Exercise 2
Celestial mechanics

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Task 1: The orbit constants

a)

Show that for the "restricted two-body problem", i.e., where only gravitational forces act upon point-like masses with one mass M much greater than the other mass, the specific angular momentum $\overrightarrow{h} = \overrightarrow{r} \times \overrightarrow{v}$ and the total energy per unit mass $\epsilon = \frac{1}{2}V^2 - \frac{\mu}{r}$ are conserved.
Solution:
We start by deriving the angular momentum:

$$\frac{d}{dt}\overrightarrow{h} = \frac{d}{dt}(\overrightarrow{r} \times \overrightarrow{v}) = \frac{d\overrightarrow{r}}{dt} \times \overrightarrow{v} + \overrightarrow{r} \times \frac{d\overrightarrow{v}}{dt}$$

And we know that $\overrightarrow{a} = \ddot{r}$, $\overrightarrow{v} = \dot{r}$ and $\overrightarrow{r} = \overrightarrow{r}$. We can change the formulation, of the equation.

$$\frac{d}{dt}\overrightarrow{h} = \dot{\overrightarrow{r}} \times \dot{\overrightarrow{r}} + \overrightarrow{r} \times \ddot{\overrightarrow{r}}$$

Than, we use that $\ddot{\overrightarrow{r}} = \frac{G(m_1+m_2)}{r^2}\frac{\overrightarrow{r}}{r}$.

$$\frac{d}{dt}\overrightarrow{h} = 0 + \overrightarrow{r} \times (-\frac{GM}{r^3}\overrightarrow{r}) = 0$$

Now we gone show that the total energy per unit mass (ϵ) is conserved as well. We start with derivation.

$$\begin{aligned}
\frac{d}{dt}\epsilon = \frac{1}{2}\frac{d}{dt}V^2 - \frac{d}{t}\frac{\mu}{r} &= \frac{1}{2}(\overrightarrow{v} \cdot \overrightarrow{v}) - \mu\frac{d}{dt}(\frac{1}{(\overrightarrow{r} \cdot \overrightarrow{r})^{1/2}}) \\
&= \frac{1}{2}(2\overrightarrow{v} \cdot \ddot{\overrightarrow{r}}) + \frac{\mu}{2(\overrightarrow{r} \cdot \overrightarrow{r})^{3/2}}\frac{d}{dt}(\overrightarrow{r} \cdot \overrightarrow{r}) \\
&= \overrightarrow{v} \cdot \ddot{\overrightarrow{r}} + \frac{\mu}{2r^3}(2\overrightarrow{r} \cdot \overrightarrow{v}) \\
&= \overrightarrow{v} \cdot (\ddot{\overrightarrow{r}} + \frac{\mu}{r^3}\overrightarrow{r}) \\
&= \overrightarrow{v} \cdot 0 = 0
\end{aligned}$$

Again we use that $\ddot{\overrightarrow{r}} = -\frac{\mu}{r^3}\overrightarrow{r}$, which we see leads the equation to **zero**.

b)

Show that the velocity of a satellite moving in an elliptical orbit is always greatest at its apogee than at its perigee. Can you relate the difference you see in the velocities at apogee and perigee to any of Kepler's laws?
Solution:
We use that the angular momentum is $h = \overrightarrow{r} \times \overrightarrow{v}$. Than re-formulate for apogee and perigee.

$$r_p v_p = r_a v_a \Longrightarrow v_p = \frac{r_a v_a}{r_p}$$

$$v_p \gg v_a$$

Since we know that r_a will always be bigger, we will have that the v_p will always be the biggest. So the velocity in perigee, is the highest which makes sense.
We can see the similar difference, by first thinking of Kepler's second law which states that a line between a sun and a planet sweeps the same area during the same time intervals. Than think of Kepler's first law that planets orbits is ellipses.

Task 2: Kepler's equation

Consider a spacecraft orbiting elliptically around Earth with a perigee radius $r_p = 7280$km, an apogee radius of $r_a = 44720$km and having an eccentricity $e = 0.72$.

a)