# Assignment 2: Fourier and Laplace transforms

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# 1 Task 1

## 1.1 1:a

We want to analyze one period of the signal, in this case for example, when $0 \leq t \leq T_0$, where the period $T_0 = 5$. Within this interval the signal x(t) can be described by two linear equations as seen in Equation 1.

$$x(t) = \begin{cases} \frac{2}{5}t & \text{for } 0 \leq t \leq 2.5, \\ \frac{2}{5}(t-5) & \text{for } 2.5 < t \leq 5. \end{cases} \tag{1}$$

Using the definition of x(t) from Equation 1 in the definition for $C_n$, which can be found in the transform table, means that $C_n$ can be described as seen in Equation 2.

$$C_n = \frac{1}{5}\left(\int_0^{2.5} \frac{2}{5}te^{-jn\omega_0 t}\,dt + \int_{2.5}^{5} \frac{2}{5}(t-5)e^{-jn\omega_0 t}\,dt\right) \tag{2}$$

When n = 0, $e^{-jn\omega_0 t} = 1$ and $C_n$ can be solved as seen in Equation 3.

$$C_n = \frac{1}{5}\left(\int_0^{2.5} \frac{2}{5}tdt + \int_{2.5}^{5} \frac{2}{5}(t-5)dt\right) = \frac{1}{5}\left(\left[\frac{t^2}{5}\right]_0^{2.5} + \left[\frac{t^2}{5} - 2t\right]_{2.5}^{5}\right) = \frac{1}{5}(1.25 - 1.25) = 0. \tag{3}$$

When n $\neq$ 0, $C_n$ can be solved for using partial integration as seen in Equation 4.

$$\begin{aligned}
C_n &= \frac{2}{25}\left(t \cdot \frac{5e^{\frac{-jn2\pi t}{5}}}{-jn2\pi} - \int_0^{2.5} \frac{5e^{\frac{-jn2\pi t}{5}}}{-jn2\pi}\,dt + (t-5)\cdot\frac{5e^{\frac{-jn2\pi t}{5}}}{-jn2\pi} - \int_{2.5}^{5} \frac{5e^{\frac{-jn2\pi t}{5}}}{-jn2\pi}\,dt\right) \\
&= \frac{2}{25}\left(\left[\frac{25e^{\frac{-jn2\pi t}{5}}}{4n^2\pi^2} - \frac{5te^{\frac{-jn2\pi t}{5}}}{jn2\pi}\right]_0^{2.5} + \left[\frac{25e^{\frac{-jn2\pi t}{5}}}{4n^2\pi^2} - \frac{(t-5)5e^{\frac{-jn2\pi t}{5}}}{jn2\pi}\right]_{2.5}^{5}\right) \\
&= \frac{2}{25}\left(\frac{25}{4n\pi}\left(\frac{e^{-jn\pi}-1}{n\pi} - \frac{e^{-jn\pi}}{j}\right) + \frac{25}{4n\pi}\left(\frac{e^{-jn2\pi} - e^{-jn\pi}}{n\pi} - \frac{e^{-jn\pi}}{j}\right)\right) \\
&= \frac{1}{2n\pi}\left(\frac{e^{-jn2\pi}-1}{n\pi} - \frac{2e^{-jn\pi}}{j}\right)
\end{aligned} \tag{4}$$

According to Euler's identity, $e^{-jn\pi}$ will equal -1 for odd n and 1 for even n. It therefore follows that $e^{-jn2\pi}$ will be 1 for all n, meaning that the first term equals 0. With further simplification, the resulting expression can be seen in Equation 5.

$$C_n = \frac{1}{2n\pi}\left(-\frac{(-1)^n 2}{j}\right) = (-1)^n j\frac{1}{n\pi} \tag{5}$$

## 1.2 1:b

The signal x(t) is a $T_0$-periodic signal, since the pattern repeats after a period $T_0$. According to the transform table, that means that the Fourier transform is given by Equation 6.

$$X(\omega) = F\{x(t)\} = 2\pi\sum_n C_n\delta(\omega - n\omega_0) \tag{6}$$

Substituting $C_n$ for its definition from Equation 5, X($\omega$) can be expressed as seen in 7.

$$X(\omega) = 2\pi\sum_n (-1)^n j\frac{1}{n\pi}\delta(\omega - n\omega_0) \tag{7}$$

## 2 Task 2

The system's transfer function can be derived from the differential equation that describes the system, seen in Equation 8, into $H(s)$, seen in Equation 9. With the transfer function $H(s)$ being the relationship between $Y(s)$ and $X(s)$.

$$\frac{d^2y(t)}{dt^2} + 2\alpha\frac{dy(t)}{dt} + \alpha^2 y(t) = \alpha^2 x(t), \qquad \alpha = 1000\pi \tag{8}$$

$$\begin{aligned} s^2Y(s) + 2\alpha sY(s) + \alpha^2 Y(s) = \alpha^2 X(s) \Longrightarrow Y(s)(s^2 + 2\alpha s + \alpha^2) = \alpha^2 X(s) \\ \frac{Y(s)}{X(s)} = \frac{\alpha^2}{s^2 + 2\alpha s + \alpha^2} = \frac{1000000\pi^2}{s^2 + 2000\pi s + 1000000\pi^2} = H(s) \end{aligned} \tag{9}$$

To obtain the frequency response from the transfer function, H(s) can be evaluated along the imaginary axis, $s = j\omega$. In practice this means substituting $s = j\omega$ for $s$ in the transfer function Equation 9 leading to the result in Equation 10, the frequency response.

This is possible because h(t) is causal and the imaginary axis lies within the region of convergence for H(s).

$$H(\omega) = \frac{\alpha^2}{-\omega^2 + 2\alpha j\omega + \alpha^2} \tag{10}$$

The system's main property is that it acts as a low pass filter. As can be seen in the magnitude spectrum of $H(\omega)$ in Figure 1, the frequency response has a higher magnitude at angular frequencies ($\omega$) closer to 0. When $\omega$ approaches 0, the magnitude approaches 1, meaning the amplitude of the original signal closely remains after passing through the system. For angular frequencies further from 0, the system will dampen the the signal, which is represented in the left plot in Figure 1 as lower values on the y-axis, approaching 0, when $\omega$ is further from 0.

The phase spectrum, seen to the right in Figure 1, indicates that the phase shift is low for frequencies close to 0, and becomes greater for frequencies further from 0. As $\omega \to \pm\infty$, the phase shift approaches $\pm\pi$. This is because when $\omega \to \pm\infty$, the real part of the denominator of $H(\omega)$ will dominate and the value is negative because of the $-\omega^2$ term, from Equation 10. In the complex plane this is located on the negative side of the real-axis. This makes the corresponding angles $\pi$ for $-\omega$ and $-\pi$ for positive $\omega$.

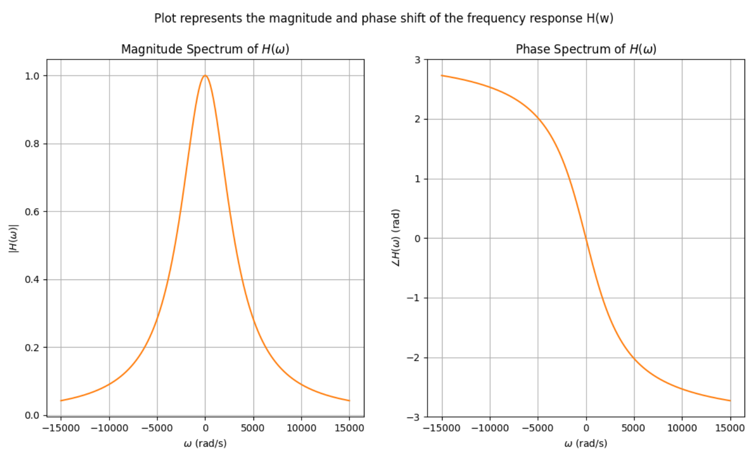


Figure 1: Plot representing the magnitude and phase of H(w)

## 3 Task 3

The Fourier transforms of the input and output signals, can be seen in Figure 2. Looking at the magnitude spectrum, the magnitude of Y($\omega$) is a a scaled version of the magnitude of X($\omega$). This happens since the magnitude of the output is calculated by multiplying the magnitudes of X($\omega$) and

H($\omega$). Since the magnitude of H($\omega$) is $< 1$ for all $\omega \neq 0$, Y($\omega$) will be a scaled down version of X($\omega$). As for the phase shift, $\angle Y(\omega)$ is the sum of the phase shift of X($\omega$) and H($\omega$), leading to a phase shift seen in Figure 2

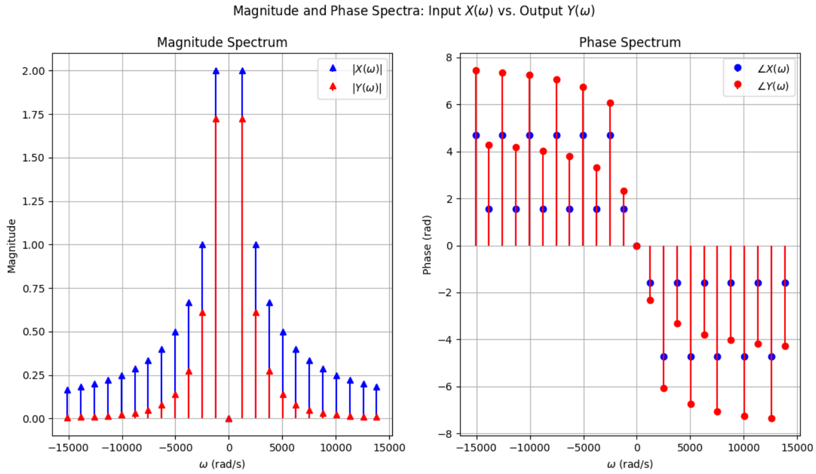


Figure 2: Plot representing the magnitude and phase of X($\omega$) and Y($\omega$)

# 4 Task 4

The resulting output signal resembles that of the input sawtooth signal. Lower N, fewer iterations, result in a more typical sinusoidal signal while a higher N results in a more sawtooth-like signal. This is because higher N introduce more harmonics, making the signal more exact. In this case, making the signal closer to a sawtooth signal.

Compared to the original sawtooth signal, which had a magnitude of 1 and 0 phase shift, the output signal has a lower magnitude and is shifted slightly, which can be seen in Figure 3. This can be explained by the low pass filter properties of the frequency response. The frequency of the input signal $x(t)$ is $\frac{1}{T_0} = 200$Hz, corresponding to $400\pi$ rad, which will dampen the magnitude by about 0.7 and shift with $-0.5$ rad, according to the frequency response 10 and Figure 1. This matches the result of the actual output signal in Figure 3.

What can also be observed in Figure 3 is the sharper peaks of $y(t)$ versus the softer turns at the minimum points. This correlates to the difference between the value of $y(t)$ at the maximum points (about 0.75), compared to the minimum points (about $-0.55$). Both of these phenomena happen because of the phase shift of the original signal introduced by the system.

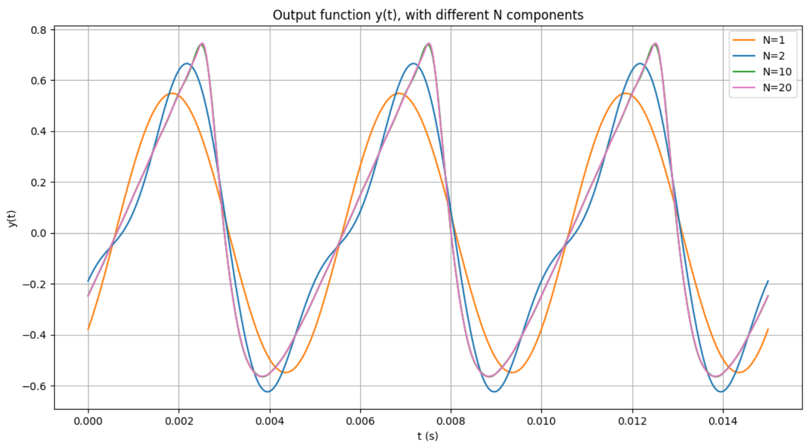


Figure 3: Plot representing the output function $y(t)$, depending of different N