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Exponential Estimates for Contagion in Financial Networks

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Abstract

In financial systems contagion occurs when the default of institutions affected by a random shock triggers a domino effect, leading to defaults of other institutions within a given set. We provide a sufficient and necessary condition for weak contagion using exponential estimates for contagion and default probabilities resulting from Sub-Gaussian shocks.

Keywords: Financial Contagion, Systemic Risk, Exponential Estimates

1. Introduction

In finance and economics, it is widely accepted that the interconnectedness of institutions, particularly banks, within a financial system is a significant factor that can potentially trigger a financial crisis. The purpose of this paper is to investigate financial contagion considering the randomness of shocks that impact the outside assets of a bank.

The loss caused by a shock to the outside assets of the ith bank is measured by a positive random variable X_i. If this loss is greater than its corresponding net worth (or equity) w_i, we say that bank i defaults due to a probability-direct shock $P(X_i \geq w_i)$. In cases where the shocks are independent, the probability that a given set of banks defaults due to simultaneous direct shocks to their outside assets is calculated as the product of the individual default probabilities of each node in the set. We refer to this as *direct default probability* of the given set of banks. Furthermore, *contagion probability* refers to the likelihood that a group of banks will default due to financial contagion originating from some bank i, which was initially affected by a shock to its outside assets.

Glasserman and Young (2015) have extensively studied these types of probability and computed an upper bound for the contagion probability. The contagion probability used with the direct default probability provided sufficient conditions for *weak contagion*. However, their calculations are strongly dependent on the knowledge of the shock distribution F. Our work presents a sufficient and necessary condition for weak contagion (definition 2) using exponential estimates for these probabilities assuming that the shock random variables X_i are Sub-Gaussian (SG) with mean μ_i and parameter σ_i.[1]

1. For the definition of SG, see Wainwright (2019), p.23 or Buldygin and Kozachenko (2000), p.3.

2. Main Results

We model a set of banks as a set $N = \{1, ..., n\}$ and denote by c_i the outside asset of the ith bank, for $i = 1, ..., n$. Assume for the shock X_i, that $X_i \in [0, c_i]$ and $c_i \leq w_i$, and define $\lambda_i = \frac{c_i}{w_i}$, while for any shock variable X_i, $\lambda_{\sigma_i} = \frac{\sigma_i}{w_i}$, with σ_i^2 the proxy variance. The two definitions and the results are given.

Definition 1 (Maximum Contagion Probability (MCP).) *Consider some bank i with net worth w_i and fix a set of banks D not containing i. Let X_i be the shock that occurs in the outside assets of the i th bank, and let $X_j = 0$ for all banks $j \neq i$. If bank i has probability of connectivity β_i, then the probability $\mathbb{P}(X_i \geq (w_i + (\frac{1}{\beta_i})\sum_{j\in D} w_j)$ is called the* maximum probability of contagion *of the event in which shock X_i occurred in bank i causes all banks in the set D to collapse.*

Definition 2 (Weak Contagion from Independent Shocks.) *We call the contagion from i to D* weak *if the following relation holds $\mathbb{P}[X_i \geq w_i + (1/\beta_i)\sum_{j\in D} w_j] \leq \mathbb{P}(X_i > w_i)\prod_{j\in D}\mathbb{P}(X_j > w_j)$. That is, it is more likely that banks in D collapse due to independent shocks rather than from contagion caused by the shock X_i.*

Theorem 3 *Fix a bank i and let D be a set of banks that do not contain i. Let X_i be a Sub-Gaussian shock with parameter σ_i and mean μ_i, i.e., $X_i \sim SG(\sigma_i)$. Set $t_{w_i} = w_i + \frac{1}{\beta_i}\cdot\sum_{j\in D} w_j$, where w_i is the net worth of the ith bank with the probability of financial connectivity β_i. Then $\mathbb{P}\Big[X_i > t_{w_i}\Big] \leq e^{-\frac{(\mu_i - t_{w_i})^2}{2\sigma_i^2}}$. a sufficient condition for weak contagion from i to D is satisfied if one of the next relations hold*

$$\mathbb{P}\Big[X_i > t_{w_i}\Big] \leq e^{-\frac{(\mu_i - t_{w_i})^2}{2\sigma_i^2}} \leq \prod_{j=1}^{d}\mathbb{P}(X_j > w_j) \leq \prod_{j=1}^{d}\Big[1 - F\Big(\frac{1}{\lambda_{\sigma_j}}\Big)\Big] \tag{1}$$

.

$$\sum_{j\in D} w_j \geq w_i\beta_i \cdot \left[\frac{\mu_i}{w_i} + \lambda_{\sigma_i}\sqrt{\sum_{j\in D}\log\frac{1}{\Big(1 - [F\big(\frac{1}{\lambda_{\sigma_j}}\big)]^2\Big)}} - 1\right]. \tag{2}$$

A necessary condition for weak contagion is satisfied if for SG shocks X_j for $j \in D$

$$\sum_{j\in D} w_j \geq w_i\beta_i \cdot \left[\lambda_{\sigma_i}\sqrt{\sum_{j\in D}\Big(\frac{1}{\lambda_{\sigma_j}} - \frac{\mu_j}{\sigma_j}\Big)^2} + \frac{\mu_i}{w_i} - 1\right]. \tag{3}$$

3. Applications and Conclusion

In Theorem 3 it is assumed that X_i for $i \in N$ are SG with mean μ_i and parameter σ_i. If X_i is bounded by c_i, σ_i can be estimated by $\frac{c_i}{2}$ (Hoeffding (1963)), while the unknown μ_i, by the methods described in Waudby-Smith and Ramdas (2024),Shafer (2021),Shafer (2019) and in the books Vovk et al. (2005), Shafer and Vovk (2019) and Shafer and Vovk (2005). These methods use the language of a game-theoretic approach to probability. We will present results for the estimation of the mean μ_i using this language in another paper .

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