统计方法与机器学习 作业3

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Q1 对于模型 $x_j = \alpha_1 x_1 + \cdots + \alpha_{j-1} x_{j-1} + \alpha_{j+1} x_{j+1} + \cdots + \alpha_p x_p + \varepsilon$

$SST = \sum_{i=1}^{n} (x_{ji} - \bar{x}_j)^2 = 1$ $\quad SSE = \sum_{i=1}^{n} (x_{ji} - \hat{x}_{ji})^2 = x_j'(I-H)x_j$ 其中 $H = X_j(X_j'X_j)^{-1}X_j'$

$X_j = (x_1 \cdots x_{j-1}, x_{j+1} \cdots x_p) = (X_1, X_2)$

$R_j^2 = 1 - \frac{SSE}{SST}$ $\quad \therefore \frac{1}{1-R_j^2} = (SSE)^{-1}$

$VIF_j = C_{jj} = (X'X)^{-1}_{jj}$ $\quad X = (X_1, x_j, X_2)$

$$\therefore (X'X)^{-1} = \left(\begin{bmatrix} X_1' \\ x_j' \\ X_2' \end{bmatrix} \begin{bmatrix} X_1 & x_j & X_2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} X_1'X_1 & X_1'x_j & X_1'X_2 \\ x_j'X_1 & x_j'x_j & x_j'X_2 \\ X_2'X_1 & X_2'x_j & X_2'X_2 \end{bmatrix}^{-1}$$

$\because A^{-1} = \frac{A^*}{|A|}$ 当 $j=p$ 时 $X_2 = 0$. $\quad \therefore (X'X)^{-1} = \begin{bmatrix} X_1'X_1 & X_1'x_j \\ x_j'X_1 & x_j'x_j \end{bmatrix}^{-1}$

$\therefore C_{jj} = (D - CA^{-1}B)^{-1} = (x_j'x_j - x_j'X_1(X_1'X_1)^{-1}X_1'x_j)^{-1}$

$\because SSE = x_j'(I-H)x_j = (x_j'x_j - x_j'X_1(X_1'X_1)^{-1}X_1'x_j)^{-1}$

2° ∵ 矩阵 X 的各列可以互换顺序

∴ 对于 j $\quad VIF_j = \frac{1}{1-R_j^2}$

Q2 $MSE(\hat\beta) = E((\hat\beta - \beta)'(\hat\beta - \beta)) = E[(\hat\beta_1 - \beta_1)^2 + \cdots + (\hat\beta_p - \beta_p)^2]$

$$Var(\hat\beta) = E((\hat\beta - E(\hat\beta))^2) = E(\hat\beta - \beta)^2 = E((\hat\beta - \beta)(\hat\beta - \beta)') = E\begin{bmatrix} (\hat\beta_1 - \beta_1)^2 & (\hat\beta_1 - \beta_1)(\hat\beta_2 - \beta_2) & \\ (\hat\beta_2 - \beta_2)(\hat\beta_1 - \beta_1) & (\hat\beta_2 - \beta_2)^2 & \\ & & \ddots & \\ & & & (\hat\beta_p - \beta_p)^2 \end{bmatrix}$$

$\therefore MSE(\hat\beta) = E(tr(Var(\hat\beta))) = tr(\sigma^2 (X'X)^{-1}) = \sigma^2 \sum_{i=1}^{p} \lambda_i^{-1}$

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