Assignment 3: Sampling and reconstruction

Group 10: Nora Reneland, Tuva Björnberg

December 2024

1 Task 1

1.1 a

The reconstructed signal does match the expectations for a cosine signal with frequency $1000Hz$, in this case $x(t) = cos(2000\pi t)$. When comparing the generated $1000Hz$ signal from the task to an actual $1000Hz$ signal they sound extremely the same.

This is because the sampling frequency f_s for a signal of this frequency ($1000Hz$) should be at least $2000Hz$, as calculated using the Nyquist-shannon sampling theorem in Equation 1, and a sampling time of $T_s = 0.1ms$ gives a sampling frequency of $10kHz$, calculated using Equation 2. The sampling frequency used is therefore higher than the minimum sampling frequency for a $1000Hz$ signal, which means no aliasing will occur and the signal will sound as expected.

$$\begin{aligned} f_b &= 1000Hz \\ f_s > 2f_b \implies f_s &> 2000Hz \end{aligned} \tag{1}$$

$$f_s = \frac{1}{T_s} \quad [Hz] \tag{2}$$

1.2 b

When decreasing the sample frequency f_s to $1100Hz$ the sampling time T_s will increase to about $0.9ms$, according to Equation 2. The samples will be fewer during the entire time span, i.e there are longer time steps between each sample. When sampling the signal with a lower frequency than the required sampling frequency of $2000Hz$, the sound becomes distorted since aliasing occurs. This means the signal is corrupted since the repeated copies of the original signal start to overlap with each other.

The reconstructed signal now has a much lower frequency because it sounds like a low electrical hum instead of the high pitch sound the $1000Hz$ played. The new frequency components of the reconstructed signal will be placed according to Equation 3. Where f_s is the sampling frequency, f_b the one-sided bandwidth frequency and f is the new frequency component, all in Hz. To find the smallest new frequency component N is an integer chosen to minimize the absolute value of f.

In this case the frequency component with the lowest frequency will be $100Hz$ (when $N = 1$), as can be seen from the calculations in 4. Since it is the lowest frequency component that decides what the signal will look like, the frequency of the reconstructed signal will also be $100Hz$.

$$f = f_s - N \times f_b \quad [Hz] \tag{3}$$

$$f = 1100 - 1 \times 1000 = 100Hz \tag{4}$$

2 Task 2

The lower bound of the sampling frequency f_s was calculated in Equation 1 to be $2000Hz$. With this it also follows that the maximum sampling time T_s will be $0.5ms$, according to Equation 2

The reconstructed signal when using the lower bound for the sampling frequency, $2000Hz$ sounds exactly the same as when using a higher frequency, like $10kHz$ in Task 1a in Section 1.1. It is possible that there is some slight distortion to this reconstructed signal due to the sampling frequency being right on the edge of what is necessary to avoid aliasing, but if there is, it's so slight that it isn't audible. However, when using a slightly lower sampling frequency, like $1999Hz$, the signal is noticeably distorted, showing that any frequency below the lower bound will cause aliasing.

When considering the sine signal $y(t) = sin(2000\pi t)$, which has the same frequency as the cosine in Task 1, the resulting signals sound the same when sampling with a frequency of $10kHz$. This is because they share the same frequency. The sinus signal is only a phase shifted version of the cosine, a phase shift that is not presented when playing the sound. A change of the amplitude in the magnitude spectra would be audibly noticeable as it controls how much the sound is amplified, but that has not been changed in this task.

However, when sampling the sinus signal with the lower bound of the sampling frequency, $2000Hz$, there is a noticeable difference between the sine and cosine. In particular, the sinus signal does not make any sound at all. This occurs because of the spacing of the sampling points.

The sampling points of the cosine coincide with the peaks and valleys of the signal, making the amplification 1, whereas for the sinus they coincide with the zero-crossings. This means that the amplitude for the sinus signal is 0 for all sampling points, that is the signal appears to be quiet at that specific point. The magnitude of a sampling point at $t = t_n$ for cosine and sine is calculated in Equation 5 and Equation 6 respectively.

$$x(t_n) = cos(2000\pi \times t_n) = cos(2000\pi \times \frac{n}{f_s}) = cos(2000\pi \times \frac{n}{2000}) = cos(n\pi) = \pm 1 \tag{5}$$

$$y(t_n) = sin(2000\pi \times t_n) = sin(2000\pi \times \frac{n}{f_s}) = sin(2000\pi \times \frac{n}{2000}) = sin(n\pi) = 0 \tag{6}$$

3 Task 3

3.1 a

To fully capture the full range of human speech, $300 - 3400Hz$, the lower bound for the sampling frequency would theoretically, according to the Nyquist-shannon sampling theorem, be twice the size of the highest frequency of interest. In this case that would be $2 \times 3400 = 6800Hz$. However, this is only the case if we assume that the spectrum of the sampled signal is always 0 above f_b. In practice this will not be the case since there will at the very least always be some noise entering the system. This means that there will always be some aliasing, unless this noise is filtered out. In practice this is done with an anti-aliasing filter.

Since an anti-aliasing filter cannot be implemented as ideal, it will have a finite roll-off. This means that there needs to be some margin between the one-sided bandwidth frequency f_b and the Nyquist frequency f_N (the maximum allowed one-sided bandwidth frequency for the chosen sampling frequency). This margin can then be used as the transition band for the anti-aliasing filter.

3.2 b

The resulting resampled sound is of worse quality than the original recording, which happens because less data is processed due to a lower sample rate. With a high sample rate, such as the original sample rate of $40kHz$, the recoding has more data to reconstruct the original sound with, compared to a low sample rate where more data between each sample is lost.

The cause of a decrease in quality of the playback sound could also be due to aliasing. If the call records signals above $4kHz$ (half of $f_s = 8kHz$) without filtering them out, these could introduce new frequency components which will cause aliasing and distort the sound.

4 Task 4

The requirements of the filter is that all frequencies below the one-sided bandwidth frequency should fall within the bandwidth of the filter, i.e. before the cut-off frequency. The cut-off frequency is defined as the frequency at which the amplitude of the signal is scaled by $\frac{1}{\sqrt{2}} \approx 0.707$. Looking at the magnitude plot of the filter in Figure 1, this frequency can be read as $3400Hz$, which is the exact f_b. This means that the filter meets the requirements precisely. The filter attenuation at the Nyquist frequency f_N can also be read from the same figure. Since $f_N = \frac{f_s}{2} = 4000Hz$, the attenuation can be read as 0.4.

The filtered version of the reconstructed signal sounds mostly the same as the unfiltered one. In theory, it should improve the quality, at least slightly, by filtering out higher frequency components. However, in this case this effect is very minor and not noticeable when playing the sounds. This could mean that the filter is not doing what is desired, or that the worsened quality stems from a different issue.

Since the aliasing problem should have been improved, the reduction of quality from the original signal that remains even after filtering, could be explained by the lower sampling frequency. There are still longer time steps between each sample and therefore less data to reconstruct the signal, which was discussed further in Section 3.2.

The filter could be further improved by increasing the roll-off rate. For a Butterworth filter this is done by designing a higher order filter, since the roll-off is reversely proportional to the order of the filter. A bigger roll-off rate would lead to a sharper separation between the pass- and stopband, and therefore less attenuation for frequencies just below $3400Hz$ and increased attenuation for higher frequencies, which would lead to more appropriate filtering.

By changing the cut-off frequency to something bigger than f_b that would also make sure that frequencies close to f_b are attenuated less. Combining this with a higher roll-off would also ensure that the higher frequencies are still attenuated to the desired extent.

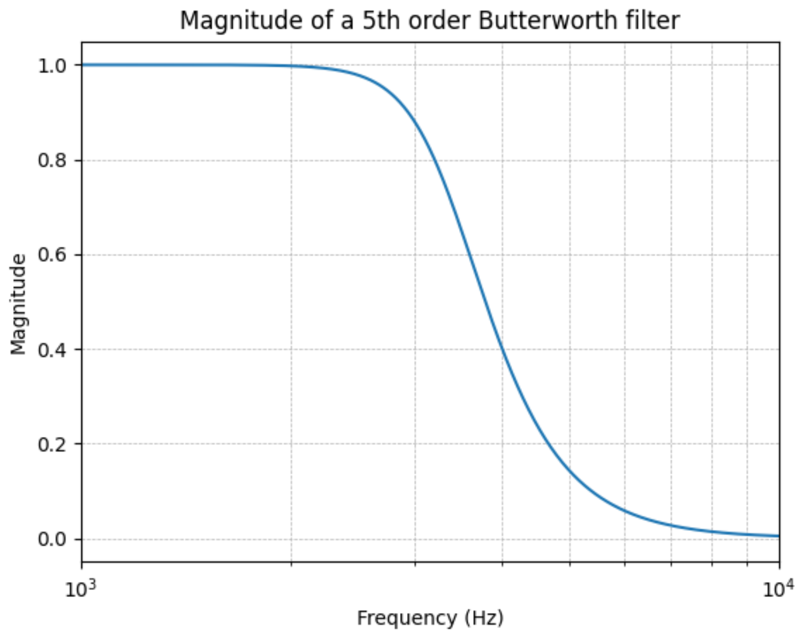

Figure 1: Plot representing the magnitude of a 5th order Butterworth filter