Algorithm of the Ore-version of equitable coloring

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Theorem 1. *If G is a graph with $\theta(G) \leq 2r+1$, then G can be equitably colored with $r+1$ colors in polynomial time.*

Proof. Let G be a graph with $\theta(G) \leq 2r+1$ and $|G| = n$. Without loss of generality, assume $n = (r+1)s$.

First order $V(G)$ such that $d(v_i) \geq d(v_{i+1})$ for $i \in [n-1]$.

Then we can coloring $V(G)$ greedily with $r+1$ colors:

1. For vertex $v$such that $d(v) \geq r+1$, it can be colored with 1.

2. For vertex $v$such that $d(v) \leq r$, it can be colored with some color not used by its neighbors.

Let the coloring above be f.

Now define a directed graph $\mathcal{H}$, where $V(\mathcal{H})$ is the color class of f and $V_1V_2 \in E(\mathcal{H})$ if and only if there is some vertex $v \in V_1$ such that $N_{V_2}(v) = \emptyset$.

Initial set up:

Let $\mathcal{A}_0$ be the family of color class U such that $|U| \leq s-1$. Note that if $\mathcal{A}_0$=$\emptyset$, then we are done.

Let $\mathcal{B}_0$ be the family of color class U such that $|U| \geq s+1$.

Let R be the family of color class U such that $|U| = s$.

Let $\mathcal{A}$ be the family of accessible color class in R, i.e., for any $U \in \mathcal{A}$, $U \in R$ and there is a path from U to some color class in $\mathcal{A}_0$.

Let $\mathcal{B} = V(\mathcal{H}) \setminus \mathcal{A}$ and $\mathcal{D} = \emptyset$. Note that $\mathcal{B}_0 \subseteq \mathcal{B}$.

Let $A = \cup_{U\in\mathcal{A}}U$, $B = \cup_{U\in\mathcal{B}}U$, $D = \cup_{U\in\mathcal{D}}U$, $a = |\mathcal{A}|$ and $b = |\mathcal{B}|$.

Note that for any $y \in B$ and $U \in \mathcal{A}$, $d_U(y) \geq 1$ and so $d_A(y) \geq a$.

Thus, for graph $G' = G[B]$, we have $\theta(G') \leq 2r+1-2a = 2b-1 = 2(b-1)+1$.

Then we will check some color class and do the following steps based on it. After each step, we will check $\mathcal{A}_0$ and stop the algorithm if $\mathcal{A}_0$=$\emptyset$.

Step 1: Check the color classes in $\mathcal{B}_0$.

- If there is some accessible color class $Y \in \mathcal{B}_0$, then we move the witness along the path from Y to some color class in $\mathcal{A}$.

- After taking care of all color classes in $\mathcal{B}_0 \cap \mathcal{A}$, we check the color class in $\mathcal{A}$. Assume $U \in \mathcal{A}$ is not accessible now.
  - If $U \in \mathcal{B}_0$, then move U to $\mathcal{B}$.
  - If U can be reached some color class in $\mathcal{B}_0$, then move U to B.
  - Otherwise, add U to D.
- Update a and b.
- Otherwise go to step 2.

Note that after step 1, the size of all color classes in $\mathcal{A}$ is at most s.

Step 2: Check the color classes in $R \cap \mathcal{A}$.

- If there is some color class $X \in R \cap \mathcal{A}$ such that X has a path to some color class $U \in \mathcal{A}_0$ with $|U| \leq s-2$, then we move the witness along the path from X to U.
- Check the colors classes in $\mathcal{A}$. If $U \in \mathcal{A}$ is not accessible anymore, then add U to D.
- Otherwise go to step 3.

This step to make each color class in $\mathcal{A}_0$ as big as possible.

Step 3: Consider the vertices in $(A-D) \cup B$.

1. Let $\mathcal{M}(U)$ be the set of color classes in $\mathcal{A}$ that is not accessible in $\mathcal{A}$−U.
   (a) Check $|\mathcal{M}(U)|$ for all $U \in \mathcal{A} - \mathcal{D}$.
   (b) Find the color class $U \in \mathcal{A} - \mathcal{D}$ such that $|\mathcal{M}(U)|$ is minimal and define $\mathcal{A}' = \mathcal{M}(U)$. Then the color class in $\mathcal{A}'$ is terminal.
   (c) Let $A' = \cup_{U \in \mathcal{A}'} U$ and $a' = |\mathcal{A}'|$. Then for any $x \in A'$, $d_A(x) \geq a - a' - 1$.
2. Find the special y and x such that
   (a) $x \in S^y$, where S^y is the set of solo neighbors of y in $A' - D$.
   (b) Let $c_y = \max\{d_B(x)|x \in S^y\}$ and $c'_y = \{d_B(x)|x \in A' - D - S^y\}$. Then either $c_y \geq b$ or $c'_y \geq 2b+1$.

Step 3.1: Check whether x is movable. Let U_x be the color class containing x. If x is movable to some color class $X \in \mathcal{A}$, then we move witness along the path in $\mathcal{A}$−U_x from $X + x$ to some color class in $\mathcal{A}_0$ and move y to U_x. Otherwise go to step 3.2.

Step 3.2: Check the vertices in $B - y$.

1. If there is some vertex $y' \in B - y$ such that $x \in S^{y'}$ and $yy' \notin E(G)$, then we move the witness along the path from U_x to $\mathcal{A}_0$. Suppose the witness of U_x be w.

   (a) If $U_x - w$ is accessible to $\mathcal{A}_0$, then we take x outside and move the witness along the path from $U_x - w - x + y + y'$ to $\mathcal{A}_0$.

   (b) If $U_x - w$ is not accessible to $\mathcal{A}_0$, then we take x outside and move y to $U - x$.

   Keeping checking until no such y' exists and then move x to B.

2. Otherwise take x out side, move y to U_x and then move x to B.

Now check the colors classes in $\mathcal{A}$ again. If $U \in \mathcal{A}$ is not accessible anymore, then add U to D.

Update $\mathcal{A}_0$, $\mathcal{B}_0$, R, $\mathcal{A}$, and $\mathcal{B}$. Repeat Step 1, 2 and 3 until $\mathcal{A}_0 = \emptyset$. $\square$