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THE PERONA–MALIK PARADOX*

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Abstract. The Perona–Malik equation is a formally ill-posed parabolic equation for which simple discretizations are nevertheless numerically found to be stable. After discussing the background of this paradox in computer vision, this paper shows the nonexistence of weak solutions even in those cases where computations are successful, and introduces a notion of generalized solutions for this equation, which do evolve smoothly and possess many of the features of numerical calculations.

Key words. Perona–Malik equation, computer vision, ill-posed equations, nonlinear diffusions, scale-space

AMS subject classifications. 35K55, 35D05, 35B65, 68U10, 68T10

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1. Introduction.

In a well-known paper, Perona and Malik [26] have proposed and investigated a preprocessing procedure for the segmentation of digital images, which consists in computing the solution of an initial-value problem:

$$u_t - \operatorname{div}(\rho(|\nabla u|^2)\nabla u) = 0, \qquad u(x, y, 0) = I(x, y), \tag{1}$$

where $I(x, y)$ is the gray-scale intensity of the given image. They found that their numerical scheme did not exhibit significant instabilities even in cases when the equation is not parabolic; in fact, when computed over a large enough time, the solution appears to tend to a piecewise constant solution, representing a simplified image with sharp boundaries—the very goal of such an algorithm. We will see, however, that this equation in fact has no weak solution in many examples where these computations are successful.

More detailed calculations show that slight oscillations in first derivatives can be generated, but none of the large-scale oscillations usually associated with ill-posedness appear.

The discovery of Perona and Malik (PM) is therefore that *there exist stable schemes for initial-value problems without a weak solution.*

Faced with this situation, several attempts at explaining this phenomenon, which are discussed in greater detail below, include

(1) trying to expose nonuniqueness or instability by changing the numerical scheme. As we saw, this approach is basically inconclusive.
(2) regularizing the equation by Gaussian smoothing on the derivative in the nonlinearity. This gives well-posed problems, which, surprisingly, can be related to mean-curvature evolution, but begs the question of the status of the PM equation itself.
(3) using a time-delay regularization. Well-posedness of the regularized equation is less clear, and the numerical results appear to depend on the mesh refinement procedure used to track sharp edges.

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The absence in all these attempts of any *violent* instability might raise the hope that there is a more general notion of solution for which certain backward parabolic equations have a well-posed initial-value problem. Also, the question of writing down a specific theorem showing the ill-posedness of the original (PM) equation has been raised several times; see the recent survey by Perona, Malik, and Shiota [27].

The purpose of this paper is to propose a solution to this paradox. It may be summarized as follows.

(1) There are nonexistence theorems which preclude the existence of weak solutions in cases where computations have been quite successful.

(2) The reason for nonexistence is, intuitively speaking, that because of interior regularity estimates, any slight irregularity in the initial I immediately causes the first derivative to blow up. The purpose of regularization is to slow down this instantaneous process so that it now takes an infinite time.

(3) In particular, it is not logically necessary to interpret the success of the computation as an indication that the initial-value problem for the unregularized equation possesses a *weak* solution.

(4) The stabilization of the numerical solutions, on the other hand, does suggest that the *large-time* limits of regularized equations are stationary solutions, in a generalized sense, of the PM equation.

(5) One can construct a theory of time-dependent generalized solutions which mimics the most salient results of the numerical calculation, and for which step functions are stationary solutions.

2. Background.

We discuss here various ways in which the segmentation problem leads naturally to the PM equation. In particular, it appears that the PM equation arises as a natural limiting case of apparently unrelated segmentation or preprocessing procedures, thus making the resolution of the paradox imperative. Also, this discussion will show what features should be expected in any reasonable resolution of the paradox.

2.1. Segmentation and PDE.

Recall that an image is identified with a gray-scale intensity function $I(x, y)$ which takes a discrete set of values on a discrete grid, but which one assimilates to a function defined on a rectangle in the x-y plane. We may scale the intensity range so that I varies between 0 and 1. *In particular, I and u lie between* 0 *and* 1 *in all subsequent models, and unbounded solutions or data are irrelevant.* We may use periodic or Neumann boundary conditions.

The goal is to define a *segmentation*, that is, a partitioning of the rectangle into finitely many regions in which the intensity function is nearly constant, or has a specific high-frequency pattern ("texture"). The boundaries of these regions are *edges*, along which, by definition, the intensity gradient is large. The definition of edges as regions of large intensity gradient is supported by standard psychophysical facts. If the segmentation is to be represented as a new intensity function, we see that it must live in a function space large enough to contain step functions.

However, large gradients can be created by spurious elements in the image, loosely referred to as "noise." In order to remove such noise, a widely used procedure in the last 15 years has been *Gaussian smoothing*, or convolution by a Gaussian kernel. A basic observation is that an equivalent way to perform such smoothing is to solve the heat equation, with I as initial value. The advantage of the PDE formulation is that the time parameter can be interpreted as a scale parameter: $u(x, t)$ can be thought of as a new image where information at scale t or less has been blurred, but coarser information has been kept. This procedure is to some extent consistent with some experimental evidence on the human retina. However, Gaussian smoothing, by indiscriminately killing all high frequencies, not only removes high-frequency noise but also blurs edges and destroys finer textures. Edges can also be displaced slightly. The ill-posedness of the backward problem makes it difficult to undo the effect of smoothing.

Two general procedures for constructing schemes that avoid these pitfalls are as follows.

(1) Replace the heat equation by a (possibly nonlinear) parabolic equation. This leads to the PM equation and to mean-curvature flow in particular.

(2) Since the heat equation can be interpreted as a gradient flow, where one tries to minimize the functional $\int |\nabla u|^2 \, dx \, dy$, it may be advisable to make up a more general functional, the minimum of which should represent the desired segmentation. The best-known example of this approach is the Mumford–Shah functional, which can be related to Bayesian estimation techniques. A simplified functional is the total variation (Osher and Rudin [25]); many variations of these functionals were proposed.

A recent reference for the variational approach is [23]. See also, on the above developments, [1, 2, 4, 9, 10, 12, 13, 17, 18, 22, 21, 24, 26, 28, 30].

2.2. The Perona–Malik equation. The first modification of the heat equation [26] consists of assuming that the equation takes the form

$$u_t - \operatorname{div}(\rho \nabla u) = 0,$$

where ρ is small near edges, so as to minimize the smearing effect found in the heat equation. One might at first try to take $\rho = (1 + |\nabla I|^2)^{-1}$ or any similar expression which is small when the intensity gradient is large. But it seems more efficient to update ρ dynamically by letting $\rho = (1 + |\nabla u|^2)^{-1}$. The problem is now apparent: upon expansion of the divergence, new second-derivative terms arise and, as we will show, actually preclude the existence of a weak solution whenever the data are not infinitely smooth.

On the other hand, intuitive arguments suggest that such an equation precisely effects a segmentation, in the sense that high gradient regions are enhanced, and small gradient regions become more uniform. Above all, computations are not unsuccessful. In [26, 21], one can see how the simplest implementations of the equation give rise to roughly acceptable segmentations.

Since this is the main point of the paradox, let us describe the upshot of these numerical calculations in more detail, both for the PM equation and for regularizations described below.

First, as we noted, the algorithm gives acceptable results for many real images.

There is, however, a typical instability, which can be made to occur under special conditions, but which seems unobtrusive in real images: *staircasing*. It is most easily explained in one dimension: an increasing solution can develop new near-constant regions ("steps") without ceasing to be nondecreasing. This means that u_x remains nonnegative but develops spikes. After letting the program run for a while, these "steps" merge, and one is left with a step function with a small number of jumps. The actual number of jumps in the final state seems to depend on the numerical procedure used to track large gradients. This is natural, since a fixed mesh implicitly imposes a gradient bound. Examples of significantly differing solutions with nearly identical initial data have also been reported. Similar results, where the final state is attained more slowly, have also been reported (Perona and Malik, Mumford, Nitzberg, and Shiota).

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Thus, it appears that the computation (which by definition effects a modification of the PM equation) first breaks up the data into a very fine staircase in the regions of high gradient and then resolves it into a particular segmentation, which depends on the details of the scheme under consideration.

In order to make the procedure more useful, one would like to make this effect fully predictable, so that the effects of the scheme may be separated from those intrinsic to the PDE.

2.3. Earlier attempts at explaining the paradox. There are essentially three types of procedures that have aimed at extracting information from the PM equation.

The first consists of a spatial regularization of the derivative term, whereby $\rho(|\nabla u|^2)$ is replaced by $\rho(|G_\sigma * \nabla u|^2)$, where G_σ is a Gaussian with small variance. This leads, as Catté et al. have shown, to a well-posed model [3]. Indeed, the reason for the nonexistence of weak solutions lies precisely in the poor regularity properties of the diffusion term. However, the reintroduction of the Gaussian smoothing is unsatisfactory, since it seems to defeat the purpose of the PM equation in the first place. Numerical implementation is satisfactory. The success of this regularization suggests that the edge sharpening properties of the PM equation are strong enough to withstand this type of regularization. Other possibilities, such as fourth-order regularizing terms, will not be considered here.

A second approach is time-delay regularization, where one replaces $|\nabla u|^2$ by an average of its values from 0 to t. It is sometimes further combined with Gaussian smoothing as well. This regularization has the advantage that if no Gaussian is introduced at all, this equation does admit step functions as stationary generalized solutions as defined later in this paper, since u_x is then independent of time. The results seem to depend on the type of mesh refinement adopted. However, the basic features of staircasing and merging are always present, with the difference that the steepening of the solution seems to be delayed by the regularization.

The third approach, which clearly follows from the other two, is to ask whether the existence of well-behaved regularizations that are arbitrarily close to the PM equation does not mean that there is a more general concept of a solution, of which the regularized solutions are approximations in a suitable sense. This seems to be in keeping with the way weak solutions emerged in the case of shock waves, for example, although we will see that the concept of weak solution is not appropriate for the PM equation. Several authors asked for a precise nonexistence theorem that would delineate what this new concept should be.

2.4. Mean curvature flow. Another natural reaction at this point might be to abandon the PM equation as unreasonable, since after all it was not derived, apparently, by the modeling of an actual physical process but by a thought experiment and appears to lead to paradoxical results.

However, equations of this type arise very naturally even without any preconceived desire to introduce nonlinear diffusion. One example occurs in the Nitzberg–Shiota–Mumford algorithm for segmentation and depth analysis. Another very striking remark, due to Mumford, connects the PM equation to our recent modification of the mean-curvature scheme for edge detection. We describe it in some detail since its result is quite unexpected.

First, we must recall that a modification of the heat equation consists of replacing it by the level set evolution for motion by mean curvature, namely:

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$$u_t = \sum_{i,j} \left(\delta_{ij} - \frac{u_i u_j}{|\nabla u|^2} \right) u_{ij}.$$

The point is that the gradient of u is now a characteristic direction, suggesting that no blurring across curves (or, in higher dimensions, hypersurfaces) $\{u = C\}$ can take place. In fact, each of the level sets of u moves with a normal speed proportional to its (mean) curvature whenever this makes sense, as was shown in detail by Evans and Spruck [8]; see also Chen, Giga, and Goto [7].

Unfortunately, this evolution by itself, if it does make level sets more regular, also shrinks them to a point, and Malladi, Sethian, and Vemuri [20] and Caselles et al. [5] proposed to replace this equation by

$$u_t = \sum_{i,j} \phi \left(\delta_{ij} - \frac{u_i u_j}{|\nabla u|^2} \right) u_{ij},$$

where $\phi = \phi(x)$ is a nonnegative, feature-dependent function that should vanish near places of interest, thereby stopping the evolution. For segmentation, a natural choice is $\phi = 1/(1 + |G_\sigma * \nabla I|^2)$.

As is well known, this evolution is especially convenient to find one particular contour in an image, by viewing it as a particular level set of the function u; see [14] for background information.

We suggested [14] that a better model would be obtained by considering mean-curvature flow with respect to the conformally Euclidean metric $\phi^2(dx^2+dy^2)$, which is naturally associated with this problem. This has the effect of changing the equation to

$$u_t = \sum_{i,j} \phi \left(\delta_{ij} - \frac{u_i u_j}{|\nabla u|^2} \right) u_{ij} + \sum_i \phi_i u_i.$$

We have analyzed this equation in some detail in [14] and proved that it has viscosity solutions, and that the level sets do approach the set $\phi = 0$ under appropriate conditions (see also [14] for the relation of these models to phase transitions). This equation was also considered in [6].

Now it turns out that this conformal model can be viewed as a regularization of an equation very similar to the PM equation.

Indeed, assume that instead of having, as above, $\phi = \rho(|G_\sigma * \nabla I|^2)$, we have $\phi = \phi(|\nabla u|^2)$; the equation then becomes

$$u_t = \sum_{i,j} \left[\phi\delta_{ij} + \left(2\phi' - \frac{\phi}{|\nabla u|^2} \right) u_i u_j \right] u_{ij}.$$

This is not quite the PM equation, but if we seek ϕ so that the right-hand side is proportional to the one in the PM equation, we find we can take $\phi(|\nabla u|^2) = |\nabla u|\rho(|\nabla u|^2)$, so that the equation takes the form

$$u_t = |\nabla u| \operatorname{div} (\rho \nabla u).$$

Observe how the gradient term, added on a purely geometric basis, is precisely the one which is needed to complete the divergence in the above equation.

Therefore, the PM equation appears ubiquitous despite its strange features. It arises as a simplified or limiting form of most of the segmentation algorithms considered recently.

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Remark 1. There is a large literature on the regularization of those ill-posed problems for which the unregularized equation exhibits violent numerical instability, unlike the present case. There is, however, one example of an ill-posed parabolic problem (see Höllig and Nohel [11]) for which it has been suggested that solutions might exist even without regularization. This problem is slightly different from PM in the sense that ρ does not tend to zero at infinity, so that convexification procedures may be helpful—for PM, the convexified functional is identically zero. It would be interesting to see whether the present concept of generalized solution helps us understand the problem of Höllig and Nohel as well. For general-purpose techniques for regularizing ill-posed problems, see [29].

Remark 2. There is another way, proposed in [16], to obviate the irreversible character of Gaussian smoothing. It consists of running a Fuchsian *hyperbolic* equation, such as the Euler–Poisson–Darboux equation, on the image. This leads to a reversible smoothing procedure which is "causal" in the sense of scale-space theory, and in which one can go back and forth from coarse to fine "scales" without difficulty. There is now an extensive theory of Fuchsian PDEs, both linear and nonlinear; see [15].

3. Nonexistence results.

3.1. Goals. The first purpose of this section is to establish to what extent the initial-value problem for the PM equation does not possess any weak solution at all unless it is regularized. The goal is therefore not to display the inadequacy of a particular set-up, since this would not prove that there isn't another more appropriate one. We must on the contrary infer from the nonexistence result what the correct notion of solution should be. We focus on the notion of weak solution, which has, so far, been the natural concept for divergence-form equations.

In this section, we limit ourselves to one space dimension, which has the advantage that the equation is then either forward or backward parabolic, depending on whether $|u_x|$ is smaller or greater than a critical value K. This restriction to one dimension is not a significant one: if the equation has no solution in this case, the only alternative would be to imagine that there is a solution which depends explicitly on y even when its initial condition does not—the equation would therefore introduce new features. Such behavior is, however, not observed numerically, and if it were, this would be an even stronger argument against the equation than the nonexistence theorems below.

We are therefore interested in a function $u(x,t)$ defined on a rectangle $Q = \{(x,t) : a \le x \le b \text{ and } 0 \le t \le T\}$, which solves, in a sense specified below, the equation

$$u_t - (\rho(u_x^2)u_x)_x = 0. \tag{2}$$

This equation can be expanded into

$$u_t - c(u_x^2)u_{xx} = 0,$$

where $c(s) = \rho(s) + 2s\rho'(s)$. The equation is said to be forward parabolic at x if $c(u_x^2) > 0$, backward parabolic if $c(u_x^2) < 0$. In the cases studied by PM,

(1) $\rho(s) > 0$ for all $s \ge 0$ and is in fact an analytic function of s;

(2) there is a cut-off value K such that $c(s) > 0$ for $s < K^2$, and $c(s) < 0$ for $s > K^2$;

(3) both $\rho(s)$ and $c(s)$ tend to zero as $s \to \infty$.

Remark. There are choices of ρ such that conditions (1) and (3) above hold, but (2) fails; $\rho(s) = (1+s)^{(p-2)/2}$ with $1 < p < 2$ is an example. In that case, the initial-value problem is well posed. However, in such a case, one can show that if the

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initial data have, say, square-integrable gradient, the solution has bounded gradient for positive time. This means that all edges are attenuated to some extent. Also, the only stationary solutions are constants, which means that we may not identify large-time limits with segmentations. For these reasons, we do not dwell on these models. The borderline case $p = 1$ contains the nonlinear term occurring in the minimal surface equation, which is similar to the nonlinearity in the mean-curvature equation or in the gradient descent for total variation minimization. Quite generally, well-posedness is in a sense antithetical to enhancement, in the sense that it predicts that the solution will never be more singular than the data. A smeared singularity is therefore permanently lost. The success of well-posed models may be thought of in terms of relative enhancement, whereby unimportant features are smeared, letting the desired features stand out. It is a classical observation that the computation of the gradient is already an ill-posed process in suitable norms.

If $|u_x|$ is initially always less than K, it is easy to show that there is a solution of the initial-value problem for which this condition is satisfied for all time. The solution actually is infinitely differentiable for $t > 0$, so that all edges are attenuated. This means that K acts as a measure of the contrast scale of the features which we expect will be destroyed.

We therefore assume that our putative solution initially satisfies, for all $x \in [A, B]$ and all $t \in [0, T]$,

$$K_1 \leq u_x \leq K_2,$$

where $K_1 > K$.

We will show in section 3.2 that this implies the continuity of u_x for $t = 0$ and $A < x < B$, and more generally, the continuity of *every* derivative $\partial_x^k u(x, 0)$, $k \geq 1$.

The implications of this result are discussed in section 3.2; in particular, we will see that the only more general notion of solution which makes sense in this application is what we shall call "generalized solution," which does not solve the initial-value problem in the ordinary sense at all but is the concept naturally suggested by the calculations of Perona and Malik.

Our nonexistence results are all based on powerful methods for proving interior regularity estimates for linear and nonlinear parabolic equations that were obtained in the late fifties and the sixties; a standard reference is the book [19]. Although many of the estimates below can be inferred from those in [19], our arguments are slightly simpler due to the special situation at hand. Since these techniques are quite elaborate and not widely used in the computer vision literature, we merely outline the main point and hope the following serves as an illustration of their upshot.

3.2. Interior estimates. The general principle of all the nonexistence theorems is that if we assume that there is a weak solution with a bounded derivative in a rectangle Q, all of its derivatives must also be bounded. Reading this result backwards, we may say that if there were a weak solution, and if we made an error on the 200th derivative of the gray-scale intensity at any point where $u_x > K$, then this would be reflected by the creation of an edge ("u_x very large") in any neighborhood of that point. This is clearly unacceptable.

Note that by the embedding theorem, u is uniformly Hölder continuous in x with exponent $1/2$; in particular, it is bounded. Also, intuitively speaking, since $u_x > K$, the equation is backward parabolic, and u is therefore determined entirely by its values on the "parabolic boundary" of Q, which consists of its upper boundary (where $t = T$) and its two sides. Therefore, its value on $t = 0$, which is given by the initial image, is strongly constrained.

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DEFINITION 1 (definition of a weak solution). *A locally integrable function $u(x,t)$ will be said to be a weak solution of the PM equation if $\int(u^2+u_x^2)dx$ is uniformly bounded for bounded t, and if for any test function $\phi(x,t)$ in C_0^1,*

$$\iint [\phi_t u - \phi_x \rho(u_x^2)u_x]\,dx\,dt = 0.$$

Recall that $c(u_x^2) = \rho(u_x^2) + 2u_x^2\rho(u_x^2)$.

Second derivative estimates. For simplicity, we limit ourselves to showing how to obtain an a priori estimate. The result itself is obtained by the method of translations. We also assume the function satisfies $c < a < 0$ in $[A,B]$. We estimate

$$\iint \eta(x)^2 u_{xx}^2\;dx\,dt,$$

where the cut-off function η is smooth and vanishes outside $[A,B]$. Differentiating the equation with respect to x and multiplying by $\eta^2 u_x$, we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int \eta^2 u_x^2\,dx &= \int \eta^2 u_{xt}u_x\,dx = -\int c u_{xx}(\eta^2 u_x)_x\,dx\\ &= \int -[c\eta^2 u_{xx}^2 + 2c\eta\eta_x u_x u_{xx}]\,dx,\end{aligned}$$

so, integrating with respect to time,

$$|a|\iint \eta^2 u_{xx}^2\;dx\,dt \le C + \iint -2c[\varepsilon\eta^2 u_{xx}^2 + (C/\varepsilon)\eta_x^2 u_x^2]\;dx\,dt$$

for some constant C and for every ε. We conclude by taking ε small enough.

$C^{1+\alpha}$ *estimate.* Differentiating the equation, we now have

$$u_{xt} - (c(u_x^2)u_{xx})_x = 0,$$

which is a uniformly backward parabolic equation for u_x. We may view it as a linear equation with bounded measurable coefficients, so that the Nash–de Giorgi theorem implies that u_x is of class $C^{\alpha_1,\alpha_1/2}$ in any rectangle bounded away from the parabolic boundary for some $\alpha_1 \in (0,1)$.

$C^{2+\alpha}$ *estimate.* Coming back to the PM equation, we have now established that the equation for u has Hölder continuous coefficients. If we view it as a linear equation, we conclude, using the parabolic version of the Schauder estimates, that u is locally $C^{2+\alpha_2,1+\alpha_2/2}$ for some $\alpha_2 \in (0,1)$.

Higher derivative estimates. Once $C^{2+\alpha,1+\alpha/2}$ estimates have been found, it is easy to derive interior estimates for higher-order derivatives. For instance, we may now differentiate the equation once more to find an equation for u_{xx}:

$$u_{xxt} - (c(u_x^2)u_{xxx})_x - (2c'(u_x^2)u_x u_{xx}^2)_x = 0,$$

which has Hölder continuous coefficients. Therefore, we conclude that u_{xx} is locally $C^{2+\alpha_2,1+\alpha_2/2}$, which gives a fourth derivative estimate. The procedure is easily iterated and continues ad infinitum since c is infinitely differentiable.

It follows that if there is a weak solution, it must be infinitely smooth initially in any region where its gradient is greater than K. If therefore the initial image is not *infinitely* differentiable, there is *no* weak solution at all. There are weak solutions with large gradient, linear functions of x being an example, but they are unstable in the sense that there are small perturbations of these solutions for which the initial-value problem has no weak solution at all.

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4. Implications of the nonexistence result. What we have established is that *there is no weak solution with a bounded derivative which stays greater than K in a rectangle $Q = (A, B) \times (0, T)$ with $A < B$ and $T > 0$.*

We must now consider whether there is a more general type of solution which solves the initial-value problem.

We show that if we keep the assumption that u_x is a *function* and not a possibly singular *measure*, we are led to unacceptable solutions.

Let us therefore assume that there is a more general solution; it should be such that in *any* such rectangle, the gradient must be either less than K or "infinite." The latter possibility is absurd if $A < B$, for it would imply that *every point* between A and B represents an edge. This cannot be the case, since computations have been successful with quite regular data with few edges.

The only remaining possibility is that the putative solution cannot have gradient larger than K in any open region. This means that the solution must, at best, consist of several regions in which it has gradient less than K in absolute value, separated by lines of discontinuity along which $|u_x|$ is "infinite." Such functions are perhaps comparable to those which arise naturally in the minimization of the Mumford–Shah functional; see, e.g., [2].

If, however, we wish to introduce functions with jump discontinuities as possible solutions, we find that the equation cannot be interpreted in the weak sense anymore, because the weak formulation requires that u_x be a *function*, while we must allow it to be a *measure*, possibly with a singular part.

4.1. Introduction of generalized solutions. Recall that in the numerical calculations, the required segmentation appears as a large-time limit of the solution, and that any step function can appear in this process. Functions with an arbitrary jump should therefore be allowed in the class of admissible functions. But this raises the following problems.

(1) Since the location of the jump is left arbitrary, we must accept that in such a theory, *there are infinitely many stationary solutions with the same boundary values*. This is natural: each such solution should represent a particular segmentation, and surely we cannot require that there be only one segmentation for an entire class of images. This rules out any definition which would imply a uniqueness theorem for the corresponding time-independent problem. Note that the mean-curvature procedure does not lead to this difficulty, because the equation itself involves a feature-dependent function ϕ which depends on the image.

(2) Since the gradient of a step function takes only two values, 0 and "$+\infty$," we cannot regularize the solution by truncation of its values, which has recently been a useful procedure.

To introduce generalized solutions, let us take for u the Heaviside function, and let $\rho(s) = 1/(1 + s/K^2)$ and $R(s) = s/(1 + s^2/K^2)$ to fix ideas. Define

$$u_\varepsilon = \begin{cases} 0 & \text{if } x < 0, \\ x/\varepsilon & \text{if } 0 < x < \varepsilon, \\ 1 & \text{if } x > \varepsilon. \end{cases}$$

Then

$$P(u_\varepsilon) := \partial_t u_\varepsilon - \partial_x(\rho(u_{\varepsilon x}^2) u_{\varepsilon x}) = \frac{\delta(x - \varepsilon) - \delta(x)}{1 + \varepsilon^2},$$

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which tends to zero in the sense of distributions as $\epsilon \to 0$. On the other hand, u_ε converges in L^1_{loc} to the Heaviside function.

We express this situation by saying that the Heaviside function is a stationary *generalized solution* of the PM. More generally, we have the following definition.

DEFINITION 2. *u is a* generalized solution *of an equation $P(u) = 0$ in an open set Ω if there is a sequence (u_n) of Lipschitz functions such that $P(u_n) \to 0$ in the sense of distributions in Ω, $u_n \to u$ in $L^1_{\text{loc}}(\Omega)$, and $R(u_{nx}) \rightharpoonup R(u_x)$.*

DEFINITION 3. *We say that a generalized solution is* admissible *if its essential variation in x is nonincreasing in time.*

Recall that a function has finite (essential) total variation if and only if the components of its gradient are bounded measures.

In practice, it is helpful to remember that in view of the patching theorem below, to check that a function is a generalized solution, it suffices to decompose its gradient into a regular and a singular part (Radon–Nikodým) and simply discard the singular part in the computation of the nonlinear term.

5. Generalized solutions. We now take on a systematic study of generalized solutions. We first construct several examples of such solutions, based on a patching theorem. We then show that in the class of increasing functions, it is always possible to perturb slightly, in the L^1 norm, any given image so that a solution with the perturbed initial value exists. It will be found that these new solutions exhibit staircasing and merging of discontinuities, indicating that their dynamics probably explains the original calculations of PM.

We again limit ourselves here to one dimension, although the basic ideas directly generalize to higher dimensions. Also, we assume we deal throughout with nondecreasing functions, since this is the case in which most of the one-dimensional experiments have been carried out. Interaction with the boundary is also not considered at all. More general cases will be considered elsewhere.

5.1. A patching theorem. This theorem explains under what circumstances one can construct a generalized solution by patching two smooth solutions defined on domains separated by a smooth curve.

THEOREM 1. *Let u be a smooth solution of the PM equation in $\Omega_1 \cup \Omega_2$, where Ω_1 and Ω_2 are two domains separated by a smooth curve Γ. Let $[f]$ denote the jump $f_{\text{right}} - f_{\text{left}}$ of any expression f, across Γ. Then u is a generalized solution provided that*

$$[u]\,dx + [\rho(u_x^2)u_x]\,dt = 0 \tag{3}$$

on the curve Γ.

They are weak solutions if $[u] = 0$, since u_x then does not have a singular part.

These solutions are not necessarily admissible; see section 5.3.

From now on, we let $R(s) = s\rho(s^2)$.

Proof. It suffices to approximate the function u by leaving it unchanged at points (x, t) which lie at a distance greater than $1/n$ from the curve Γ, and to make it vary linearly along the normal to Γ in the remaining region. In the limit $n \to \infty$, we find we only need to check that $u_t - R_x = 0$ in the sense of distributions, where R is the piecewise smooth function equal to $R(u_x)$ on each side of Γ. The result then follows from Stokes's theorem. □

Remark. The above patching result also applies to piecewise smooth solutions which are defined on domains bounded by a piecewise smooth curve, as is seen by inspection of the above argument. A special case which we will meet later is the case

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of a collapsing jump, which follows a curve Γ: $x = x(t)$ where $t < 0$. If we have a generalized solution with a jump on Γ, and another generalized solution for $t > 0$, then we see that they can be patched to form a new generalized solution defined for t close to zero, provided only that the two solutions match on $t = 0$. We may apply the patching to the three domains bounded by Γ and $\{t = 0\}$. Clearly, in this case, the boundary is piecewise smooth. We may allow $[R] \neq 0$ for $t = 0$ because $dt = 0$ on the horizontal portion of the boundary. It is not necessary to require $dx(t)/dt$ to remain finite as $t \to 0-$.

If the values u_1 and u_2 of u in Ω_1 and Ω_2, respectively, are known, then the theorem furnishes an ODE for the computation of x:

$$\frac{dx(t)}{dt} = -\frac{R(u_2(x(t),t)) - R(u_1(x(t),t))}{u_2(x(t),t) - u_1(x(t),t)}.$$

5.2. Stationary solutions. An immediate consequence of the patching theorem is that any piecewise constant function is a stationary solution. Indeed, in that case, the jumps in the derivative are all zero, so the theorem shows that the location of the jumps do not move.

A more complicated example is the Cantor staircase function; it is a stationary generalized solution, as can be seen by approximating it by a sequence of piecewise linear functions. In fact, the usual definition proceeds by considering a function u_n which is constant except for 2^n intervals $[x_i, y_i]$ of length $(1/3)^n$, where it is linear with slope $(3/2)^2$. Let $P_n = \partial_t u_n - \partial_x(\rho(u_{nx}^2)u_{nx})$. We then find

$$P_n = \sum_i \sigma_n(\delta_{x_i} - \delta_{y_i}),$$

where $\sigma_n \approx K^2(2/3)^n$. Since $|x_i - y_i| = (1/3)^n$, for any test function ϕ, $\int P_n \phi\, dx \leq C2^n(2/3)^n(1/3)^n = C(4/9)^n \to 0$. $\square$

5.3. Piecewise linear solutions. Assume that u_1 and u_2 are linear with slopes L_1 and L_2, respectively. The jump $j(t)$ of u at time t is then given by

$$j(t) = j(0) + (L_2 - L_1)(x(t) - x(0)). \tag{4}$$

It is convenient to write $[L] = L_2 - L_1$ and $[R] = R(L_2) - R(L_1)$. We find

$$(j(0) + [L](x(t) - x(0)))x'(t) = -[R].$$

This is readily integrated:

$$[L](x - x(0))^2/2 + j(0)(x - x(0)) + t[R] = 0.$$

The evolution $x(t)$ is then given by

$$x(t) = x(0) - \frac{j(0)}{[L]}(1 - \sqrt{1 - 2t[R][L]/j(0)^2})$$

if $j(0) \neq 0$, and by

$$x(t) = x(0) \pm \sqrt{-2t[R]/[L]}$$

otherwise.

Interpretation. We see that an initial jump moves to the right or the left, depending on the sign of $-[R]/[L]$, in accordance with Theorem 1. Since $R(s)$ is increasing for $0 \leq s < K$ and decreasing for $s > K$, we need in principle to distinguish four cases depending on the position of L_1 and L_2 with respect to K.

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However, not all of the above solutions are admissible, for we must exclude solutions which create a *negative jump* and in the process increase their total variation. For instance, this removes the sign ambiguity in the case $j(0) = 0$. Let us explain how this is done, assuming that both slopes are (positive and) on the same side of K. First we must have $[R]/[L] < 0$ for the square root to exist; this means we are in the ill-posed region. Further, since $j(t) = j(0) + [L](x(t) - x(0))$, and $j(0) = 0$, we must take the square root to be of the same sign as $[L]$ to ensure that we have a positive jump.

Note also that when both slopes are on the same side of K, we find that a jump is disallowed if we are in the forward region, and allowed otherwise. This is reasonable: otherwise we would have two conflicting ways of computing the solution in the forward region.

Therefore we see that, for piecewise linear admissible solutions, with slopes both large or small, the following theorem applies.

THEOREM 2. *The speed of a jump discontinuity is* $-[R]/[u]$. *If* $[u] = 0$ *initially, a jump can develop only if the slopes are large. It can then move to the left or the right. If* $[R][L] > 0$*, the jump collapses in finite time.*

We also list a few properties easily checked from the above rules, for the situation where L_1 and L_2 are both less than K. Enumerating possible cases, we find that

(1) if $[u] = 0$ initially, a jump is disallowed, and we must use the (well-posed) equation $u_t = R_x$ to compute u;

(2) jumps tend to decrease and eventually collapse, at which point the considerations of (1) must be applied.

In the latter case, we must decide how to continue a solution after the jump has collapsed to zero. Similarly, if the jump is initially zero and a discontinuity cannot form, how do we find admissible generalized solutions? There are the issues addressed in section 5.4.

5.4. Interaction and general initial data. Let us now give prescriptions for the solution derived from more general initial data.

First, we replace the initial condition by a function having jump discontinuities at points $x_k(t)$, between which it is smooth and has small derivative; this approximation should be close to the given data in L^1. We will consider elsewhere the question of the construction of such an approximation.

There are two methods for going further; each of them leads, whenever applicable, to a consistent theory of generalized solutions, and the one which must be chosen depends on the desired effect, as explained in the next paragraph.

Method of extension. Construct, for every k, an extension u_k of u, such that it solves the PM equation for $t \geq 0$, and initially agrees with the initial condition for x between x_k and x_{k+1}. Then solve the ODE

$$\frac{dx_k}{dt} = -\frac{R(\partial_x u_{k+1}(x_k(t), t)) - R(\partial_x u_k(x_k(t), t))}{u_{k+1}(x_k(t), t) - u_k(x_k(t), t)}$$

for $x_k(t)$ until interaction or collapse occurs. This requires that one be sure that the ODE never forces $x_k(t)$ out of the domain of u_k or u_{k-1}. There are, however, simple ways to ensure that this always holds.

At each interaction, jumps merge. If a jump collapses, we simply continue the procedure with fewer discontinuity points, the solution becoming smooth between x_{k-1} and x_{k+1}. If jumps merge, it means that for some k, $x_k = x_{k+1}$ at that time, and we simply assume that $x_k = x_{k+1}$ from then onwards.

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Method of interfaces. This consists of directly solving a free-boundary problem that directly gives the u_k and the interfaces $x_k(t)$ simultaneously. This problem has the general form

$$\partial_t u_k - R(\partial_x u_k)_x = 0 \qquad \text{for } x_k < x < x_{k+1},$$
$$dx_k/dt = -[R](x_k)/j_k(t),$$
$$R(\partial_x u_k)(x_k\pm) = L_{k\pm},$$

where $j_k(t) = u(x_k(t), t) - u(x_{k-1}(t), t)$, and $L_{k\pm}$ is specified in terms of the jumps.

We have shown that such problems have a unique stable solution in several cases which will be reported elsewhere.

Note that in both methods, the number of jumps never increases after the initial approximation.

Examples. We turn to a few concrete examples of the above constructions.

We begin with the method of extension. The construction of the u_k can be achieved by analytic continuation (as for linear functions) or linear extension of the data followed by the solution of the equation in the well-posed case.

The first possibility is the one which occurs in the case of piecewise linear solutions.

As an illustration of this case, let us consider a piecewise constant image on $[0, 1]$, increasing from 0 to 1 by N equal steps. Extend it to be zero for $x < 0$, and 1 for $x > 1$; let us work with Neumann conditions on, say, $[-1, 2]$. This gives a constant generalized solution. If, however, we give a small positive slope to the middle step, it is easy to see that, if we use the linear extension, it will grow and eventually merge with all the other jumps, until the solution becomes continuous. It then continues as a weak solution of the PM equation, since it is now Lipschitz continuous with small slope. If, on the contrary, we give the middle jump a small *negative* slope, we find that its endpoints now travel towards each other to finally merge. The result is a piecewise constant stationary solution with only $N - 1$ steps.

The second possibility leads naturally to a version of the porous media equation. Indeed, let us consider one interval $[x_k, x_{k+1}]$. We extend the initial condition so that it is linear of slope K outside this interval, while we are assuming that $u_x < K$ inside the interval. We then find that $K - u_x$ then solves a version of the porous media equation. This gives a definition of u_k for all values of x, from which the x_k can be computed using the jump conditions. Attractive features of this method are that (1) the porous media equation has "finite speed of propagation" of support; (2) this provides a procedure to grow individual regions of an image independently of each other; (3) if R' vanishes for $s = K$, we expect that the region $u_x = K$ does not grow for some time ("waiting time phenomenon"). This effect is enhanced if R' vanishes to a high order at K. On the negative side, we expect v to tend to zero at infinity, which means that the slope of the solution u would tend to K and not to zero. This is not necessarily a drawback if K is very small.

The simplest example of the method of interfaces is the case $L_{k\pm} = 0$, for which we find $dx_k/dt = 0$ for every k. This case can be related to the calculations of PM: if the parameter K is taken to be small enough so that the slope of a jump (which is never infinite due to the finiteness of the mesh) is large compared to K, it is found that such a jump does not move at all in the course of the computation.

We thus see that, quite generally,

(1) generalized solutions exhibit merging of jumps similar to those found in computations;

(2) the long-time limit of generalized solutions is extremely sensitive to the initial condition;

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(3) however, the evolution over finite ranges of time is quite regular and is extremely slow if the data are close to a stationary solution.

The procedure therefore appears to have good predictive power for small times.

6. Concluding remarks. We have shown the limitations of the notion of weak solution for the PM equation and introduced a notion of generalized solution which enables us to provide a constructive procedure for solving the initial-value problem approximately, even in the backward case, by replacing the initial condition by a piecewise smooth one. The results seem to provide an explanation for the numerical results found so far and suggest further numerical procedures.

If there is, as we showed, an embryo of a theory of solutions of the PM equation, what is the role of regularizations? The answer is that the first step in the construction was to mimic the breakup of the initial data, consecutive to the nonexistence of a weak solution, into a piecewise linear function that could then evolve. The well-posedness for small time of regularized equations means that such an instability is *slowed down* by regularization, and therefore avoids the problem of choosing a decomposition. The fact that such a choice is implicit is found in the fact that the long-time behavior for discretized or regularized equations appears to depend in an essential way on the parameters of the computation and other features, such as the procedure used to perform mesh refinement.

Regularizations of the PM equation are therefore still useful, providing a means to mimic and control in a specific way the initial construction of piecewise smooth data. One might compare the difference between the PM equation and any of its regularizations to that between an atom bomb and a nuclear power plant: the former exhibits potentially destructive, uncontrolled instability, while the latter, by harnessing its mechanism, puts its power to use in a controlled environment. This mechanism is consistent with existing numerical calculations on the PM equation, as well as with what we have learned from simulated annealing techniques.

However, one must stress that the most noteworthy aspect of this development is that there should exist a notion of generalized solutions for ill-posed evolution equations, without regularization, which is entirely consistent with, and indeed is implied by, all our current understanding of nonlinear PDEs.

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Note added in proof. Since the completion of this paper, we have found that the scheme initially considered by PM has the additional property that $R(u_x)$ is continuous even when u is not. This leads to a special case of the method of interfaces, where $R(u_x) = R([u]/\Delta x)$ at each jump. It can be interpreted as a gradient descent and has a good existence-uniqueness theory, which will be developed elsewhere. This confirms our solution of the paradox.

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