

# Montecarlo methods

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## The basic problem

Integrate a hard (for example high dimensional) function

$$\int_a^b g(x) dx$$

# Idea

See the integral as an expectation

## Approach

Assume we have a function  $f(x)$  and a density  $p(x)$  in  $[a, b]$  such that  $g(x) = f(x)p(x)$ , we may write

$$\int_a^b g(x)dx = \int_a^b f(x)p(x)dx = E_{p(x)}[f(x)]$$

and approximate this value through the mean of  $n$  values  $f(x_1), \dots, f(x_n)$  sampled from  $p(x)$ :

$$E[f(x)] \approx \frac{1}{n} \sum_{i=1}^n f(x_i)$$

## Approach

- 1 Sample a sequence  $x^{(1)}, x^{(2)}, \dots, x^{(n)}$  of values from distribution  $p(x)$ , that is such that  $Pr(X = x^{(i)}) = p(x^{(i)})$
- 2 Apply function  $f(x)$  to such values
- 3 Average the set of values obtained

## Sampling for expectations

### Problem

But how to sample values from  $p(x)$ ?

$$E_p[f(x)] = \int_x f(x)p(x)dx$$

where  $p(x)$  is hard to derive analitically.

### Hypothesis

Assume a (pseudo) random generator  $\mathcal{R}$  is available which returns a sequence of values (approximately) uniformly distributed in the interval  $[0, 1]$ .

Given a distribution  $p(x)$ , the sequence of values provided by  $\mathcal{R}$  can be exploited to derive a different (possibly shorter) sequence of values with distribution  $p(x)$

## General issue

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Problem underlying this method:

How can we sample from any distribution, especially if we do not have an analytical representation of it?

## Sampling: easy case

Assume we know  $p(x)$ : we wish to find a function  $f(z)$  such that if  $z \sim U(0, 1)$ , then  $f(z) \sim p(z)$ .

- This is equivalent to saying that for each  $z \in [0, 1]$  the cumulative probability of  $z$  wrt to the uniform distribution, that is  $z$  itself, should be equal to the cumulative probability of  $f(z)$ , that is  $Pr(\zeta \leq f(z))$ , wrt to  $p(z)$
- that is,

$$z = \int_0^z d\zeta = \int_{-\infty}^{f(z)} p(\zeta) d\zeta = P(f(z))$$

as a corollary, since it results  $z = P(f(z))$ , we have that  $f(z) = P^{-1}(z)$

## Sampling in the easy case: an example

### Example

Given  $\mathcal{R}$ , we use it to produce exponentially distributed values, that is values distributed according to

$$p(x|\lambda) = \lambda e^{-\lambda x}, 0 \leq x < \infty$$

The exponential cumulative distribution is

$$P(x) = \int_0^x \lambda e^{-\lambda \xi} d\xi = 1 - e^{-\lambda x}$$

by setting  $z = P(f(z)) = 1 - e^{-\lambda f(z)}$  we get

$$e^{-\lambda f(z)} = 1 - z$$

$$-\lambda f(z) = \ln(1 - z)$$

$$f(z) = -\frac{1}{\lambda} \ln(1 - z)$$



## Sampling on general distributions

### Approaches

Applying the above method is possible only for simple distributions. In most cases, you cannot immediately derive values distributed according to  $p(x)$

Many sampling methods have been introduced

- rejection sampling
- importance sampling
- adaptive rejection sampling
- sampling-importance-sampling
- ...

# Rejection sampling

## Context

- $p(x)$  is difficult to sample, for example we have no analytical definition
- there exists  $q(x)$ 
  - easier to sample
  - there exists  $K$  such that  $Kq(x) \geq p(x), \forall x$

## Method

- 1 Sample  $\bar{x} \sim q(x)$
- 2 Sample  $u^* \sim U(0, Kq(\bar{x}))$
- 3 If  $u^* \leq p(\bar{x})$  accept and return the sample, otherwise discard it

# Importance sampling

## Context

Assume we have to approximate the expectation of  $f(x)$  wrt  $p(x)$

$$E_{p(x)}[f(x)] = \int_x f(x)p(x)dx$$

If we are able to sample  $n$  values  $x^{(i)}$  from  $p(x)$  then we may apply the approximation

$$\int_x f(x)p(x)dx \approx \frac{1}{n} \sum_{i=1}^n f(x^{(i)})$$

Assume  $p(x)$  hard to sample, but easy to evaluate for all  $x$ . Let  $q(x)$  be some easy distribution to sample. Then,

$$\int_x f(x)p(x)dx = \int_x \frac{f(x)}{q(x)}p(x)q(x)dx \approx \frac{1}{n} \sum_{i=1}^n \frac{p(x^{(i)})}{q(x^{(i)})} f(x^{(i)})$$

The ratios  $p(x^{(i)})/q(x^{(i)})$  are sampling weights, associated to samples from  $q(x)$  (hence avoiding sampling from  $p(x)$ )

## Applying Markov chains

- Markov chain Montecarlo (**MCMC**) is a powerful approach to drawing samples from any distribution  $p(x)$  that we can't sample from directly, but that can be evaluated
- An important property of MCMC is that it is sufficient that  $p(x)$  can be evaluated up to a constant value, that is, it is sufficient  $\pi(x) = Kp(x)$  can be evaluated, where  $K$  is unknown

# Markov chains

## Definition

Given a (possibly infinite) sequence of random variables  $\mathbf{X} = (X_0, X_1, \dots)$  and a **state space**  $\mathcal{X}$  of possible values for all  $X_i \in \mathbf{X}$ , a **Markov chain** on  $\mathbf{X}$  is a stochastic process which defines for each ordered pair  $\langle x_i, x_j \rangle \in \mathcal{X}^2$ , a probability  $p_{ij}$  of transition from  $x_i$  to  $x_j$  such that  $p(X_t = x_i | X_{t-1} = x_j) = p_{ij}$ , for all  $t > 0$ .

## State probability

Given an initial state, that is a value assigned to  $X_0$ , the distribution  $p(X_t = x_j | X_0 = x_k)$  of each random variable  $X_t$  on the set of state can be easily obtained (by matrix multiplication).

# Markov chains

## Stationary distribution

Under suitable conditions on its structure, a Markov chain is **ergodic**, that is the probability  $p(X_t = x_j | X_0 = x_k)$ , as  $n \rightarrow \infty$ ,

- is independent from the initial state

$$p(X_t = x_j | X_0 = x_k) = p(X_t = x_j)$$

- is stationary

$$p(X_t = x_j) = p(X_{t+1} = x_j)$$

# Markov chain Montecarlo (MCMC)

## Idea

Given a hard to sample distribution  $p(x)$ , derive an ergodic Markov chain such that:

- its transition probability  $q(x_i|x_{i-1})$  is easy to sample
- stationary distribution is  $p(x)$

# Markov chain Montecarlo (MCMC)

## Ho to use it?

Given the Markov chain,

- a sequence of random transitions is performed, starting from any initial state (value of  $x$ ).
- apply a certain number of initial transitions (**burn-in time**)
- after that, at each step the value  $\bar{x}$  reached by the MC is tested wrt a predefined criterion: if the test is positive, the value is returned

The returned values are (approximately) distributed as  $p(x)$ : hence their sequence can be used as a sequence of samplings from  $p(x)$

## MCMC methods

Several MCMC methods have been defined, differing each other by the structure of the chain and the acceptance criterion applied.



# Metropolis algorithm

## Idea

After the burning time, let  $x^{(i-1)}$  be the current state and let  $\bar{x}$  be the value produced by a random transition from  $x^{(i-1)}$ , obtained by sampling  $q(x|x^{(i-1)})$

$\bar{x}$  is accepted, and returned as a sample, with probability

$$A(\bar{x}, x^{(i-1)}) = \min \left( 1, \frac{p(\bar{x})}{p(x^{(i-1)})} \right)$$

Notice that if  $\bar{x}$  has higher probability than  $x^{(i-1)}$  with respect to the target distribution  $p(x)$ , it is accepted, while if its probability is smaller, it is accepted with probability equal to the ratio between them.

If  $\bar{x}$  is accepted, then  $x^i = \bar{x}$  becomes the current state, otherwise the current state is not modified, that is  $x^{(i)} = x^{(i-1)}$

## Note

Observe that the same holds if  $\pi(x) = Kp(x)$  is applied in the definition of  $A(\bar{x}, x^{(i-1)})$ ; observe also that the value of  $K$  needs not being known

# Metropolis algorithm

## Structure

At the  $i$ -th iteration:

- 1 Sample a value  $\bar{x}$  from  $q(x|x^{(i-1)})$
- 2 With probability  $A(\bar{x}, x^{(i-1)})$ 
  - let  $x^{(i)} = \bar{x}$ , return  $\bar{x}$
  - else let  $x^{(i)} = x^{(i-1)}$

## Note

For any pair  $x, x'$ , the real probability of transition from  $x$  to  $x'$  is given by the **transition kernel**

$$T(x|x') = q(x|x')A(x, x')$$

# Metropolis algorithm

## Detailed balance

Given the target distribution  $p(x)$ , a Markov chain has the **detailed balance** property with respect to  $p(x)$ , if for each  $x, x'$ ,

$$p(x)q(x'|x) = p(x')q(x|x')$$

that is the probability that at a certain step the current state is  $x$  and the following state is  $x'$  is equal to the one that the current state is  $x'$  and the next state is  $x$ .

In this case,  $p(x)$  is the stationary distribution of the Markov chain. In fact, let us remind that if  $p^*(x)$  is the stationary distribution then by definition

$$p^*(x) = \sum_{x'} q(x|x')p^*(x')$$

and for  $p(x)$  we have

$$p(x) = \sum_{x'} q(x|x')p(x') = \sum_{x'} q(x'|x)p(x) = p(x) \sum_{x'} q(x'|x) = p(x)$$

# Metropolis algorithm

## Uniqueness of the stationary distribution

Even in the case that  $p(x)$  is a stationary distribution, we must be sure that the Markov chain tends to  $p(x)$  for any initial state, that is that it is ergodic.

A sufficient condition for ergodicity is that for all pairs  $x, x'$  the transition probability is positive, that is  $q(x|x') > 0$

# Metropolis algorithm

## Why it does work

- Assume the transition probability distribution is

- symmetric :  $q(x|x') = q(x'|x)$ ,  $\forall x, x'$
- positive:  $q(x|x') > 0$ ,  $\forall x, x'$

- then

- for the probability  $T(x|x') = q(x|x')A(x, x')$  the detailed balance property holds wrt  $p(x)$

$$\begin{aligned} p(x)T(x'|x) &= p(x)q(x'|x)A(x', x) = \min \left( p(x)q(x'|x), \frac{p(x)q(x'|x)p(x')}{p(x)} \right) \\ &= \min (p(x)q(x'|x), p(x')q(x'|x)) = \min (p(x)q(x|x'), p(x')q(x|x')) \\ &= \min \left( \frac{p(x')q(x|x')p(x)}{p(x')}, p(x')q(x|x') \right) = p(x')q(x|x')A(x, x') \\ &= p(x')T(x|x') \end{aligned}$$

hence  $p(x)$  is a stationary distribution

- all transition probabilities are positive, hence the chain is ergodic and always tends to  $p(x)$

# Metropolis-Hastings algorithm

## Idea

- Applied for non symmetric  $q(x|x')$
- In this case, the transition kernel  $T'(x|x') = q(x|x')A'(x, x')$  refers to

$$A'(x, x') = \min \left( 1, \frac{p(x)q(x'|x)}{p(x')q(x|x')} \right)$$

## Why does it works

The detailed balance property still holds for the transition kernel

$$\begin{aligned} p(x)T'(x'|x) &= p(x)q(x'|x)A'(x', x) = \min \left( p(x)q(x'|x), \frac{p(x)q(x'|x)p(x')q(x|x')}{p(x)q(x'|x)} \right) \\ &= \min (p(x)q(x'|x), p(x')q(x|x')) = p(x')q(x|x')A'(x, x') = p(x')T'(x|x') \end{aligned}$$

# Gibbs sampling

## Use

Gibbs sampling is a MCMC applied in cases when:

- $x$  has dimensionality at least 2,  $\mathbf{x} = (x_1, \dots, x_m)$ , with  $m > 1$
- for all  $i = 1, \dots, m$ , the conditional distribution  $p(x_i | \mathbf{x}_{-i})$  is easy to sample, where  $\mathbf{x}_{-i} = \{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_m\}$

## Idea

Instead of sampling the next state in a single step from  $q(\mathbf{x} | \mathbf{x}')$ , a sequence of  $m$  transitions is sampled, each wrt a component  $x_i$  of  $\mathbf{x}$  and to distribution  $p(x_i | \mathbf{x}_{-i})$ .

The basic idea in Gibbs sampling is that rather than probabilistically picking the next state of all at once, a separate probabilistic choice is performed for each of the  $m$  dimensions, with each choice depending on the other  $k - 1$  dimensions.

# Gibbs sampling

## Algorithm structure

- Sample  $m$  values for the initial state  $\mathbf{x}^{(0)} = (x_1^{(0)}, \dots, x_m^{(0)})$
- For  $i = 1, \dots, T$ 
  - For  $k = 1, \dots, m$  sample  $x_k^{(i)}$  from

$$\begin{aligned}
 p(x_k | \mathbf{x}_{-k}^{(i)}) &= p(x_k | x_{-1}^{(i)}, \dots, x_{k-1}^{(i)}, x_{k+1}^{(i-1)}, \dots, x_m^{(i-1)}) \\
 &= \frac{p(x_1^{(i)}, \dots, x_{k-1}^{(i)}, x_k^{(i-1)}, x_{k+1}^{(i-1)}, \dots, x_m^{(i-1)})}{p(x_1^{(i)}, \dots, x_{k-1}^{(i)}, x_{k+1}^{(i-1)}, \dots, x_m^{(i-1)})}
 \end{aligned}$$

- set  $\mathbf{x}^{(i)} = (x_1^{(i)}, \dots, x_m^{(i)})$



# Gibbs sampling

## Why it does work

- it is possible to prove that  $p(\mathbf{x})$  is a stationary distribution of the Markov chain
- also, if distributions  $p(x_i|\mathbf{x}_{-i})$  are never equal to zero, the chain is ergodic, and tends to  $p(\mathbf{x})$

## MCMC and bayesian models

- MCMC can be applied (as it frequently happens) in bayesian inference by observing that the posterior distribution is defined as

$$p(\theta|\mathbf{X}) = \frac{p(\mathbf{X}|\theta)p(\theta)}{p(\mathbf{X})} = \frac{p(\mathbf{X}|\theta)p(\theta)}{Z}$$

where  $Z$  is usually hard to compute

- Let us remind that MCMC is able to sample a distribution  $p(\mathbf{x})$  assuming that a proportional function  $\pi(\mathbf{x}) = Kp(\mathbf{x})$  can be evaluated, for any unknown  $K$
- Thus, samples of the posterior distribution of parameters can be obtained if both the prior  $p(\theta)$  and the likelihood  $p(\mathbf{X}|\theta) = \prod_i p(\mathbf{x}_i|\theta)$  can be evaluated for any value  $\theta$

# Sampling the evidence

- Actually, the evidence

$$p(\mathbf{X}) = \int p(\mathbf{X}|\theta)p(\theta)d\theta$$

could be explicitly evaluated, if necessary, as the average of a set of  $m$  values

$$p(\mathbf{X}|\theta_i) \quad i = 1, \dots, m$$

computed from the set of samples  $\theta_1, \dots, \theta_m$  of  $p(\theta)$

## Sampling the predictive distribution

- For what regards the predictive distribution

$$p(\mathbf{x}|\mathbf{X}) = \int_{\theta} p(\mathbf{x}|\theta)p(\theta|\mathbf{X})d\theta$$

the same considerations apply, averaging the set of values

$$p(\mathbf{x}|\theta_i) \quad i = 1, \dots, m$$

computed from the set of samples  $\theta_1, \dots, \theta_m$  of the posterior distribution  $p(\theta|\mathbf{X})$