Problem 1. *Prove that, given any set C and a point $x \in C$, the normal cone $N_C(X) = \{g : G^T x \geq g^T y, \forall y \in C\}$ is a convex set (C not necessarily convex)*

Solution. Let $g_1, g_2 \in N_C(x)$, and since normal cones are convex cones,

$$(t_1 g_1 + t_2 g_2)^T = t_1 g_1^T x + t_2 g_2^T x \geq t_1 g_1^T y + t_2 g_2^T y = (t_1 g_1 + t_2 g_2)^T y, \ \forall \ t_1, t_2 \geq 0$$

■

Problem 2. *Prove that for any set C (convex or not), its support function $I_C^*(x) = \max_{y \in C} x^T y$ is a convex function.*

Solution. For every $y \in C, x^T y$ is a linear function of x, so I_C^* is a pointwise supremum of a family of linear functions, therefore convex. ■

Problem 3. *Consider a closed set defined by $C = \{(x,y) | y \geq \frac{1}{1+x^2}\}$ where $(x,y) \in \mathbb{R}^2$*

1. *Is the set C convex? (Hint: you can draw the set C on the plane to explain your answer.)*
2. *Is the convex hull of set C also a closed set? You can also explain your answer by the plot.*

Solution.

1. *Consider the function $f(x) = \frac{1}{1+x^2}$, the set $C =$ **epi**(f), i.e. $\forall \ y \in C, f(x) \leq y$*
   *Now consider $\frac{\partial}{\partial^2 x} f(x) = \frac{6x^2-2}{(1+x^2)^3}$, which is not strictly nondecreasing, which means $f(x)$ is not convex, implying **epi**(f) is not convex.*
2. *The closure of a the convex hull H of a closed set C will be H itself, making H closed as well.*

■

Problem 4. *Adapt the proof of log-deterainant function to show that $f(x) = tr(X^{-1})$ is convex on **dom**$f = S_{++}^n$*

Solution. Let $\lambda_1, ..., \lambda_n$ be eigenvalues of X and Y be another matrix in S_{++}^n, and its eigenvalues be $\mu_1, ..., \mu_n$. Both X^{-1} and Y^{-1} so their eigenvalues are $\frac{1}{\lambda 1}, ..., \frac{1}{\lambda n}$, and $\frac{1}{\mu_1}, ..., \frac{1}{\mu_n}$, respectively.

Consider the matrix $\theta X + (1-\theta)Y$ for $0 \leq \theta \leq 1$, which is also positive definite because $($X) and Y are positive definite, so and similar to above, the eigenvalues of $(\theta X + (1-\theta)Y)^{-1}$ are $\frac{1}{\theta \lambda_i + (1-\theta)\mu_i}$.

So

$\mathbf{tr}(\theta X + (1 - \theta Y)) = \sum_{i=1}^{n} \frac{1}{\theta\lambda_i + (1-\theta)\mu_i}$,

and

$\theta\mathbf{tr}(X^{-1}) + (1-\theta)\mathbf{tr}(Y^{-1}) = \sum_{i=1}^{n}(\theta\frac{1}{\lambda_i} + (1-\theta)\frac{1}{\mu_i})$

And given that the function $g(x) = \frac{1}{x}$ is convex in $\mathbb{R}_+$, then for some $a, b \in \mathbb{R}_+,\ 0 \leq \theta \leq 1$,

$\frac{1}{\theta a + (1-\theta)b} \leq \theta(\frac{1}{a}) + (1-\theta)(\frac{1}{b})$

Applying this to the above functions, it follows that

$\sum_{i=1}^{n} \frac{1}{\theta\lambda_i + (1-\theta)\mu_i} \leq \sum_{i=1}^{n}(\theta\frac{1}{\lambda_i} + (1-\theta)\frac{1}{\mu_i})$
■

Problem 5. *1. Prove that $f(x_1, x_2) = \frac{1}{x_1 x_2}$ on $\mathbb{R}^2_{++}$ is convex.*

2. Prove that $f(x_1, x_2) = \frac{x_1}{x_2}$ on $\mathbb{R} \times \mathbb{R}^2_{++}$ is convex.

Solution. ■

Problem 6. *Prove that $\log\left(\frac{e^x}{1+e^x}\right)$ on $\mathbb{R}$ is concave.*

Solution. ■

Problem 7. *Prove that, by applying Jensen's inequality, an arithmetic mean is greater than or equal to its geometric mean, i.e., $(\prod_{i=1}^{n} x_i)^{1/n} \leq \frac{1}{n}\sum_{i=1}^{n} x_i$ where $\boldsymbol{x} \in \mathbb{R}^{\ltimes}_{++}$*

Solution. ■

Problem 8. *Prove that $\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1$*

Solution. ■

Problem 9. *Show that the following function $f(x)$ is convex:*

$$f(x) = x^T(A(x))^{-1}x,\ \boldsymbol{dom} f = \{x | A(X) \succ 0\}$$

Solution. ■

Problem 10. *Prove that the maximum eigenvalue of a symmetric matrix is convex.*

Solution. ■

Problem 11. *Let $f(x) = \sum_{i=1}^{r} |x|_{[i]}$ on $\mathbb{R}^n$, where $|x|_{[i]}$ is the i-th largest component of teh absolute value of $|x|$, and $r \leq n$ is a positive number.*

1. *Prove that $f(x)$ is convex.*

2. *Show that $f(x) = \min_t(rt) + \sum_{i=1}^{n} \max(0, |x_i - t|)$ is convex.*

Solution. ■

Problem 12. *Let $f(\lambda) = \boldsymbol{x}^T D(\lambda)(D(\lambda) + \boldsymbol{I})^{-1}\boldsymbol{x} - \boldsymbol{1}^T\lambda$ on $\mathbb{R}^n_+$, where $D(\lambda)$ is a diagonal matrix and $\boldsymbol{1}^T = [1, 1, ..., 1]$.*

1. *Prove that $f(\lambda)$ is concave.*

2. *Find the optimal value of $f(\lambda)$.*

Solution. ■

Problem 13. *Is the function $f(x) = ln(1 + \frac{x^2}{2\tau_0^2})$, where $\tau_0 > 0$, convex?*

Solution. ■

Problem 14.

1. *Determine whether or not $\sqrt{x_1 x_2}$ is convex on $\mathbb{R}^2_{++}$.*

2. *Determine whether or not $g(\boldsymbol{y}) = \max_{x \in \mathbb{R}_{++}2} x_1 y_1 + x_2 y_2 - \sqrt{x_1 x_2}$ is convex on $\mathbb{R}^2$*

Solution. ■