

# Neural networks

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Master Degree in Computer Science

Course of Machine Learning

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- Up to now, only models with a single level of parameters to be learned were considered.
- The model has a generalized linear model structure such as  $y = f(\mathbf{w}^T \phi(\mathbf{x}))$ : model parameters are directly applied to input values.
- More general classes of models can be defined by means of sequences of transformations applied on input data, corresponding to multilayered networks of functions.

## Multilayer network structure: first layer

For any  $d$ -dimensional input vector  $\mathbf{x} = (x_1, \dots, x_d)$ , the first layer of a **neural network** derives  $m_1 > 0$  **activations**  $a_1^{(1)}, \dots, a_{m_1}^{(1)}$  through suitable linear combinations of  $x_1, \dots, x_d$

$$a_j^{(1)} = \sum_{i=1}^d w_{ji}^{(1)} x_i + w_{j0}^{(1)} = \bar{\mathbf{x}}^T \mathbf{w}_j^{(1)}$$

where  $M$  is a given, predefined, parameter and  $\bar{\mathbf{x}} = (1, x_1, \dots, x_d)^T$ .

## Multilayer network structure: first layer

Each activation  $a_j^{(1)}$  is transformed by means of a non-linear **activation function**  $h_1$  to provide a vector  $\mathbf{z}^{(1)} = (z_1^{(1)}, \dots, z_{m_1}^{(1)})^T$  as output from the layer, as follows

$$z_j^{(1)} = h_1(a_j^{(1)}) = h_1(\bar{\mathbf{x}}^T \mathbf{w}_j^{(1)})$$

here  $h_1$  is some approximate threshold function, such as a sigmoid

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

or a hyperbolic tangent

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{1 + e^{-2x}} - \frac{1}{1 + e^{2x}} = \sigma(2x) - \sigma(-2x)$$

Observe that this corresponds to defining  $m_1$  units, where unit  $j$  implements a GLM on  $\mathbf{x}$  to derive  $z_j^{(1)}$ .

## Multilayer network structure: inner layers

Vector  $\mathbf{z}^{(1)}$  provides an input to the next layer, where  $m_2$  hidden units compute a vector  $\mathbf{z}^{(2)} = (z_1^{(2)}, \dots, z_{m_2}^{(2)})^T$  by first performing linear combinations of the input values

$$a_k^{(2)} = \sum_{i=1}^{m_1} w_{ki}^{(2)} a_i^{(1)} + w_{k0}^{(2)} = (\bar{\mathbf{z}}^{(1)})^T \bar{\mathbf{w}}_k^{(2)}$$

and then applying function  $h_2$ , as follows

$$z_k^{(2)} = h_2((\bar{\mathbf{z}}^{(1)})^T \bar{\mathbf{w}}_k^{(2)})$$

## Multilayer network structure: inner layers

The same structure can be repeated for each inner layer, where layer  $r$  has  $m_r$  units which, from input vector  $\mathbf{z}^{(r-1)}$ , derive output vector  $\mathbf{z}^{(r)}$  through linear combinations

$$a_k^{(r)} = (\bar{\mathbf{z}}^{(r-1)})^T \bar{\mathbf{w}}_k^{(r)}$$

and non linear transformation

$$z_k^{(r)} = h_r((\bar{\mathbf{z}}^{(r-1)})^T \bar{\mathbf{w}}_k^{(r)})$$

## Multilayer network structure: output layer

For what concerns the last layer, say layer  $t$ , an output vector  $\mathbf{y} = \mathbf{z}^{(t)}$  is again produced by means of  $m_t$  **output units** by first performing linear combinations on  $\mathbf{z}^{(t-1)}$

$$a_k^{(t)} = (\bar{\mathbf{z}}^{(t-1)})^T \bar{\mathbf{w}}_k^{(t)}$$

and then applying function  $h_t$

$$y_k = z_k^{(t)} = h_t((\bar{\mathbf{z}}^{(t-1)})^T \bar{\mathbf{w}}_k^{(t)})$$

where:

- $h_t$  is the identity function in the case of regression
- $h_t$  is a sigmoid in the case of binary classification
- $h_t$  is a softmax in the case of multiclass classification

### 3 layer networks

A sufficiently powerful model is provided in the case of 3 layers (input, hidden, output).

For example, applying this model for  $K$ -class classification corresponds to the following overall network function for each  $y_k, k = 1, \dots, K$

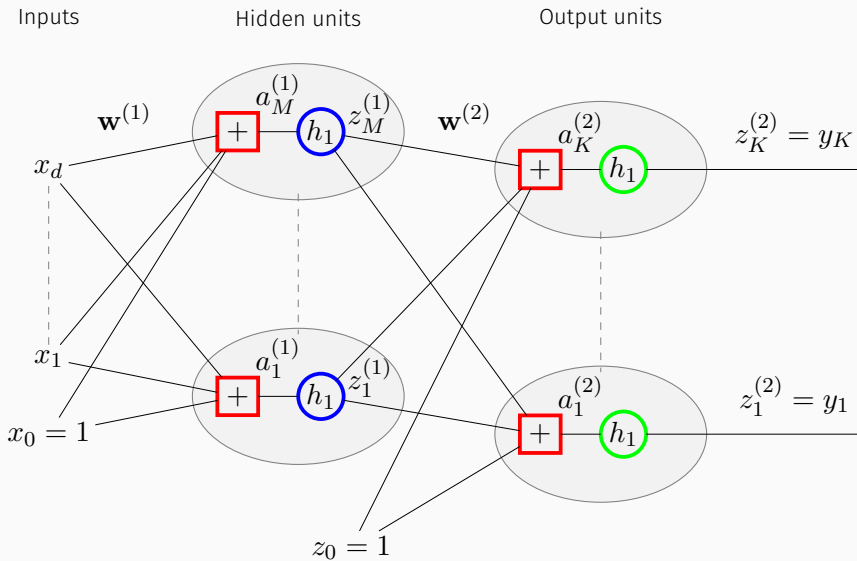
$$y_k = \sigma \left( \sum_{j=1}^M w_{kj}^{(2)} h \left( \sum_{i=1}^d w_{ji}^{(1)} x_i + w_{j0}^{(1)} \right) + w_{k0}^{(2)} \right)$$

where the number  $M$  of hidden units is a model structure parameter.

The resulting network can be seen as a GLM where base functions are not predefined wrt to data, but are instead parameterized by coefficients in  $\mathbf{w}^{(1)}$ .



### 3 layer networks



# Approximating functions with neural networks

Neural networks, despite their simple structure, are sufficient powerful models to act as **universal approximators**.

It is possible to prove that any continuous function can be approximated, at any by means of two-layered neural networks with sigmoidal activation functions. The approximation can be indefinitely precise, as long as a suitable number of hidden units is defined.

## 3-layered neural network training

The training phase of a neural network implies learning the values of all parameters from a training set

$(\mathbf{X}, \mathbf{t}) = \{(\mathbf{x}_1, \mathbf{t}_1), (\mathbf{x}_2, \mathbf{t}_2), \dots, (\mathbf{x}_n, \mathbf{t}_n)\}$ . In the case of 3-layered networks, this corresponds to learning  $\mathbf{w} = \mathbf{w}^{(1)} \cup \mathbf{w}^{(2)}$ .

As usual, learning can be performed by minimizing some cost function, in dependance of the problem considered and the assumed probabilistic model.

In the case of maximum likelihood, the minimization of the cost function is equivalent to the maximization of the likelihood of the training set, given the model and its parameters.

Probabilistic model: for each element  $(\mathbf{x}_i, t_i)$  of the training set, the value  $y_i = y(\mathbf{x}_i, \mathbf{w})$  returned by the network is normally distributed around the target value  $t_i$  with variance  $\sigma^2$  to be determined.

This is equivalent to assuming that, given an element  $\mathbf{x}$ , its unknown target value  $t$  is normally distributed around the returned value  $y = y(\mathbf{x}, \mathbf{w})$  with same variance  $\sigma^2$ : that is,

$$p(t|\mathbf{x}, \mathbf{w}, \sigma^2) = \mathcal{N}(t|y(\mathbf{x}, \mathbf{w}), \sigma^2)$$

The likelihood of the training set is

$$L(\mathbf{t}|\mathbf{X}, \mathbf{w}, \sigma^2) = \prod_{i=1}^n p(t_i|\mathbf{x}_i, \mathbf{w}, \sigma^2) = \prod_{i=1}^n \mathcal{N}(t_i|y(\mathbf{x}_i, \mathbf{w}), \sigma^2)$$

and the log-likelihood

$$\begin{aligned} l(\mathbf{t}|\mathbf{X}, \mathbf{x}, \sigma^2) &= \log L(\mathbf{t}|\mathbf{x}, \mathbf{w}, \sigma^2) \\ &= -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (y(\mathbf{x}_i, \mathbf{w}) - t_i)^2 \end{aligned}$$

## ML and regression in the case $K = 1$

As well known, maximizing the log-likelihood wrt  $\mathbf{w}$  is equivalent to minimizing the cost function

$$E(\mathbf{w}) = \frac{1}{2} \sum_{i=1}^n (y(\mathbf{x}_i, \mathbf{w}) - t_i)^2$$

Differently than in the case of linear regression,  $y(\mathbf{x}, \mathbf{w})$  is now not linear and, in general, has several local minima.

The variance can be estimated once maximum (or at least local maximum ) likelihood values are available by setting the corresponding derivative to zero

$$\frac{\partial l(\mathbf{t}|\mathbf{X}, \mathbf{w}_{ML}, \sigma^2)}{\partial \sigma^2} = 0$$

thus obtaining

$$\sigma_{ML}^2 = \frac{1}{n} \sum_{i=1}^n (y(\mathbf{x}_i, \mathbf{w}_{ML}) - t_i)^2$$

## ML and regression in the case $K > 1$

Probabilistic model: if we assume that target variables are conditionally independent given  $\mathbf{x}$  and  $\mathbf{w}$ , with common variance  $\sigma^2$ , the overall distribution of  $\mathbf{t}$  given  $\mathbf{x}$  is a multivariate gaussian with mean  $\mathbf{y} = \mathbf{y}(\mathbf{x}, \mathbf{w})$ : that is,

$$p(\mathbf{t}|\mathbf{x}, \mathbf{w}, \sigma^2) = \mathcal{N}(\mathbf{t}|\mathbf{y}(\mathbf{x}, \mathbf{w}), \sigma^2\mathbf{I}) = \prod_{j=1}^K \mathcal{N}(t_j|y_j(\mathbf{x}, \mathbf{w}), \sigma^2)$$

This results in the training set likelihood,

$$L(\mathbf{T}|\mathbf{X}, \mathbf{w}, \sigma^2) = \prod_{i=1}^n p(\mathbf{t}_i|\mathbf{x}_i, \mathbf{w}, \sigma^2) = \prod_{i=1}^n \prod_{j=1}^K \mathcal{N}(t_{ij}|y_j(\mathbf{x}_i, \mathbf{w}), \sigma^2)$$

and log-likelihood

$$l(\mathbf{T}|\mathbf{X}, \mathbf{w}, \sigma^2) = -\frac{nK}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n \sum_{j=1}^K (y_j(\mathbf{x}_i, \mathbf{w}) - t_{ij})^2$$



## ML and regression in the case $K > 1$

The cost function is now

$$E(\mathbf{w}) = \sum_{i=1}^n \sum_{j=1}^K (y_j(\mathbf{x}_i, \mathbf{w}) - t_{ij})^2$$

and the variance, given a maximum likelihood estimation  $\mathbf{w}_{ML}$  of the parameters, can now be estimated as

$$\sigma_{ML}^2 = \frac{1}{nK} \sum_{i=1}^n \|\mathbf{y}(\mathbf{x}_i, \mathbf{w}_{ML}) - \mathbf{t}_i\|^2$$

## ML and regression in the case $K > 1$

For each element  $(\mathbf{x}, \mathbf{t})$  of the training set, the derivative of the error function  $E(\mathbf{W})$  wrt the linear combination  $a_k^{(2)} = (\overline{\mathbf{w}}_k^{(2)})^T \mathbf{z}^{(1)}$  computed by the  $k$ -th output unit is

$$\frac{\partial E(\mathbf{W})}{\partial a_k^{(2)}} = \frac{\partial}{\partial a_k^{(2)}} \left( \frac{1}{2} \sum_{j=1}^K (y_j - t_j)^2 \right)$$

In linear regression,  $y_j = a_j^{(2)}$  for all  $j = 1, \dots, K$ , hence

$$\frac{\partial E(\mathbf{W})}{\partial a_k^{(2)}} = \frac{\partial}{\partial a_k^{(2)}} \left( \frac{1}{2} \sum_{i=1}^K (a_i^{(2)} - t_i)^2 \right) = a_k^{(2)} - t_k = y_k - t_k$$

Let  $y(\mathbf{x}, \mathbf{W}) = p(C_1|\mathbf{x})$ : we refer to a probabilistic model where the conditional probability of the target value, given the feature values, is distributed according to a Bernoulli

$$p(t|\mathbf{x}, \mathbf{W}) = y(\mathbf{x}, \mathbf{W})^t(1 - y(\mathbf{x}, \mathbf{W}))^{1-t}$$

The likelihood of the training set is then

$$L(\mathbf{t}|\mathbf{X}, \mathbf{W}) = \prod_{i=1}^n y(\mathbf{x}_i, \mathbf{W})^{t_i}(1 - y(\mathbf{x}_i, \mathbf{W}))^{1-t_i}$$

with log-likelihood

$$l(\mathbf{t}|\mathbf{X}, \mathbf{W}) = \sum_{i=1}^n (t_i \ln y_i + (1 - t_i) \ln(1 - y_i))$$

The cost function is  $E(\mathbf{W}) = -l(\mathbf{t}|\mathbf{X}, \mathbf{W})$  and its derivative wrt  $a_k^{(2)}$

$$\frac{\partial E(\mathbf{W})}{\partial a_k^{(2)}} = -t_k \frac{1}{y_k} \frac{\partial y_k}{\partial a_k^{(2)}} + (1 - t_k) \frac{1}{1 - y_k} \frac{\partial y_k}{\partial a_k^{(2)}}$$

Since  $y_k = \sigma(a_k^{(2)})$ , we get

$$\frac{\partial y_k}{\partial a_k^{(2)}} = \frac{\partial \sigma(a_k^{(2)})}{\partial a_k^{(2)}} = \sigma(a_k^{(2)})(1 - \sigma(a_k^{(2)})) = y_k(1 - y_k)$$

As a consequence,

$$\begin{aligned}\frac{\partial E(\mathbf{W})}{\partial a_k^{(2)}} &= -t_k \frac{1}{y_k} y_k (1 - y_k) + (1 - t_k) \frac{1}{1 - y_k} y_k (1 - y_k) \\ &= -t_k (1 - y_k) + (1 - t_k) y_k \\ &= y_k - t_k\end{aligned}$$

Again, as in the case of regression, the derivative is equal to the difference between value computed by the network and corresponding target.

It is possible, also in this case, to prove that

$$\frac{\partial E(\mathbf{W})}{\partial a_k^{(2)}} = y_k - t_k$$

## Parameter optimization

## Iterative methods to minimize $E(\mathbf{w})$

The error function  $E(\mathbf{w})$  is usually quite hard to minimize:

- there exist many local minima
- for each local minimum there exist many equivalent minima
  - any permutation of hidden units provides the same result
  - changing signs of all input and output links of a single hidden unit provides the same result

hence, if the network has  $M$  hidden units, for each local minimum there exists a set of  $M!2^M$  equivalent minima

Analytical approaches to minimization cannot be applied: resort to iterative methods (possibly comparing results from different runs).

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} + \Delta \mathbf{w}^{(k)}$$



At each step, two stages:

1. the derivatives of the error functions wrt all weights are evaluated at the current point
2. weights are adjusted (resulting into a new point) by using the derivatives

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta \frac{\partial E(\mathbf{w})}{\partial \mathbf{w}} \Big|_{\mathbf{w}^{(k)}}$$

- at each step the weight vector is moved in the direction of greatest decrease
- the whole training set is used at each step: quite expensive
- more efficient methods can be applied (conjugate gradient, quasi-newton)

## On-line gradient descent

We exploit the property that the error function is the sum of a collection of terms, each characterizing the error corresponding to each observation

$$E(\mathbf{w}) = \sum_{i=1}^n E_i(\mathbf{w})$$

the update is based on one training set element at a time

$$\mathbf{w}^{(k+1)} = \mathbf{w}^{(k)} - \eta \left. \frac{\partial E_i(\mathbf{w})}{\partial \mathbf{w}} \right|_{\mathbf{w}^{(k)}}$$

- at each step the weight vector is moved in the direction of greatest decrease only wrt the error for a specific data element
- only one training set element is used at each step: less expensive at each step (more steps may be necessary)
- makes it possible to escape from local minima

## Backpropagation

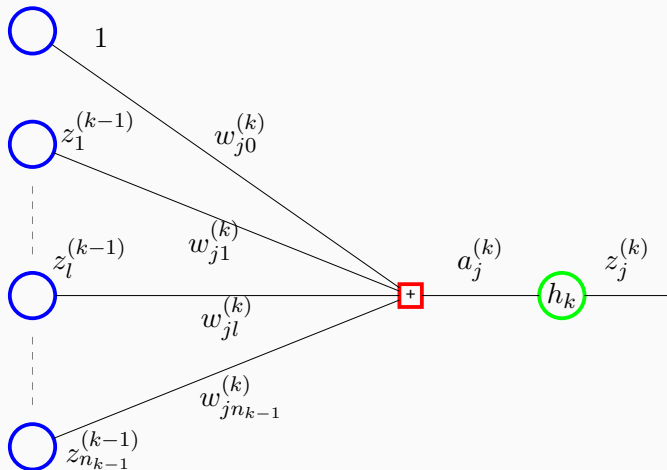
Algorithm applied to evaluate derivatives of the error wrt all weights

It can be interpreted in terms of backward propagation of a computation in the network, from the output towards input units.

It provides an efficient method to evaluate derivatives wrt weights. It can be applied also to compute derivatives of output wrt to input variables, to provide evaluations of the Jacobian and the Hessian matrices at a given point.

# Backpropagation

Assume a feed-forward neural network with arbitrary topology and differentiable activation functions and error function.



# Backpropagation

- All variables  $z_i$  could be either an input variable to the network or the output from a unit in the preceding layer
- The variable  $a_j$  could also be directly returned ( $h_k$  being the identity function)

Assumption on the error function: it may be expressed, given a training set, as the sum of the errors corresponding to single elements of the training set

$$E(\mathbf{w}) = \sum_{i=1}^n E_i(\mathbf{w})$$

If  $E_i$  is differentiable, so is  $E$ , with derivative given by the sum of the derivatives of functions  $E_i$ .

# Backpropagation

- Assume that, for each element  $(\mathbf{x}_i, \mathbf{t}_i)$  of the training set, the feature values  $\mathbf{x}_i$  have been given as input to the network and both the activation values for each unit and the output values are available: this step is denoted as **forward propagation**
- We wish to evaluate the derivative of  $E_i$  wrt to parameter  $w_{jl}^{(k)}$ , which associates a weight to the contribution of  $z_l^{(k-1)}$  to the unit computing  $a_j^{(k)}$
- $E_i$  is a function of  $w_{jl}^{(k)}$  only through the following sum

$$a_j^{(k)} = \sum_{r=1}^m w_{jr}^{(k)} z_r^{(k-1)}$$



# Backpropagation

Let us define  $\delta_j^{(k)}$  as follows:

$$\delta_j^{(k)} = \frac{\partial E_i}{\partial a_j^{(k)}}$$

Since

$$\frac{\partial a_j^{(k)}}{\partial w_{jl}^{(k)}} = \frac{\partial}{\partial w_{jl}^{(k)}} \sum_{r=1}^m w_{jr}^{(k)} z_r^{(k-1)} = z_l^{(k-1)}$$

it results

$$\frac{\partial E_i}{\partial w_{jl}^{(k)}} = \delta_j^{(k)} z_l^{(k-1)}$$

To compute the derivatives of  $E_i$  wrt to all parameters, it is necessary to compute  $\delta_j^{(k)}$  for all network units.

Let us first consider the output, that is  $z_j^{(k)} = y_j$ .

As observed before, in this case we have

$$\delta_j^{(k)} = \frac{\partial E_i}{\partial a_j^{(k)}} = y_j - t_j$$

Hidden unit.

- any change of  $a_j^{(k)}$  has effect on  $E_i$  by inducing changes for all variables  $a_l^{(k+1)}$
- the effect on  $E_i$  is a function of the sum of the effect of the change of  $a_j^{(k)}$  on all variables  $a_r^{(k+1)}$

$$\delta_j^{(k)} = \frac{\partial E_i}{\partial a_j^{(k)}} = \sum_{r=1}^{n_{k+1}} \frac{\partial E_i}{\partial a_r^{(k+1)}} \frac{\partial a_r^{(k+1)}}{\partial a_j^{(k)}} = \sum_{r=1}^{n_{k+1}} \delta_r^{(k+1)} \frac{\partial a_r^{(k+1)}}{\partial a_j^{(k)}}$$

Since by definition

$$a_r^{(k+1)} = \sum_l w_{rl}^{(k+1)} z_l^{(k)}$$
$$z_j^{(k)} = h_k(a_j^{(k)})$$

it results

$$\frac{\partial a_r^{(k+1)}}{\partial a_j^{(k)}} = \frac{\partial a_r^{(k+1)}}{\partial z_j^{(k)}} \frac{\partial z_j^{(k)}}{\partial a_j^{(k)}} = w_{rj}^{(k+1)} h'_k(a_j^{(k)})$$

and

$$\delta_j^{(k)} = h'_k(a_j^{(k)}) \sum_{r=1}^{n_{k+1}} \delta_r^{(k+1)} w_{rj}^{(k+1)}$$

$$\delta_j^{(k)} = h'_k(a_j^{(k)}) \sum_{r=1}^{n_{k+1}} \delta_r^{(k+1)} w_{rj}^{(k+1)}$$

can the be evaluated if the following are known

- $w_{rj}^{(k+1)}, r = 1, \dots, k + 1$ : this are assumed as known for any single back propagation step
- $a_j^{(k)}$ : this is computed, during forward propagation, from the current  $\mathbf{w}$  and the input values
- $\delta_r^{(k+1)}, r = 1, \dots, k + 1$ : these can be computed from the network output and the target values by applying a backward propagation of the values from the last to the first network layers (that is, in opposite sense wrt to the output computation)

Example of backpropagation on a 3-layered network:

1. The feature values  $\mathbf{x}_i$  of a training set item are provided as input to the network: all values  $a_j^{(1)}, a_j^{(2)}, z_j^{(1)}, z_j^{(2)} = y_j$  are derived and made available
2. Starting from output and target values, the  $\delta$  values for each output variables is derived, as  $\delta_j^{(2)} = y_j - t_j$

3. For each hidden unit, the corresponding  $\delta$  value is computed, as

$$\delta_j^{(1)} = h_1'(a_j^{(1)}) \sum_{i=1}^K w_{ij}^{(2)} \delta_i^{(2)} = h_1'(a_j^{(1)}) \sum_{i=1}^K w_{ij}^{(2)} (y_j - t_j)$$

which, in the usual case  $h_1(x) = \sigma(x)$ , results into

$$\delta_j^{(1)} = \sigma(a_j^{(1)}) (1 - \sigma(a_j^{(1)})) \sum_{i=1}^K w_{ij}^{(2)} (y_j - t_j) = z_j^{(1)} (1 - z_j^{(1)}) \sum_{i=1}^K w_{ij}^{(2)} (y_j - t_j)$$

4. For each parameter  $w_{jl}^{(k)}$ , where  $k = 1, 2$ , the value of the derivative of the function error wrt  $w_{jl}^{(k)}$  at the current value  $\mathbf{w}$  of all weights is computed as

$$\frac{\partial E_i}{\partial w_{jl}^{(k)}} = \delta_j^{(k)} z_l^{(k-1)}$$

which results into

$$\begin{aligned}\frac{\partial E_i}{\partial w_{jl}^{(2)}} &= z_l(y_j - t_j) \\ \frac{\partial E_i}{\partial w_{jl}^{(1)}} &= x_l z_j (1 - z_j) \sum_{i=1}^K w_{ij}^{(2)} (y_j - t_j)\end{aligned}$$



Iterate the preceding steps on all items in the training set (or a subset of them). In fact, since

$$E(\mathbf{w}) = \sum_{i=1}^n E_i(\mathbf{w})$$

it is

$$\frac{\partial E}{\partial w_{jl}^{(k)}} = \sum_{i=1}^n \frac{\partial E_i}{\partial w_{jl}^{(k)}}$$

This provides an evaluation of  $\frac{\partial E(\mathbf{w})}{\partial \mathbf{w}}$  at the current point  $\mathbf{w}$ .

Once  $\frac{\partial E(\mathbf{w})}{\partial \mathbf{w}}$  is known, a single step of gradient descent can be performed

$$\mathbf{w}^{(i+1)} = \mathbf{w}^{(i)} - \eta \frac{\partial E(\mathbf{w})}{\partial \mathbf{w}} \Big|_{\mathbf{w}^{(i)}}$$

The whole process can be made more efficient through on-line descent, that is by considering a single training set element at a time.

A single evaluation of error function derivatives requires  $O(|\mathbf{w}|)$  steps

Alternative approach: finite differences. Perturb each weight  $w_{ij}$  in turn and approximate the derivative as follows

$$\frac{\partial E_i}{\partial w_{ij}} = \frac{E_i(w_{ij} + \varepsilon) - E_i(w_{ij} - \varepsilon)}{2\varepsilon} + O(\varepsilon^2)$$

This requires  $O(|\mathbf{w}|)$  steps for each weight, hence  $O(|\mathbf{w}|^2)$  steps overall.