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Relations: Domain and Range, Identity, and Inverse Relations

1 Introduction: Relations

- If two ordered pair of elements are written, separated by comma and enclosed by parentheses like (a, b), they form a binary relation. In a binary relation (a, b), a-coordinate is called the left component or the domain and the b-coordinate is called the right component or the range.
- Relations can be used to store information in the computer databases. Relationships between people, numbers, events, letters, sets, and many other entities can be formalized in the idea of a binary relation. It is a binary relation because it relates two objects.

Definition 3.1

A binary relation or simply a relation from a set A to a set B is a subset R of the cartesian product $A \times B$. If $(a, b) \in R$, we write aRb and say that a is related to b. On the other hand, $a \not{R} b$ or $(a, b) \not\in R$ means a is not related to b.

Definition 3.2

The Domain of R (denoted by D_R) is the set of all left components of the elements of R.

$$D_R = \{a \in A \,|\, (a,\, b) \in R \text{ for some } b \in B\} = \{a \,|\, aRb\}$$

Definition 3.3

The Range of R (denoted by R_R) is the set of all right components of the elements of R.

$$R_R = \{b \in B \,|\, (a,\, b) \in R \text{ for some } a \in A\} = \{b \,|\, aRb\}$$

Example

Suppose $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, $R = \{(1, a), (3, b), (2, b)\}$ and R is a relation from A to B since R is a subset o f $A \times B$ with respect to relation R, $1Ra$, $3Rb$ but $\cancel{1Rb}$, $\cancel{1Rc}$, $\cancel{2Ra}$, $\cancel{2Rc}$, $\cancel{3Ra}$, $\cancel{3Rc}$

Methods of Describing Relations

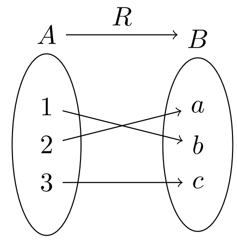
1. Relations through Ordered Pairs

$$R = \{(1, x), (2, y), (3, z), (4, w)\}$$

2. Relations through Rule Form

$$R = \{(a,\, b) \,|\, b = a^2\}$$
$$R = \{(a,\, b) \,|\, b < a + 2\}$$

3. Relation through Arrow Diagrams



4. Relations through Table

x	1	2	3
$y = f(x)$	1	4	9

5. Relation through Rectangular Array Matrix

	x	y	z
1	0	1	1
2	0	0	0
3	0	1	0

$$R = \{(1, y), (1, z), (3, y)\}$$

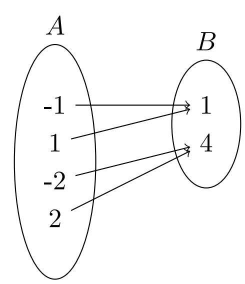
Note:

The rows of the matrix are considered as the elements of A and the columns are considered as the elements of B. Put “1” if $a \in R$ and “0” if $a \not\in R$

Example 1

Suppose the relation R is described by the diagram at the right, express R using:

1. A list of ordered pairs
2. Rule form



Solution:

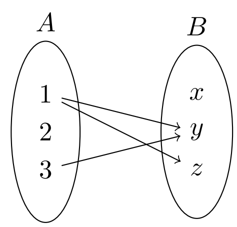
1. $R = \{(-1, 1), (1, 1), (2, 4), (-2, 4)\}$
2. In the figure, you have noticed that $(-1)^2 = 1, 1^2 = 1, (-2)^2 = 4$, and $2^2 = 4$.

   This means that number in the first set is paired with its square in the second set. Hence, $b = a^2$ and therefore we can describe the relation R as $R = \{(a, b) \,|\, b = a^2,\, a = -1,\, 1,\, -2,\, 2\}$

Example 2

Suppose $R = \{(1, y), (1, z), (3, y)\}$ is a relation from $A = \{1, 2, 3\}$ to $B = \{x, y, z\}$. Write down the elements of A and elements of B in two disjoint disks, and then draw from $a \in A$ to $b \in B$ whenever a is related to b.

Solution:



Definition 3.4

Let R be a relation from a set A to itself. A relation on the set A is a subset of $A \times A$.

Example 3

Let $A = \{1, 2, 3, 4\}$, $R = \{(a, b) \mid a \text{ divides } b\}$. Which ordered pairs are in the relation R?

Note:

"a divides b" means b can be divided by a. However, the other way around is not applicable.

Solution:
Since (a, b) is in the relation R if and only if a and b are positive integeres exceeding 4 such that a divides b. The following are the elements of $A \times A$:

$A \times A =$
{(1,1),(1,2),(1,3),(1,4),
(2,1),(2,2),(2,3),(2,4),
(3,1),(3,2),(3,3),(3,4),
(4,1),(4,2),(4,3),(4,4)}

R = {(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)}

Example 4

Suppose $A = 1, 2, 3, 4$ and the following relations from $A \times A$

$R_1 = \{(a, b) : a \geq b\}$
$R_2 = \{(a, b) : a < b\}$
$R_3 = \{(a, b) : a < b\}$
$R_4 = \{(a, b) : a + 1 = b\}$
$R_5 = \{(a, b) : a + b \geq 7\}$

Which of the following contain each of the pairs $(1, 1)$, $(2, 2)$ $(3, 3)$ and $(4, 4)$?

Solution:
The relation on a set A is simply a subset of $A \times A$. i.e.

$A \times A =$
{(1,1),(1,2),(1,3),(1,4),
(2,1),(2,2),(2,3),(2,4),
(3,1),(3,2),(3,3),(3,4),
(4,1),(4,2),(4,3),(4,4)}

2 Identity and Inverse Relation

Definition 3.5

Let A be a set. The **identity relation** is denoted by I_A, and is given by the symbols: $I_A = \{(a, a) \mid a \in A\}$

Note:

Identity relation is also called *equality* or *diagonal relation on A*. It is denoted by '=' sometimes "$\triangle_A$" or simply "$\triangle$".

Definition 3.6

Let R be a relation from A to B. The *inverse of R,* denoted by R_{-1}, is the relation from B to A given by $bR_{-1}a$ if and only if aRb. In symbols, $R^{-1} = \{(b, a) \mid (a, b) \in R\}$

Remark:

If R is any relation, then $(R^{-1})^{-1} = R$. The domain of R^{-1} is the range of R and vice versa. If R is a relation to A, i.e. R is a subset of $A \times A$, R^{-1} is also a relation on A.

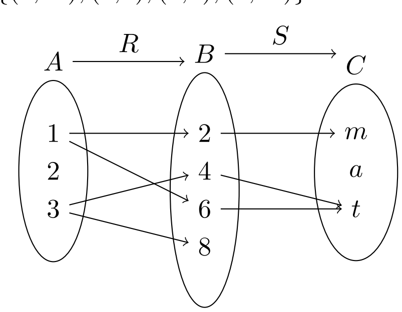
3 Composition of Relations

Definition: Let R be a relation from A to B and S be a relation from B to C. The composition of R and S, denoted by RoS, is the relation from A to C defined by: $RoS = \{(a, b) \mid (a, b) \in R \, and \, (b, c) \in S \, for \, some \, a \in A, b \in B, c \in C\}$

Example 1

Suppose $A = \{1, 2, 3\}; B = \{2, 4, 6, 8\}; C = \{m, a, t\}$; and Let $R = \{(1, 2), (1, 6), (3, 4), (3, 8)\}$ be a relation from set A to set B.
and $S = \{(2, m), (4, t), (6, t), (8, m)\}$ be a relation from set B to set C.

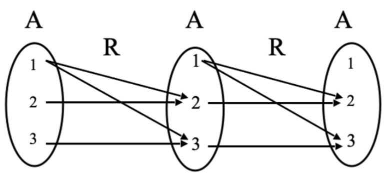
Solution:
Using the diagram the composition of
$RoS = \{(1, m), (1, t), (3, t), (3, m)\}$.



Suppose R is a relation from a set A to itself. The RoR, the composition of R to itself, is always defined, and RoR is sometimes denoted by R^2. Similarly, $R^3 = RoR = RoRoR$, and so on. ThusR is defined for all positive numbers.

Example 2

Suppose $A = \{1, 2, 3\}$, and let
$R = \{(1, 2), (1, 3), (2, 2), (3, 3)\}$ be a relation from set A to set A. Then, the composition of the relation R^3 is the set $\{(1, 2), (1, 3), (2, 2), (3, 3)\}$.



Digraph of Relations

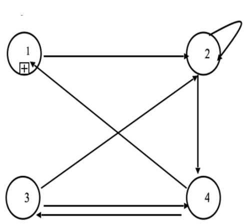
If a is a finite set and R is a relation on A, we can represent R pictorially as follows. Draw a small circle for each element of A and label the circle with corresponding element of A. These circle are called vertices, Draw a directed line, called an edge, from a_i to a_j and only if $a_i R a_j$. The resulting pictorial representation of R is called a directed graph or digraph of R. Thus, if R is a relation on A, the edges in the digraph of R correspond exactly to the pairs (a_i,a_j) in R, and the vertices correspond exactly to the elements of set A.

Note:

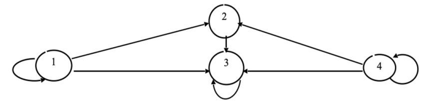
Arrangement of the vertex or vertices is arbitrary, you can also make your own composition, unless otherwise provided in the given problem.

Example 1

Let $A = \{1, 2, 3, 4\}$;
$R = \{(1, 2).(2, 2), (2, 4), (3, 2), (3, 4), (4, 1), (4, 3)\}$.
Then the digraph of R is shown below. Observe that there is an arrow from 2 to itself, it is because 2 is related to 2 under R.



Example 2

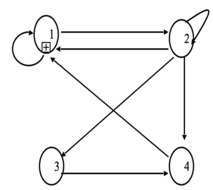


Solution:
$R = \{(1, 2).(2, 2), (1, 3), (2, 3), (3, 3), (4, 2), (4, 3), (4, 4)\}$

Definition:

If R is a relation on a set A, and $a \in A$, the in-degree of a vertex is the number of edges terminating at the vertex and the out-degree is the number of edges leaving the vertex.

Example 1

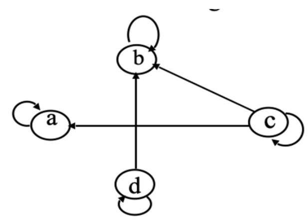


Example 2

Let $A = \{a, b, c, d\}$, and let R be the relation on A that has the matrix. Determine R, construct the digraph of R and list in-degrees and out-degrees of all vertices.

$$M_R = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{vmatrix}$$

Solution:
The digraph of $R = \{(a, a), (b, b), (c, a), (c, b), (c, c), (d, b), (d, d)\}$ and the table of the in-degrees and out-degrees is shown below:



Path of Length (n)

Definition 1

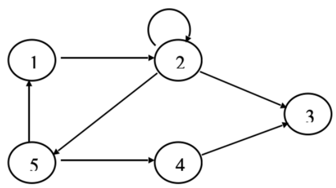
Suppose that R is a relation on a set A. A path of the length n in R from a to b is a finite sequence $\pi = a, x_1, x_2, ..., x_{n-1}, b$ beginning with a and ending with b, such that $aRx_1, aRx_2, ..., x_{n-1}Rb$.

Definition 2

A path that begins and ends at the same vertex is called a **cycle**. A path of length n involves **n+1** elements of A, although they are not necessarily distinct. Simply put, **the length of a path is the number of edges in the path,** where the vertices need not all be distinct.

Example 1

Consider the digraph below. The $\pi_1 = 1, 2, 3, 4, 5$ is a path length 4 from vertex 1 to vertex 3, $\pi_2 = 1, 2, 5, 1$ is a path length of 3 from vertex 1 to itself, and $\pi_3 = 2, 2$ is path length of 1 from vertex 2 to itself.



4 Properties of Relations

Definition 3.11

A relation R is reflexive if for every $a \in A (a, a) \in R$. A relation is not reflexive if there exists $a \in A$ such that $(a, a) \not\in R$.

Example

Consider the following relations on $\{1, 2, 3, 4\}$.

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}
R_2 = {(1, 1), (1, 2), (2, 1)}
R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}
R_4 = {(3, 1), (2, 1), (3, 2), (4, 1), (4, 2), (4, 3)}
R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}
R_6 = A x A, the universal relations

Definition 3.12

A relation R is symmetric if whenever $(a, b) \in R$, then $(b, a) \in R$. A relation is not symmetric if there exists $(a, b) \in A$ such that $(a, b) \in R$ but $(b, a) \not\in R$.

Example

Consider the following relations on $\{1, 2, 3, 4\}$.

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}
R_2 = {(1, 1), (1, 2), (2, 1)}
R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}
R_4 = {(3, 1), (2, 1), (3, 2), (4, 1), (4, 2), (4, 3)}
R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}
R_6 = A x A, the universal relations

Definition 3.13

A relation R is antisymmetric if whenever (a, b) or (b, a) belong to R then $a = b$. Otherwise, it is not antisymmetric.

Example

Consider the following relations on $\{1, 2, 3, 4\}$.

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}
R_2 = {(1, 1), (1, 2), (2, 1)}
R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}
R_4 = {(3, 1), (2, 1), (3, 2), (4, 1), (4, 2), (4, 3)}
R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}
R_6 = A x A, the universal relations

Definition 3.14

A relation R on a set A is transitive if whenever $(a, b), (b, c) \in R$ then $(a, c) \in R$. A relation is not transitive if there exists $a, b, c \in A$ such that $(a, b), (b, c) \in R$, but $(a, c) \in R$.

Example

Consider the following relations on $\{1, 2, 3, 4\}$.

R_1 = {(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)}
R_2 = {(1, 1), (1, 2), (2, 1)}
R_3 = {(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)}
R_4 = {(3, 1), (2, 1), (3, 2), (4, 1), (4, 2), (4, 3)}
R_5 = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)}
R_6 = A x A, the universal relations

5 Closure Relations

The reflexive, symmetric and transitive closures of a relation R is to be denoted by respectively by: reflexive (R), symmetric (R), and transitive (R),

Definition:

Reflexive (R) can be obtained by adding to R those identity elements (a,a) which do not belong to R.

Example 1. Suppose A ={1, 2, 3, 4} and let R = {(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3)}
Solution: *Reflexive* (R) = $R \cup I_A$
= $R \cup$ {(2, 2), (4, 4)}
= {(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3), (2, 2), (4, 4)}

Example 2. What is the *reflexive closure* of the relation $R = \{(a, b) : a < b\}$ on the set of integers?
Solution: *Reflexive* (R) is $R \cup I_A = \{(a, b) : a < b\} \cup \{(a, a) : a \in Z\}$
$= \{(a, b) : a \leq b\}$

Definition:

Symmetric (R) can be obtained by adding to R all pairs of (b,a) whenever (a,b) belong to R.

Example 1. Suppose A ={1, 2, 3, 4} and let R = {(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3)}
Solution: *Symmetric* (R) = $R \cup R^{-1}$
= $R \cup$ {(4, 2), (3, 4)}
= {(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 3), (4, 2), (3, 4)}

Example 2. What is the *symmetric closure* of the relation $R = \{(a, b) : a > b\}$ on the set of positive integers?
Solution: *Symmetric* (R) is $R \cup R^{-1} = \{(a, b) : a > b\} \cup \{(b, a) : a > b\}$

5.1 Definition:

Let R be a relation on a set A with n elements. Recall that $R^2 = RoR$ and $R^n = R^{n-1}oR$, Then, transitive (R) $-R \cup R^2 \cup ... \cup R^n$.

Example. Let R be the relation on A = { 1, 2, 3 }; and R = { (1, 2), (2, 3), (3, 3) }.
Then $R^2 = RoR$ = { (1, 3), (2, 3), (3, 3) }
and $R^3 = R^2oR$ = { (1, 3), (2, 3), (3, 3) }
Accordingly,
Transitive (R) = $R \cup R^2 \cup R^3$ = { (1, 2), (2, 3), (3, 3), (1, 3) }