Assignment 4: Discrete Fourier transform

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1 Task 1

The signals for Task 1, $x_1(t)$ and $x_2(t)$, can be seen in Equation 1 and their respective Fourier transforms can be seen in Equation 2.

$$x_1(t) = cos(2\pi \times 21 \times t), \quad x_2(t) = cos(2\pi \times 22 \times t) \tag{1}$$

$$X_1(f) = \frac{1}{2}[\delta(f+21) + \delta(f-21)], \quad X_2(f) = \frac{1}{2}[\delta(f+22) + \delta(f-22)] \tag{2}$$

2 Task 2

When sampling the signals in Equation 1 at a sampling frequency of $f_s = 128Hz$, the equation for the discrete-time sampled signals can be obtained using Equation 3. For the signals in Equation 1, $A = 1$, $\varphi = 0$, and the sampling time T_s relates to the sampling frequency as $T_s = \frac{1}{f_s}$. The discrete-time signals, $x_1[k]$ and $x_2[k]$ therefore become as can be seen in Equation 4 and are plotted in Figure 1.

$$x[k] = Acos(\omega_0 T_s k + \varphi) \tag{3}$$

$$x_1[k] = cos(\frac{2\pi 21}{128}k) = cos(\frac{21\pi}{64}k), \quad x_2[k] = cos(\frac{2\pi 22}{128}k) = cos(\frac{11\pi}{32}k) \tag{4}$$

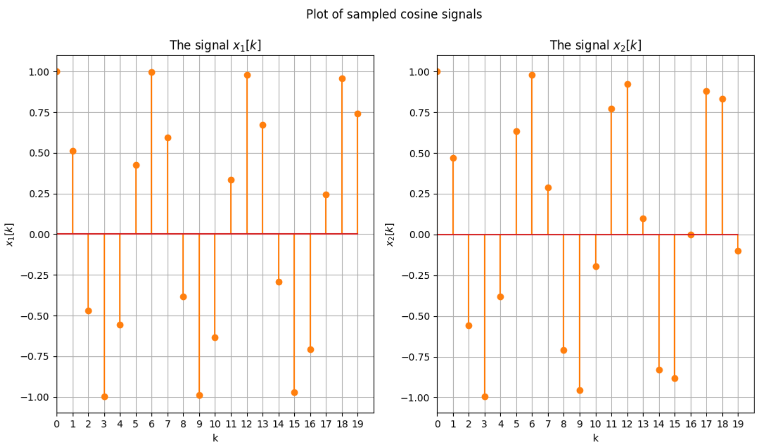

Figure 1: Plot representing 20 samples taken from the signals $x_1[k]$ and $x_2[k]$

The signals are both periodic discrete-time signals. This is because discrete-time signals are periodic when the fraction $\frac{f_0}{f_s}$ is a rational number. In this case, $\frac{21}{128}$ and $\frac{22}{128}$ are both rational numbers and therefore the signals are periodic. Their periods, K_0, are given by the relationship to the sampling and original frequencies, and thereby also to the continuous-time periods, as seen in Equation 5, where both K_0 and n must be natural numbers.

$$n = \frac{T_s}{T_0}K_0 = \frac{f_0}{f_s}K_0 \tag{5}$$

$$\frac{n}{K_0} = \frac{f_{1,2}}{128} \implies K_0(x_1) = 128 \text{ samples}, \quad K_0(x_2) = 64 \text{ samples} \tag{6}$$

As can be seen in Equation 6, the smallest natural number of the fundamental period K_0 for $x_1[k]$ is 128, with n being 21. Similarly, for $x_2[k]$, K_0 is 64 and n is 11.

This being said, even though both signals are periodic, neither sampled signal is equivalent to a discrete-time cosine signal. This is because the ratio $\frac{f_0}{f_s}$ is not a natural number (for either signal), a condition that must hold for a sampled cosine to be equivalent to a discrete-time cosine.

3 Task 3

3.1 a

The resulting spectra is a discrete spectra where $X_{1,2}(f)$ is 0 for all frequencies except one, $f = 21Hz$ and $f = 22Hz$ respectively. This corresponds to what we would expect to see from the calculation of $X_{1,2}(f)$ that were made in Task 1 in Section 1, since those calculations express dirac deltas with a weight of $\frac{1}{2}$ at those same frequencies. The best guess for the frequencies f_1 and f_2, from looking at the spectra, would therefore be $21Hz$ and $22Hz$, the frequencies at which there are non-zero components.

The reason why the estimated frequencies for $X_{1,2}(f)$ match the original frequencies of $x_{1,2}(t)$ is because K was chosen equal to L. Where K is the number of samples for each given signal as calculated in Task 2, and L is the number of frequency bins. This means that having $K = L$ is the ideal relationship to extract all information from $x_{1,2}[k]$.

3.2 b

When applying zero-padding, that is, choosing $L > K$, we add more frequency samples to the spectra which improves the graphical resolution of the DFT. Since more frequency bins are used, more points of the spectra are calculated, meaning that the result is a more accurate representation of the underlying signal, which can be seen in the difference between the padded and non-padded graphs in Figure 2.

Zero-padding does not, on the other hand, have any actual effect on the spectral resolution, that is how accurate the calculated Fourier transform is, since no new information is added. This can be concluded from the fact that the DFT is made up of a sum, to which zero-padding only adds zeros, which doesn't change the sum. This can be seen in Figure 2 as all plots align at the frequency $21Hz$ and $22Hz$, respectively, regardless of the number of frequency bins.

Furthermore, the spectral resolution is given by Equation 7, which shows that the only way to improve the spectral resolution is by increasing the number of samples K, which can be done either by changing the measurement window or increasing the sampling frequency.

$$\Delta\Omega = \frac{2\pi}{K} \tag{7}$$

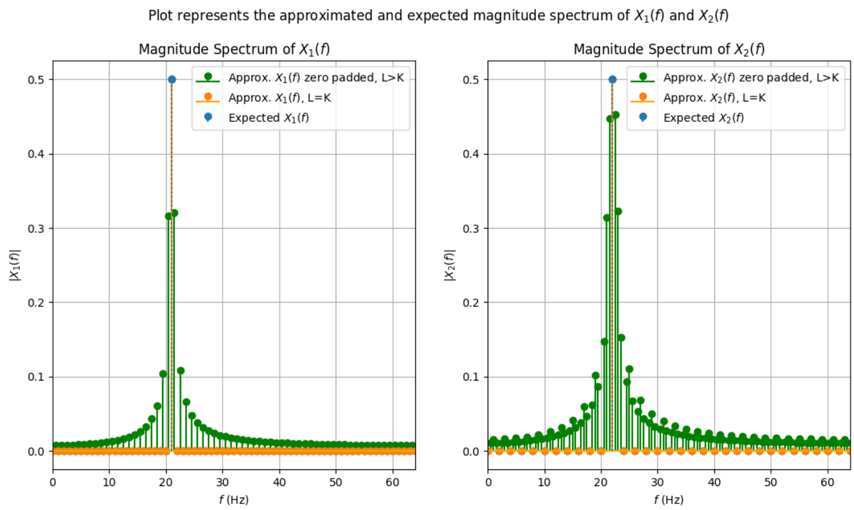

Figure 2: Plot representing the magnitude spectra of $X_{1,2}(f)$, approximated and calculated, with $L = K$ and $L = 256$

4 Task 4

4.1 a

There seems to be only one peak in the spectra, which would mean that there is one frequency component, i.e. one cosine that make up the signal $x_3[k]$. This does not align with the definition of $x_3[k]$ as the sum of two sampled cosines. The frequency of the cosine should lie around the highest peak in the spectra, which roughly speaking looks to be just above $20Hz$, as seen in Figure 3.

This frequency does roughly correspond to either of the expected frequencies, which would be the frequencies of the signals $x_1[k]$ and $x_2[k]$ that make up $x_3[k]$, that is, $21Hz$ and $22Hz$. The reason why only one frequency component seems to be visible is most likely due to smearing caused by the windowing done to obtain the DFT.

When applying zero-padding to the DFT, as discussed in Section 3.2, only the graphical resolution is improved, not the spectral one. This means that the issue caused by smearing will persist even with zero-padding. On the other hand, with zero-padding increasing the graphical resolution, the view of the underlying signal is clearer, which can be seen by comparing the two plots in Figure 3.

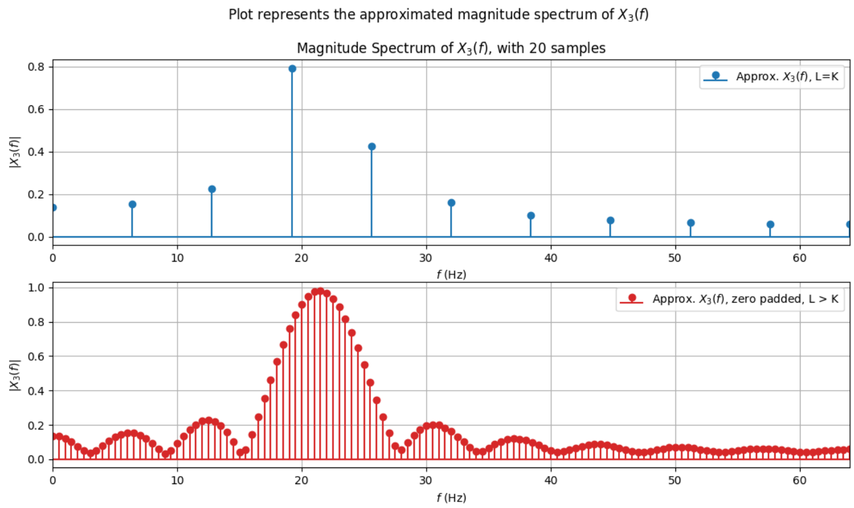

Figure 3: Plot representing the magnitude spectra of $X_3(f)$, approximated, with $L = K$ and $L = 256$

4.2 b

To improve the result, one can, as discussed in Section 3.2, either increase the number of samples or change how the windowing is done. To maximize spectral resolution, low smearing is needed, which is the result of having a narrow mainlobe. This is best achieved by using a rectangular window, as opposed to any other type of window. Using a rectangular window, the resolution can also be improved by extending the window, which would make the mainlobe more narrow. However, the mainlobe being narrower comes at the expense of higher leakage, since more sidelobes are introduced, in turn introducing new frequency components. This can have other undesired effects, but doesn't impact the spectral resolution.

Since a rectangular window was already used to achieve the result in Section 4.1, the number of frequency samples can instead be increased in an attempt to improve the spectral resolution. The effect of this can be seen in Figure 4. In this figure the number of samples was increased from 20 to 128 and the figure shows a higher spectral resolution in that the frequency peaks are more narrow and now possible to distinguish. Both of the original frequency components that make up the signal $x_3[k]$ are now visible at $21Hz$ and $22Hz$.

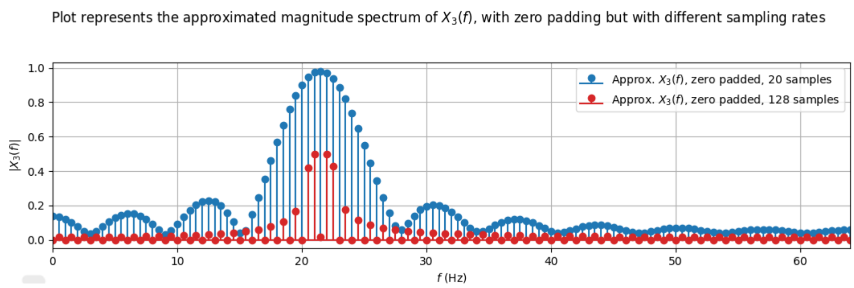

Figure 4: Plot representing the magnitude spectra of $X_3(f)$, approximated, with 20 and 128 samples, and zero-padding used