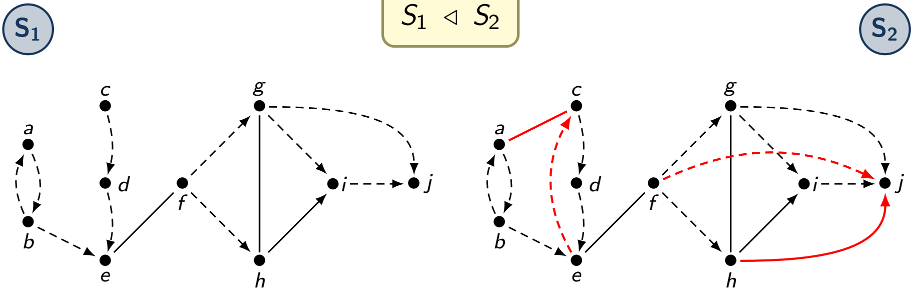


Containment relationship ($S_1 \lhd S_2$)

$S_1 = (X, \rightleftharpoons_1, \sqsubset_1)$ is contained in $S_2 = (X, \rightleftharpoons_2, \sqsubset_2)$ (or S_2 extends S_1) if:

$$\rightleftharpoons_1 \subseteq \rightleftharpoons_2 \quad \text{and} \quad \sqsubset_1 \subseteq \sqsubset_2 .$$

$S_1 \lhd S_2$



(Reflexive) generalized mutex order structure (*gmos*)

A relational structure $G = (X, \rightleftharpoons, \sqsubset)$ where $\forall_{a,b,c,d \in X}$:

G1: $a \rightleftharpoons b \implies b \rightleftharpoons a.$

G2: $a \sqsubset a \;\wedge\; a \not\rightleftharpoons a.$

G3: $a \sqsubset b \sqsubset c \implies a \sqsubset c.$

G4: $a \sqsubset b \sqsubset a \;\wedge\; a \rightleftharpoons c \implies b \rightleftharpoons c.$

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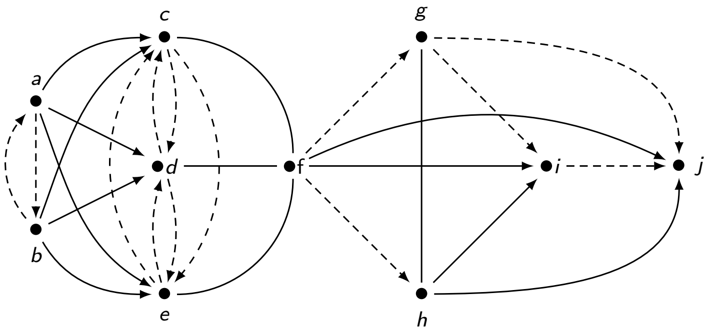

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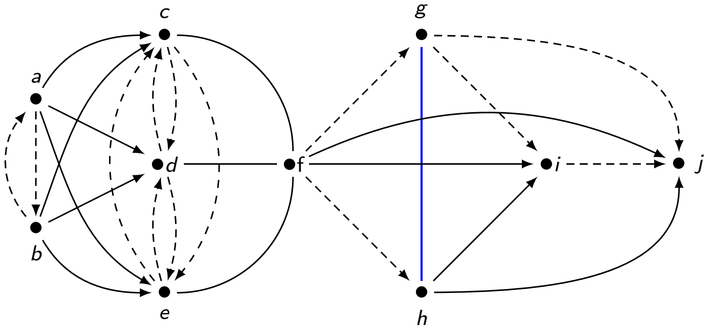

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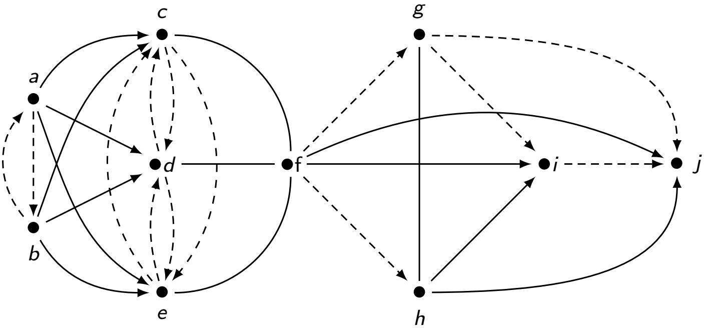

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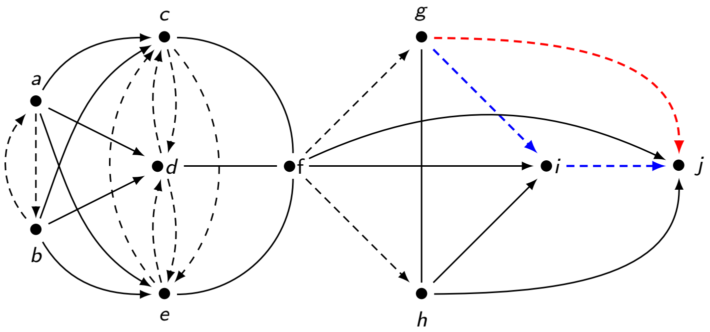

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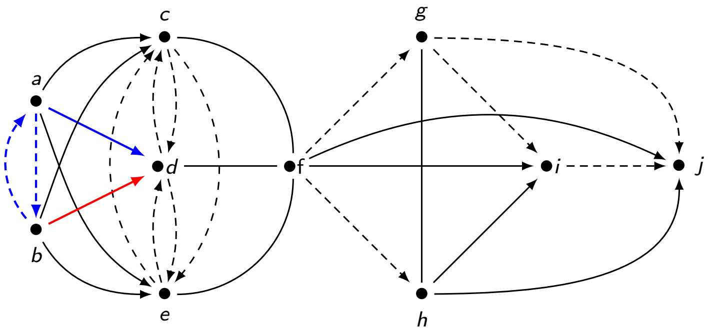

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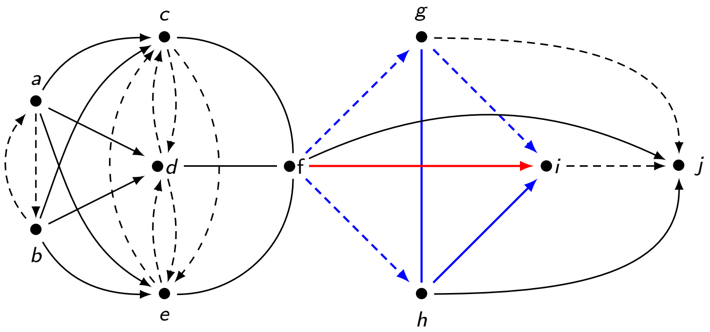

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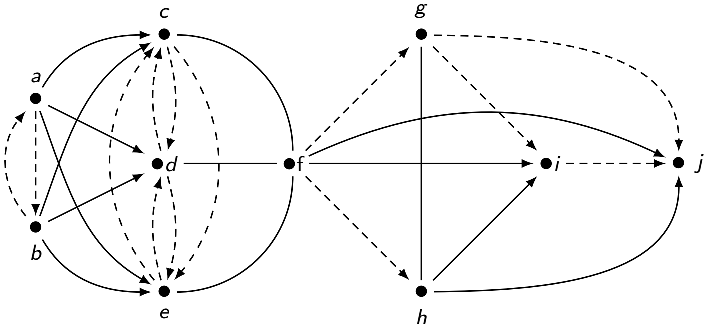
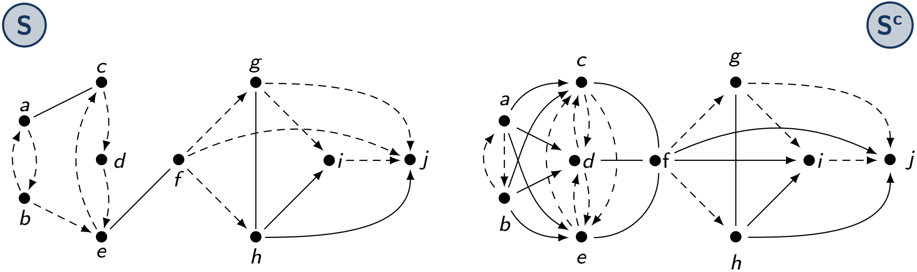


Closure of $S = (X, \rightleftharpoons, \sqsubset)$

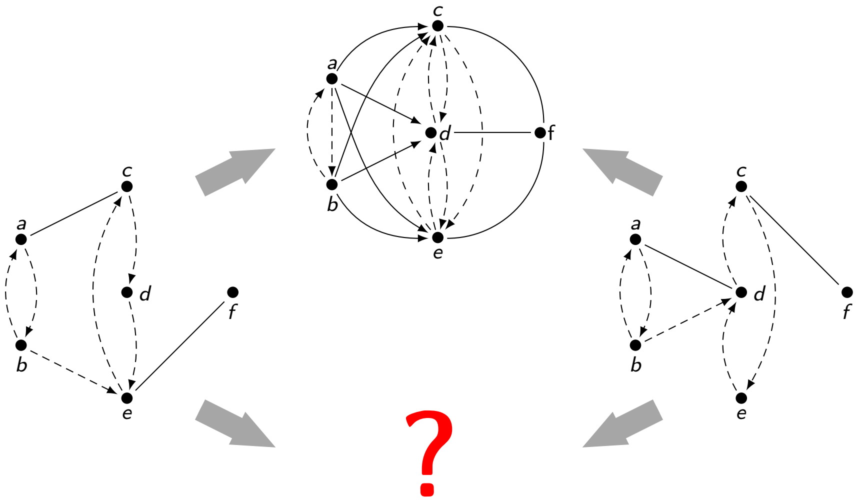
A relational structure $S^c = (X, \rightleftharpoons^c, \sqsubset^c)$ such that:

- $S \lhd S^c$,
- $S_1 \in GMOS \;\wedge\; S \lhd S_1 \implies S^c \lhd S_1$.



Equivalence relation: $S \equiv_{gmos} T \iff S^c = T^c$

How to define the reduction?



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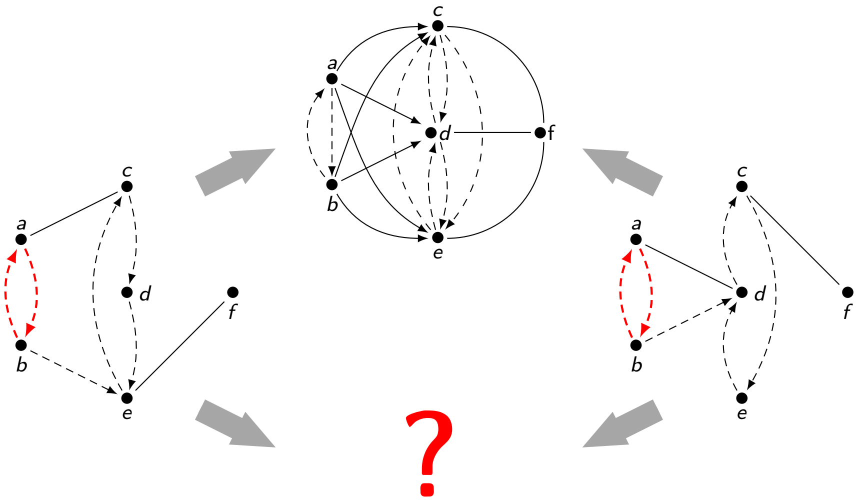

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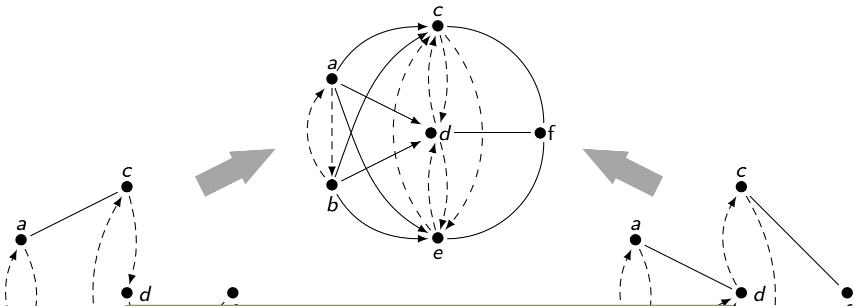
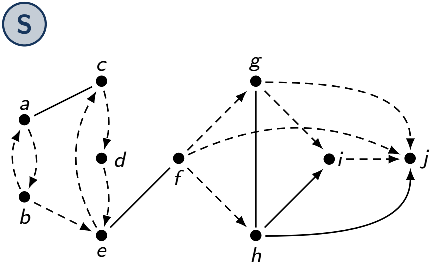


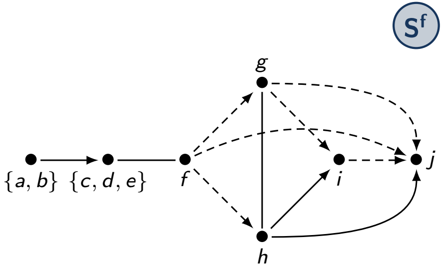
The reduction is a **unique** effect of a simplifying procedure that is an **invariant** for all structures with equal closure.

Folding of *ros* $S = (X, \rightleftharpoons, \sqsubset)$

A relational structure $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$, where:

- X^f – the set of all equivalence classes of $\sqsubset^* \cap (\sqsubset^{-1})^*$
- $\forall_{x,y \in X^f} \;\; x \rightleftharpoons^f y \iff \exists_{a \in x,\, b \in y} \; a \rightleftharpoons b$
- $\forall_{x,y \in X^f} \;\; x \sqsubset^f y \iff \exists_{a \in x,\, b \in y} \; a \sqsubset b$





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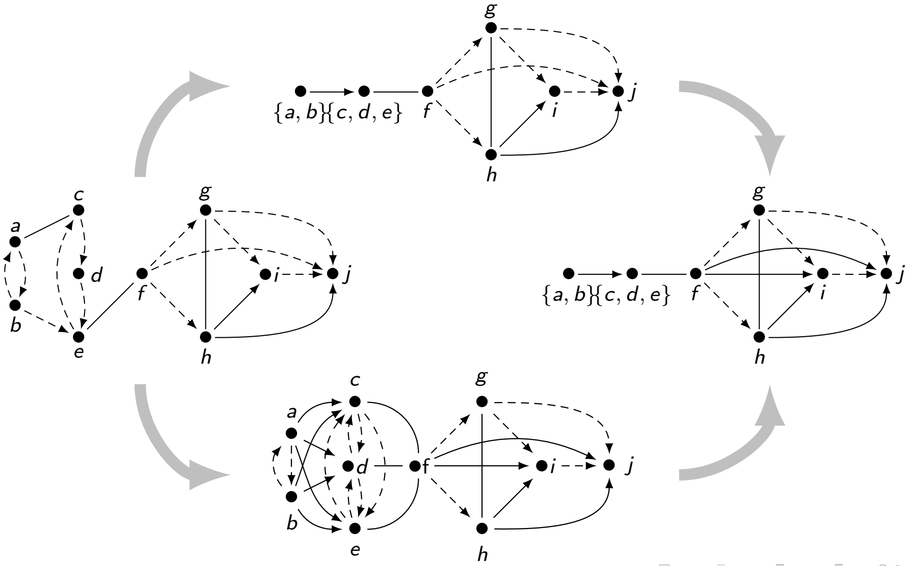
Folding of *gmos* $G = (X, \rightleftharpoons, \sqsubset)$

A relational structure $G^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$, where:

- $\forall_{a,b \in X} \;\; a \equiv_f b \iff a \sqsubset b \sqsubset a$
- $X^f = {}^{X}/_{\equiv_f}$
- $\forall_{a,b \in X} \;\; [a] \rightleftharpoons^f [b] \iff a \rightleftharpoons b$
- $\forall_{a,b \in X} \;\; [a] \sqsubset^f [b] \iff a \sqsubset b$

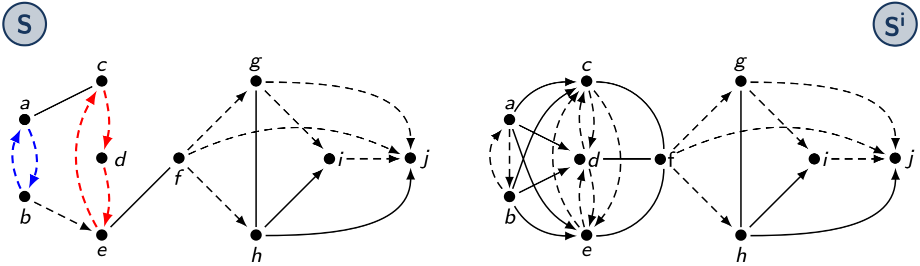
The folding and closure operations commute: $(S^c)^f = (S^f)^c$.



Integration of *ros* $S = (X, \rightleftharpoons, \sqsubset)$

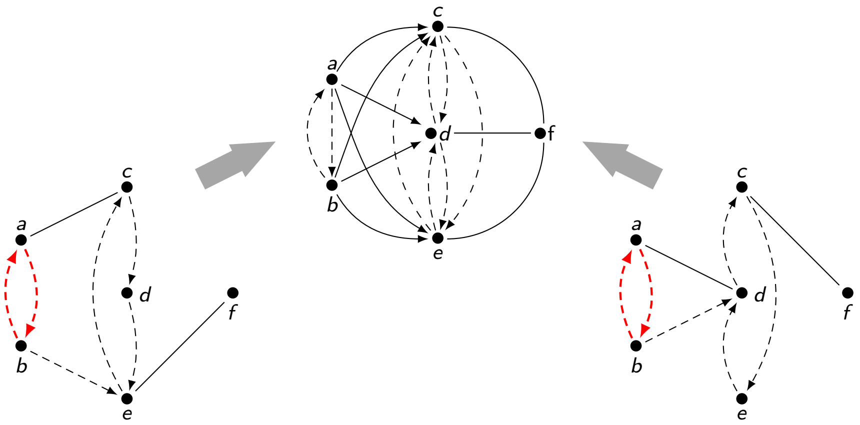
A relational structure $S^i = (X, \rightleftharpoons^i, \sqsubset^i)$, where:

- $\forall_{a,b \in X} \quad a \rightleftharpoons^i b \quad \text{if} \quad [a]_f \rightleftharpoons^f [b]_f$
- $\forall_{a,b \in X} \quad a \sqsubset^i b \quad \text{if} \quad [a]_f \sqsubset^f [b]_f$

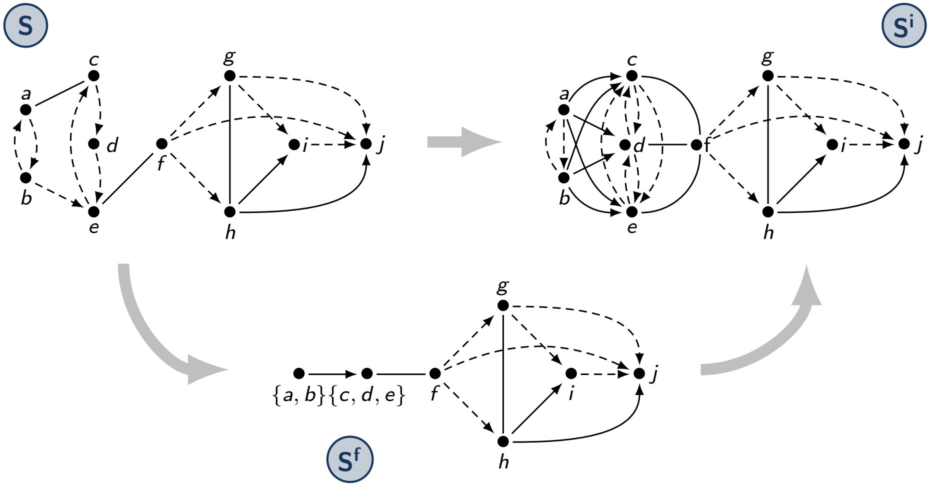


$S^c \in [S]_{gmos}$ and $S^i \in [S]_{gmos}$.

If $T \in [S]_{gmos}$ then $(S^i \cap T^i) \in [S]_{gmos}$.



$$S^f + S^{(f^{-1})} = S^i$$



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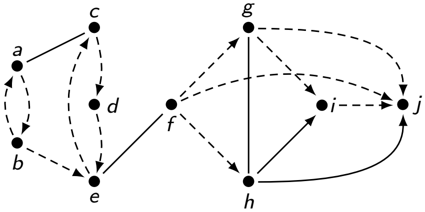
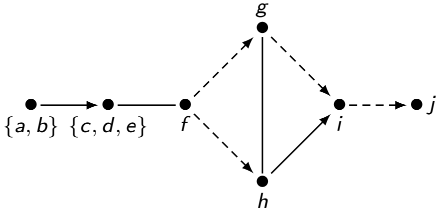
Reduction of *ros* $S = (X, \rightleftharpoons, \sqsubset)$

A relational structure $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$, which is defined as the folding of $\bigcap T^i$, where $T \in [S]_{gmos}$.

S



S^r



The reduction is a unique effect of a simplifying procedure that is an invariant for all structures with equal closure.

Reduction of *ros* $S = (X, \rightleftharpoons, \sqsubset)$

A relational structure $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$, which is defined as the folding of $\bigcap T^i$, where $T \in [S]_{gmos}$.

Reduction of *gmos* $G = (X, \rightleftharpoons, \sqsubset)$

A relational structure $G^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$ where:

- $X^r \overset{\text{df}}{=} X^f$
- $\rightleftharpoons^r \overset{\text{df}}{=} \upharpoonleft\downharpoonright^f \cup \prec^d \cup (\prec^d)^{-1}$
- $\prec^d \overset{\text{df}}{=} \prec^f \setminus \Big(\beta(\prec^f) \cup \alpha(\upharpoonleft\downharpoonright^f)\Big)$
- $\sqsubset^r \overset{\text{df}}{=} \sqsubset^f \setminus \Big((\sqsubset^f \setminus Id_{X^f}) \circ (\sqsubset^f \setminus Id_{X^f})\Big)$

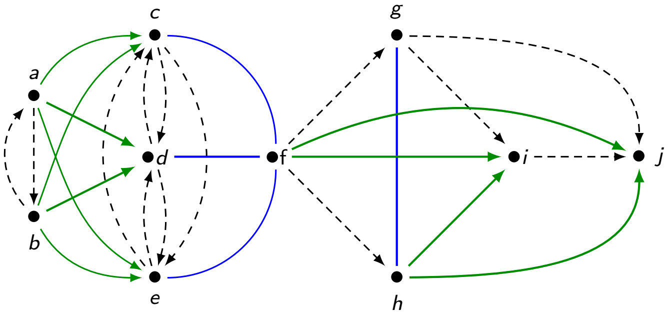
$S = (X, \rightleftharpoons, \sqsubset)$

Crossing mutexes

$\upharpoonleft\!\downharpoonright \;\overset{\mathrm{df}}{=}\; \rightleftharpoons \setminus (\prec \cup \prec^{-1})$

Aligned mutexes

$\prec \;\overset{\mathrm{df}}{=}\; \sqsubset \cap \rightleftharpoons$



! For every *gmos* $G = (X, \rightleftharpoons, \sqsubset)$: $(X, \prec)$ and $(X^F, \prec^f)$ are strict partial orders, while $(X^F, \sqsubset^f)$ is a weak partial order.

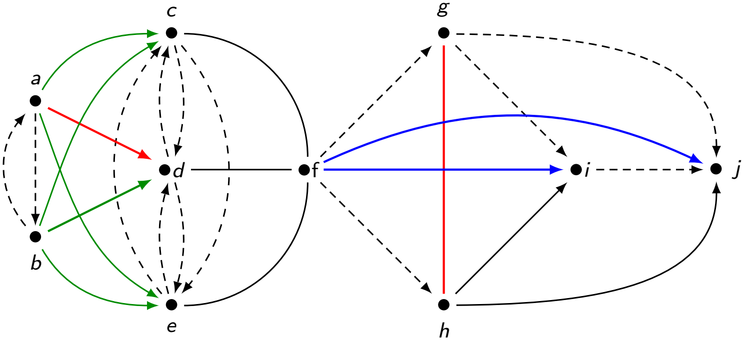
$$\bullet x = \{z \mid z \sqsubset x\} \quad \text{and} \quad x\bullet = \{z \mid x \sqsubset z\}$$

Crossing mutex $m = (x, y)$

$$\alpha(m) = (\bullet x \;\cap\; \bullet y) \times (x\bullet \;\cap\; y\bullet)$$

Aligned mutex $m = (x, y)$

$$\beta(m) = \bullet x \times y\bullet \;\setminus\; \{(x, y)\}$$



$$\bullet x = \{z \mid z \sqsubset x\} \quad \text{and} \quad x\bullet = \{z \mid x \sqsubset z\}$$

Crossing mutex $m = (x, y)$

$$\alpha(m) = (\bullet x \ \cap \ \bullet y) \times (x\bullet \ \cap \ y\bullet)$$

Aligned mutex $m = (x, y)$

$$\beta(m) = \bullet x \times y\bullet \ \setminus \{(x, y)\}$$

Proposition

$G = (X, \rightleftharpoons, \sqsubset)$ – $gmos$

$m_1 = (a, b),\ m_2 = (c, d) \in X \times X$

1. $m_1 \in \upharpoonleft\downharpoonright \wedge m_2 \in \alpha(m_1) \implies m_2 \in \prec \wedge \beta(m_2) \subseteq \alpha(m_1)$
2. $m_1 \in \prec \wedge m_2 \in \beta(m_1) \implies m_2 \in \prec \wedge \beta(m_2) \subseteq (\beta(m_1) \cup \{m_1\})$

$$\bullet x = \{z \mid z \sqsubset x\} \quad \text{and} \quad x\bullet = \{z \mid x \sqsubset z\}$$

Crossing mutex $m = (x, y)$

$$\alpha(m) = (\bullet x \ \cap \ \bullet y) \times (x\bullet \ \cap \ y\bullet)$$

Aligned mutex $m = (x, y)$

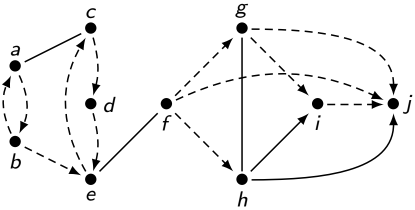
$$\beta(m) = \bullet x \times y\bullet \ \setminus \{(x, y)\}$$

Proposition

$G = (X, \rightleftharpoons, \sqsubset) - gmos$

$m_1 = (a, b),\ m_2 = (c, d) \in X^f \times X^f$

1. $m_1 \in \upharpoonleft\downharpoonright^f \wedge m_2 \in \alpha(m_1) \implies m_2 \in \prec^f \wedge \beta(m_2) \subseteq \alpha(m_1)$
2. $m_1 \in \prec^f \wedge m_2 \in \beta(m_1) \implies m_2 \in \prec^f \wedge \beta(m_2) \subseteq \beta(m_1)$

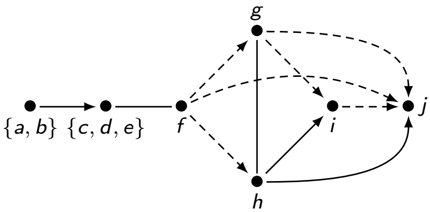


Reduction algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. Compute the folded transitive reduction $\sqsubset^{fr}$
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex reduction



Reduction algorithm

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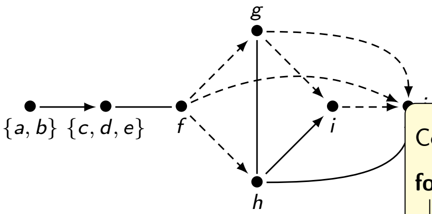

Compute X^f; $\Big\}\ O(|X| + |\sqsubset|)$

foreach $(x, y) \in \rightleftharpoons$ **do**

  add $([x]_f, [y]_f)$ to $\rightleftharpoons^f$; $\Big\}\ O(|\rightleftharpoons|)$

foreach $(x, y) \in \sqsubset$ **do**

  add $([x]_f, [y]_f)$ to $\sqsubset^f$; $\Big\}\ O(|\sqsubset|)$

Reduction algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

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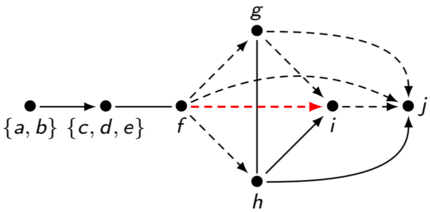

Reduction algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

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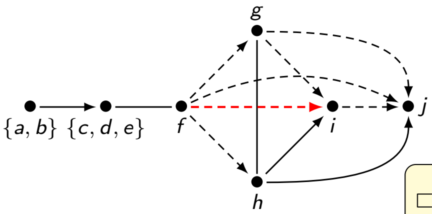


BFS of the graph $(X^f, \sqsubset^f)$

$\sqsubset^{fc} = \sqsubset^f$;

foreach $x \in X^f$ **do**
  foreach $y \in X^f$ **do**
    add (x, y) to $\sqsubset^{fc}$

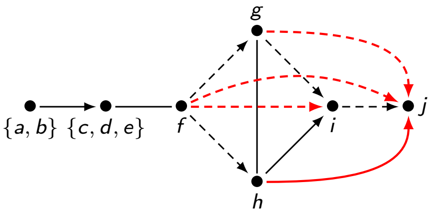
$O(|X^f| \cdot |\sqsubset^f|)$

Reduction algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. Compute the folded transitive reduction $\sqsubset^{fr}$
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex reduction

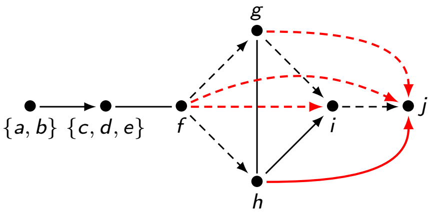


Reduction algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. Compute the folded transitive reduction $\sqsubset^{fr}$
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex reduction



Reduction algo

Input: a relational st

Output: a reduction

1. Compute the fo
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. Compute the folded transitive reduction $\sqsubset^{fr}$
4. Precompute $(\rightleftharpoons^{f} \cap \sqsubset^{fc})$
5. Compute the mutex reduction

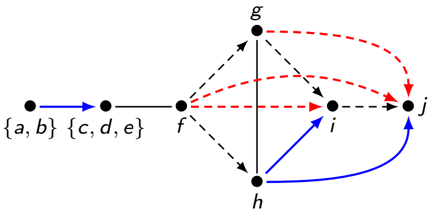
$\sqsubset^{fr} = \emptyset;$

foreach $(x, y) \in \sqsubset^{f}$ **do**

  if $\forall_{z \in X^f \setminus \{x,y\}}\ (x, z) \notin \sqsubset^{fc}$ or $(z, y) \notin \sqsubset^{fc}$ **then**

    add (x, y) to $\sqsubset^{fr}$

$O(|X^f| \cdot |\sqsubset^f|)$

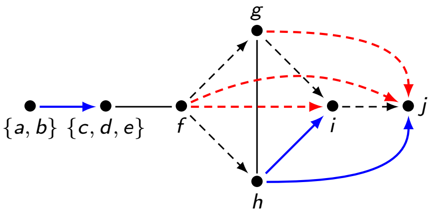


Reduction algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. Compute the folded transitive reduction $\sqsubset^{fr}$
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex reduction



Reduction algorithm

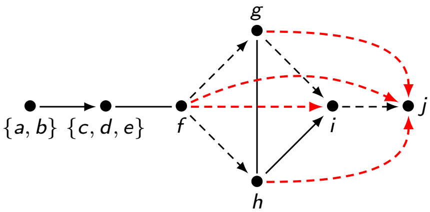
Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset$
2. Compute the folded transitive closure $\sqsubset$
3. Compute the folded transitive reduc
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex reduction

foreach $(x, y) \in \rightleftharpoons^f$ **do**
  if $(x, y) \in \sqsubset^{fc}$ **then**
    add (x, y) to $\rightleftharpoons^f \cap \sqsubset^{fc}$

$O(|\rightleftharpoons^f|)$

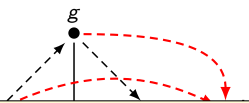


Reduction algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. Compute the folded transitive reduction $\sqsubset^{fr}$
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex reduction



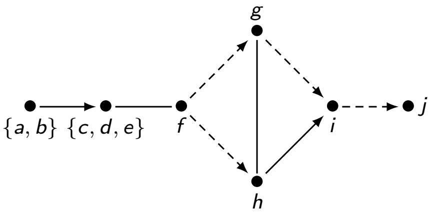
Initialize $\rightleftharpoons^{fr}$ as $\rightleftharpoons^{f}$; $\left.\right\}O(|\rightleftharpoons^{f}|)$

foreach *mutex* $(u, v) \in \rightleftharpoons^{f}$ **do**
  foreach *aligned mutex* $(x, y) \in \rightleftharpoons^{f} \cap \sqsubset^{fc}$ **do**
    if $(u, v) \in \sqsubset^{fc}$ **then**
      if $x \sqsubset^{fc} u \;\wedge\; v \sqsubset^{fc} y \;\wedge\; (u, v) \neq (x, y)$ **then**
        remove (x, y) from $\rightleftharpoons^{fr}$;
    else
      if $x \sqsubset^{fc} u \;\wedge\; x \sqsubset^{fc} v \;\wedge\; u \sqsubset^{fc} y \;\wedge\; v \sqsubset^{fc} y$ **then**
        remove (x, y) from $\rightleftharpoons^{fr}$;

$\left.\right\}O(|\rightleftharpoons^{f}| \cdot |\rightleftharpoons^{f} \cap \sqsubset^{fc}|)$

Inp
Ou

1
2
3
4. Precompute $(\rightleftharpoons^{f} \cap \sqsubset^{fc})$
5. Compute the mutex reduction



Reduction algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. Compute the folded transitive reduction $\sqsubset^{fr}$
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex reduction

$$\boxed{O\Big(|S| + |X^f| \cdot |\sqsubset^f| + |\rightleftharpoons^f| \cdot |\rightleftharpoons^f \cap \sqsubset^{fc}|\Big)}$$

Reduction algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a reduction $S^r = (X^r, \rightleftharpoons^r, \sqsubset^r)$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. Compute the folded transitive reduction $\sqsubset^{fr}$
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex reduction

Closure algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a folded closure $S^{fc} = (X^f, \rightleftharpoons^{fc}, \sqsubset^{fc})$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. ~~Compute the folded transitive reduction $\sqsubset^{fr}$~~
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex closure

Initialize $\rightleftharpoons^{fc}$ as $\rightleftharpoons^{f}$; $\Big\} O(|\rightleftharpoons^{f}|)$

foreach *mutex* $(u,v) \in \rightleftharpoons^{f}$ **do**

  foreach *aligned mutex* $(x,y) \in \rightleftharpoons^{f}$ **do**

    if $(u,v) \in \sqsubset^{fc}$ **then**

      if $x \sqsubset^{fc} u \wedge v \sqsubset^{fc} y \wedge (u,v) \neq (x,y)$ **then**

        add (x,y) to $\rightleftharpoons^{fc}$;

    else

      if $x \sqsubset^{fc} u \wedge x \sqsubset^{fc} v \wedge u \sqsubset^{fc} y \wedge v \sqsubset^{fc} y$ **then**

        add (x,y) to $\rightleftharpoons^{fc}$;

$\Bigg\} O(|\rightleftharpoons^{f}| \cdot |\sqsubset^{fc}|)$

Clo

Input: a

Output:

1. Co
2. Co
3. ~~Co~~
4. Precompute $(\rightleftharpoons^{f} \cap \sqsubset^{fc})$
5. Compute the mutex closure

$$O\Big(|S| + |X^f| \cdot |\sqsubset^f| + |\rightleftharpoons^f| \cdot |\sqsubset^{fc}|\Big)$$

Closure algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a folded closure $S^{fc} = (X^f, \rightleftharpoons^{fc}, \sqsubset^{fc})$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. ~~Compute the folded transitive reduction $\sqsubset^{fr}$~~
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex closure

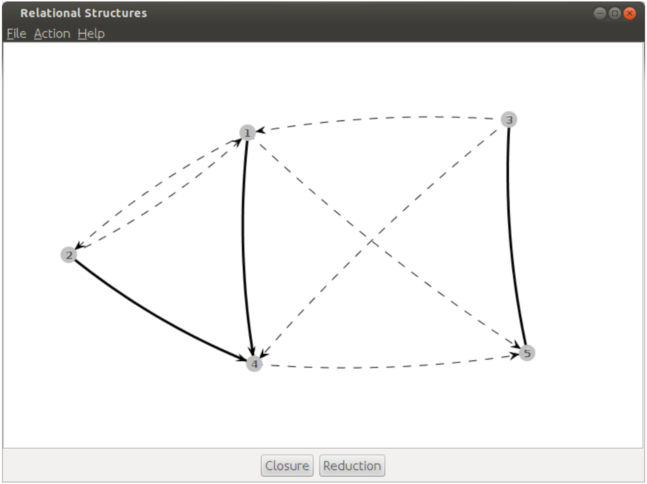
$$O\Big(|S| + |X^f| \cdot |\sqsubset^f| + |\rightleftharpoons^f| \cdot |\sqsubset^{fc}| + |X|^2\Big)$$

Closure algorithm

Input: a relational structure $S = (X, \rightleftharpoons, \sqsubset)$

Output: a closure $S^c = (X, \rightleftharpoons^c, \sqsubset^c)$

1. Compute the folding $S^f = (X^f, \rightleftharpoons^f, \sqsubset^f)$
2. Compute the folded transitive closure $\sqsubset^{fc}$
3. ~~Compute the folded transitive reduction~~ $\cancel{\sqsubset^{fr}}$
4. Precompute $(\rightleftharpoons^f \cap \sqsubset^{fc})$
5. Compute the mutex closure
6. Expand folding



http://folco.mat.umk.pl/rs/

Conclusions

- Analogue of transitive reduction for order structures
- Mutual exclusion decomposition (crossing/aligned)
- Algorithm for reduction and closure
- Software tool with the implementation

Future work

- Further optimization of the algorithm
- Deeper study the distinction between crossing and aligned mutexes

 Thank you

 Dziękuję

 Спасибо

 Ыракмат

 Gracias